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A NUCLEONIC DIFFUSION INVERSE PROBLEM

by

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ABSTRACT

This paper is concerned with the problem of the indirect determination of the nucleonic spectrum at the top of the atmosphere, starting from the knowledge of its flux measured at some atmosphere depth $x(g/cm^2)$.

The differential energy spectrum of the primary cosmic ray nucleons at the atmospheric depth $x(g/cm^2)$ is given in a paper of G. Brooke, P.J. Haymann, Y. Kamya and A.W. Wolfendale, published about twenty years ago[1].

Their result is expressed by the following relation

$$F(x,E) = e^{-x/\lambda} \sum_{n=0}^{\infty} \frac{(x/\lambda)^n}{n!} \frac{1}{(1-K)^n} G(\frac{E}{(1-K)^n}) \qquad (1)$$

where G(E)dE is the primary energy spectrum (in the top of the atmosphere); F(x,E)dE is the spectrum of the nucleons at the atmospheric depth $x(g/cm^2)$; K is the average inelasticity and λ is the average interaction length of the nucleon's collisions with the air nuclei. The function G(E) is assumed to be a reasonable good function in order to ensure the convergence of the series[2]. In the top of the atmosphere we have $x = 0$ and F(0,E) reduces to G(E).

About two years ago, Prof. C.M.G. Lattes proposed me the question of finding G(E) when the function F(x,E) is assumed to be known. For this purpose we give to $e^{x/\lambda}$ F(x,E) the infinite sequence of values $C_\nu = e^{x/\lambda} F(x, \frac{E}{(1-K)^\nu})$ where $\nu = 0,1,2,\ldots$, and establish the following system of infinite equations with the infinite unknowns $G(\frac{E}{(1-K)^\nu})$ where $\nu = 0,1,2,\ldots$, and $y = \frac{x}{\lambda(1-K)}$:

$$\begin{array}{llll} G(E) + \frac{y}{1!} G(\frac{E}{1-K}) & + \frac{y^2}{2!} G(\frac{E}{(1-K)^2}) & + \ldots & = C_0 \\ 0 \qquad G(\frac{E}{1-K}) & + \frac{y}{1!} G(\frac{E}{(1-K)^2}) & + \ldots & = C_1 \\ 0 \qquad\qquad 0 & \quad G(\frac{E}{(1-K)^2}) & + \ldots & = C_2 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{array} \qquad (2)$$

Now we write this system in the matrix form

$$AX = C \tag{3}$$

The matrix elements of the matrix A are

$$a_{i\kappa} = \begin{cases} 0 & \text{for } \kappa < i \\ \dfrac{y^{\kappa-i}}{(\kappa-i)!} & \text{for } \kappa \geq i \\ i,\kappa = 0,1,2,\ldots \end{cases}$$

C is the vector of components C_ν, and X is the vector of components $X_\nu = G(\frac{E}{(1-K)^\nu})$, where $\nu = 0,1,2,\ldots$

To solve the system (3), first we remember that "if A admits a right hand reciprocal B, (that is AB = I, where I is the identical matrix), then X = BC is a solution of AX = C, provided the product ABC is associative". In fact, if the product ABC is associative, and B is a right hand reciprocal of A, we have

$$A(BC) = (AB)C = C \tag{4}$$

and equation (4) shows that X = BC is a solution of AX = C.

Now we proceed to show that the matrix

$$B = \begin{bmatrix} 1 & -y & y^2/2! & -y^3/3!\ldots\ldots \\ 0 & 1 & -y & y^2/2!\ \ldots\ldots \\ 0 & 0 & 1 & -y\ \ldots\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \end{bmatrix} = |b_{i\kappa}| \tag{5}$$

$i,\kappa = 0,1,2,\ldots$

is a right reciprocal of A.

Note that

$$b_{i\kappa} = \begin{cases} 0 & \text{for } \kappa < i \\ y^{\kappa-i}(-1)^{\kappa-i}/(\kappa-i)! & \text{for } \kappa \geq i \end{cases}$$

$$i,\kappa = 0,1,2,\ldots$$

The matrix element of the product AB is

$$c_{ij} = \sum_{\kappa=i}^{\kappa=j} a_{i\kappa} b_{\kappa j} = \sum_{\kappa=i}^{\kappa=j} \frac{y^{j-i}}{(\kappa-i)!(j-\kappa)!} (-1)^{j-\kappa}$$

If we put $j-i = n$, $K-i = \nu$ it results $j-\kappa = n-\nu$ and

$$c_{ij} = \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu} y^n}{\nu!(n-\nu)!} = \frac{y^n}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{n-\nu}$$

But, for $n \neq 0$ the right hand side of this equality is zero because it is the product of $y^n/n!$ by $(1-1)^n$.
For $n = 0$, we have $j = i$ and it reduces to one.

Therefore we have

$$\begin{cases} c_{ij} = 0 & \text{for } i \neq j \\ c_{jj} = 1 & \text{for } j = 0,1,2,\ldots \end{cases}$$

This prove that $AB = I$ so that B is a right reciprocal of A. The first condition of the theorem is then satisfied.

Now we proceed to show that the second condition of the theorem is also satisfied, that is we shall prove the associativity of the product ABC. The product BC is the vector of components

$$S_i = \sum_{\kappa=0}^{\infty} b_{i\kappa} C_\kappa = \sum_{\kappa=i}^{\infty} (-1)^{\kappa-i} \frac{y^{\kappa-i}}{(\kappa-i)!} C_\kappa$$

The product A(BC) is the vector of components

$$L_j = \sum_{i=0}^{\infty} a_{ji} S_i = \sum_{i=j}^{\infty} \frac{y^{i-j}}{(i-j)!} \sum_{\kappa=i}^{\infty} (-1)^{\kappa-i} \frac{y^{\kappa-i}}{(\kappa-i)!} C_\kappa$$

$$= \sum_{i=j}^{\infty} \sum_{\kappa=i}^{\infty} \frac{y^{\kappa-j}(-1)^{\kappa-i}}{(i-j)!(\kappa-i)!} C_\kappa$$

If we put: $\kappa-j = n$, $\kappa-i = n-\nu$ it results that $\kappa = n+j$ and $i-i = \nu$, so that

$$L_j = \sum_{n=0}^{\infty} \sum_{\nu=0} \frac{y^n(-1)^{n-\nu}}{\nu!(n-\nu)!} C_{n+j} =$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{n-\nu} C_{n+j} \quad .$$

But, for $n \neq 0$ the right hand side of this equality is zero, according to the same identitity which was used before. For $n = 0$, it reduces to $L_j = C_j$, $(j = 0,1,2,\ldots)$. This proves that $A(BC) = C$. Since we have shown that $AB = I$, we have also $(AB)C = C$. Then the product ABC is associative, the second condition of the theorem is satisfied and we can state that $X = BC$ is a solution of the equation (3).

To prove that the above solution is unique, we proceed as follows. First we must prove that B is also a left reciprocal of A, that is $BA = I$. The proofs is the same to that we used to demonstrate that $AB = I$, thus we can omit it.

Now to show that the solution $X = BC$ is unique, suppose that the system $AX = C$ has another solution X', such that $AX' = C$. Then we have $A(X-X') = 0$. But B is a left reciprocal of A, therefore $BA(X-X') = X-X' = 0$. Since B is different from the zero matrix this implies $X = X'$. This prove the unicity of the solution. From the solution $X = BC$ we have

$$X_i = \sum_{\nu=0}^{\infty} b_{i\nu} C_\nu \qquad i = 0,1,2,... \tag{6}$$

Now note that, the component $X_0 = G(E)$ gives us the solution of the physical problem which is the object of this paper, so that we arrive to the following explicit solution of our problem:

$$G(E) = \sum_{\nu=0}^{\infty} b_{0\nu} C_\nu = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu y^\nu}{\nu!} C_\nu$$

$$= e^{x/\lambda} \sum_{\nu=0}^{\infty} (-1)^\nu \frac{(x/\lambda)^\nu}{\nu!} \frac{1}{(1-K)^\nu} F(x \frac{E}{(1-K)^\nu}) \tag{7}$$

With this formula we can calculate the primary energy spectrum of the cosmic radiation on the top of the atmosphere, provided we know the values of $F(x, \frac{E}{(1-K)^\nu})$ for $\nu = 0,1,2,...$ at the atmospheric depth $x(g/cm^2)$, which are supposed to be known.

Clearly to obtain the solution for any other component X_j of the vector X it is sufficient to substitute E in equation (7) by $E/(1-K)^j$.

Let we now consider the question of convergence. The flux $F(x,E)$ is a <u>datum</u> of the problem. The observation shows that it decreases very rapidly with the energy. From the very nature of the physical problem, the integral $\int_a^\infty F(x,E)dE$ must exist in some interval $[a,\infty)$, $a > 0$. Clearly the condition that $F(x,a)$ be <u>bounded</u> in the interval $[a,\infty)$ is less restrictive than the above conditions, and is sufficient to ensure the convergence of all the X_j in $[a,\infty)$. In fact, the condition $|F(x,E)| < M$, for $M > 0$, and $a \leq E < \infty$, implies $|F(x, \frac{E}{(1-K)^\mu})| \leq M$, whatever be $\mu = 0,1,2,...,$ for any fixed $x \geq 0$ and $a \leq E < \infty$.

Thus we have

$$|X_j| \leq e^{+x/\lambda} \sum_{\nu=0}^{\infty} \left|\frac{(-x/\lambda)^{\nu}}{\nu!} \frac{1}{(1-K)^{\nu}}\right| \times M$$

$$= M\, e^{\frac{x}{\lambda}\frac{2-K}{1-K}}$$

for every $x \geq 0$ and $0 < K < 1$. (Note that K is the average inelasticity of the collisions).

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