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Comments on Dirac-like Monopole, Maxwell and Maxwell-Chern-Simons Electrodynamics in D=(2+1)

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Abstract

Classical Maxwell and Maxwell-Chern-Simons (MCS) Electrodynamics in (2+1)D are studied in some details. General expressions for the potential and fields are obtained for both models, and some particular cases are explicitly solved. Conceptual and technical difficulties arise, however, for accelerated charges. The propagation of electromagnetic signals is also studied and their reverberation is worked out and discussed. Furthermore, we show that a Dirac-like monopole yields a (static) tangential electric field. We also discuss some classical and quantum consequences of the field created by such a monopole when acting upon an usual electric charge. In particular, we show that at large distances, the dynamics of one single charged particle under the action of such a potential and a constant (external) magnetic field as well, reduces to that of one central harmonic oscillator, presenting, however, an interesting angular sector which admits energy-eigenvalues. For example, the quantisation of these eigenvalues yields a Dirac-like condition on the product of the charges. Moreover, such eigenvalues are shown to feel (and respond) to discrete shift of the angle variable. We also raise the question on the possibility of the formation of bound states in this system.

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Introduction

Field-theoretic models defined in a (2+1)-dimensional space-time have been studied for nearly two decades[1, 2]. Actually, lower-dimensional models have provided many interesting results which do not take place in the (3+1)D world, e.g., Schwinger' mechanism in QED_2[3] and fractional statistics in three dimensions [4]. Consequently, lower-dimensional theories cannot be considered as mere lower limits of four-dimensional ones; they have rather revealed characteristics that are intrinsic to its dimensionality.

On the other hand, some (2+1)D theories, whenever supplemented by a Chern-Simons' term, turn out to exhibit a new interesting physical content, as for example, Maxwell and Einstein-Hilbert actions [2, 9]. Furthermore, it has been claimed that such models (mainly those in the context of MCS) have relevance for a deeper understanding of some Condensed Matter phenomena, like the Quantum Hall Effect (QHE)[5] and High-Tc Superconductivity [6] (see also, Ref. [7, 8]).

Although Maxwell and Maxwell-Chern-Simons (mainly the latter, in both Abelian and non-Abelian frameworks) have attracted a great deal of efforts, it is curious that one has not provided an "electrodynamical body" (Liénard-Wiechert-type potentials, Larmor-like formula and so forth) for such (say, Abelian) theories which would be similar to the one we have for (3+1)D Maxwell[3]. Thus, we shall try to draw the attention to the fact that the "lack" of a complete "electrodynamical body" is related to some serious difficulties, for instance, in calculating A_μ (and $F_{\mu\nu}$) for a single accelerated point-like charge. In view of that, a Larmor-like expression relating energy-flux (radiation) and the acceleration of the sources is still missing.

We start the present work by studying the Maxwell (massless) case. Some results are dis-

[3]Although in a different approach, a classical analysis of the non-Abelian case (SU(2), more precisely) was performed by D'Hoker and Vinet[10].

cussed and a number of difficulties are pointed out. Following, we add a Chern-Simons term to the former model and some consequences of such a procedure are worked out. Going on, we analyse the issue concerning the introduction of a Dirac-like monopole within both models and some properties of its field. Some effects of its potential on an usual electric charge are discussed in both classical and quantum (non-relativistic) frameworks. We close this paper by pointing out some Conclusions and Prospects.

i) Classical Maxwell Electrodynamics in D=(2+1)

Let us consider the D=(2+1) Maxwell Electrodynamics (MED$_3$) Lagrangian: [4]

$$\mathcal{L}_{MED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu \,. \tag{1}$$

The invariance of the action under local Abelian gauge transformations, $A_\mu(x) \to A_\mu(x) - \partial_\mu\Lambda(x)$, is ensured by the conservation of the 3-current, say, $\partial_\mu j^\mu = 0$. Moreover with the usual definition of the field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we get $F_{\mu\nu} = (F_{0i} = +(\vec{E})_i; F_{12} = B)$. Next, the field-strength clearly satisfies $\partial_\mu F^{\mu\nu} = j^\nu$ and $\partial_\mu \tilde{F}^\mu = 0$, whence there follow:

$$\nabla B = \partial_t \vec{E}^* + \vec{j}^* \,, \quad \nabla\cdot\vec{E} = \rho \quad \text{and} \quad \nabla\cdot\vec{E}^* = \partial_t B \,,$$

where we have defined $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\nu\kappa}F_{\nu\kappa} = (+B; -\vec{E}^*)$, with the components of a dual-vector given by $(\vec{U}^*)_i = \epsilon_{ij}U_j$.

The dynamical equation for the more basic quantity, A_μ, reads (in the gauge $\partial_\mu A^\mu = 0$):

$$\Box A_\mu(x) = j_\mu(x) \,. \tag{2}$$

The solutions to this wave-equation may be readily obtained by means of the well-known Green's function method (or by applying the Hadamard's *Descent Method*, see Ref. [11] for further details). However, as we shall see, in (2+1)D the Green functions, $G^{2+1}(x-y)$, present a quite different behaviour respect to their (3+1)D-counterpart: the support of G^{2+1} lies no longer only on the surface of the light-cone, where $(x-y)^2 = 0$; it rather spreads throughout the whole internal region, $(x-y)^2 > 0$ (blowing up as $(x-y)^2 \to 0_+$). As we shall see, this

[4] Our conventions read: $diag(\eta_{\mu\nu}) = (+,-,-)$, greek letters running 0,1,2; the 2-D spatial coordinates are labeled by latin letters running 1,2; and $\epsilon_{012} = \epsilon^{012} = \epsilon_{12} = \epsilon^{12} = +1$.

will lead to profound modifications in planar electrodynamics whith respect to its 3-spatial counterpart, say, (3+1)D Maxwell theory.

We solve eq. (2) by taking: $\Box_x G^{2+1}(x-y) = \delta^{2+1}(x-y) \Rightarrow \tilde{G}^{2+1}(k) = -1/k^2$, and hence:

$$G^{2+1}(x-y) = \frac{1}{(2\pi)^3}\int d^{2+1}k\,\frac{e^{ik(x-y)}}{k^2}\,,$$

which, after a suitable choice of the integration contour on the k^0-complex plane and the subsequent integration yields (the advanced function is easily got by introducing a $\Theta(-\tau)$; Θ is the usual step-function):

$$G^{2+1}_{ret}(x-y) = -\frac{\Theta(\tau)}{2\pi}\int_0^\infty J_0(kr)\sin(k\tau)dk = -\frac{\Theta(\tau)}{2\pi}\frac{\Theta(\tau^2-r^2)}{\sqrt{\tau^2-r^2}}\,, \tag{3}$$

where $\tau = x^0 - y^0$ and $r = |\vec{x}-\vec{y}|$. The integral above may be found, for example, in Ref.[13] (on page 731 and eq. 6.671-7). The final form of such a function confirms what we have already stated about its support: instead of a Dirac delta, we get a step-function (times a rational one) of the space-time interval. Furthermore, we shall see later that such an aspect will lead us to some new interesting properties of this model whenever compared to the (3+1)D Maxwell theory, namely, the reverberation of signals and the "lack"of a Larmor-like formula for the radiated power.

Next, by taking a single point-like charge, $j^\mu(y) = q\int_{-\infty}^{+\infty}\dot{z}^\mu(s)\delta^{2+1}(y-z(s))ds$, we get the general form for its potential (we have omitted the homogeneous part of the potential):

$$A^\mu_{ret}(x) = +\frac{q}{2\pi}\int_{-\infty}^{+\infty}\Theta(x^0-z^0(s))\frac{\Theta[(x-z(s))^2]}{\sqrt{(x-z(s))^2}}\,\dot{z}^\mu(s)\,ds\,, \tag{4}$$

with $(x-z)^2 = [(x^0-z^0)^2 - |\vec{x}-\vec{z}|^2]$. The expression for the field-strength is also obtained in the usual way, and reads:

$$F_{\mu\nu}(x) = \frac{q}{2\pi}\int_{-\infty}^{+\infty}\frac{\Theta(x^0-z^0)\Theta((x-z)^2)}{P^2\sqrt{(x-z)^2}}\left[\ddot{z}_\nu\,(x-z)_\mu\,P + \dot{z}_\nu\,(x-z)_\mu(1-Q) - \mu\leftrightarrow\nu\right]ds. \tag{5}$$

Here, it is worthy noticing that, in general, we do not get to solve the expressions above. Actually, we have tried to solve elementary accelerated motions, say parabolic and hyperbolic ones. Unfortunately, we have found serious difficulties in performing some integrals that are highly non-trivial and plagued with serious divergences that have to be suitable handled[5]. In (3+1)D, the scenario is quite different, because we have a $\delta^{3+1}((x-z)^2)$ (instead of $\Theta((x-z)^2)/\sqrt{(x-z)^2}$) which, in turn, implies in a straightforward factorisation of the integral in s-variable, by picking up only those points for which $(x-z)^2 = 0$.

Hence, we conclude that the lack of closed analytic expressions for A_μ (and $F_{\mu\nu}$) in the case of an arbitrary motion (Liénard-Wiechert-type expressions) is deeply related to the failure of the Huyghens' principle, since the solutions to the $\Box$-operator in (2+1)D, G^{2+1}, do not satisfy such a principle (indeed, the same happens for any G^{n+1}, n even. See, for example, Ref.[11, 12, 15, 16]).

On the other hand, even the static case (the constant motion may be easily got by a Lorentz' boost) reveals some of the new characteristics of the model. Thus, by taking $z^\mu = (s, \vec{0}) \Rightarrow \dot{z}^\mu = (1, \vec{0})$, we get:

$$A^\mu(x) = \begin{cases} A^0(\vec{r},t) = -\frac{q}{2\pi}\ln|\vec{r}| + \frac{q}{2\pi}\lim_{\tau\to+\infty}\left(\ln|\tau + \sqrt{\tau^2 - r^2}|\right) \\ \vec{A}(\vec{r},t) = 0 \end{cases} \tag{6}$$

$$F_{\mu\nu}(x) = \begin{cases} F_{0i}(\vec{r},t) = +\frac{q}{2\pi}\frac{r^i}{r^2} - \frac{q}{2\pi}r^i \lim_{\tau\to r^+}\left(\frac{\tau}{r^2\sqrt{\tau^2 - r^2}}\right) \\ F_{ij}(\vec{r},t) = 0 \end{cases} . \tag{7}$$

Here, we notice that, besides the well-known $\ln|\vec{x}|$-behaviour of the potential in planar Electrodynamics, there is an extra term which explicitly diverges. Such a term clearly represents the asymptotic value of the potential as $|\vec{x}| \to +\infty$ and is directly related to the *infrared divergence* of the theory. Indeed, by calculating $A_\mu(x)$ by means of $\tilde{A}_\mu(k)$ (its Fourier transform), we may clearly see that such a term arises when the mass term is set to zero. On the other hand, the explicitly divergent term appearing in the $F_{\mu\nu}$ above may be removed by a suitable subtraction procedure, which is possible because of such a quantity vanishes asymptotically. [Among others, such subtleties will appear in Ref.[17]].

Still concerning the general $F_{\mu\nu}$-form, eq. (5), there remains an interesting issue to be pointed

[5]It was already pointed out in the literature that (2+1)D Electrodynamics indeed imposes additional troubles in calculating some quantities; for example, in Ref.[14], the author discusses some difficulties brought about by the logarithmic behaviour of the potential.

out. By taking into account the terms proportional to the acceleration, $\ddot{z}(s)$, which are those that effectively contribute to the energy-flux and so, to a Larmor-like formula, we notice that such terms are proportional to $\int ds/R(s)$ [$R(s)$ being essentially $\sqrt{(x-z)^2}$] and might surprisingly lead us to the result that radiation in (2+1)D no longer falls off with r^{-1} (as we know from (3+1)D electrodynamics); instead, it increases proportionally to $\ln |\vec{r}|$! As far as we have seen, this seems to be not an impossibility, it rather might come to be another peculiar property of (2+1)D Electrodynamics.

Another peculiar characteristic of the model concerns the propagation of electromagnetic signals. In order to see this, let us start by considering some well-known results from (3+1)D. First, let us take the following charge configuration: $\rho(\vec{y}, t') = q\delta^3(\vec{y})\Theta(t')$. Its scalar potential reads:

$$\Phi_{sup}(\vec{x}, t) = -\frac{q}{4\pi}\frac{\Theta(t - |\vec{x}|)}{|\vec{x}|},$$

which is clearly the *linear superposition* of Φ-pulses,

$$\Phi_{pulse}(\vec{x}, t) = -\frac{q}{4\pi}\frac{\delta(t - |\vec{x}|)}{|\vec{x}|},$$

produced by the ("hypothetical") configuration $\rho(\vec{y}, t') = q\delta^3(\vec{y})\delta(t')$ (we placed the charge at the origin in the time $t' = 0$, and we imediately took it away). The previous results state us that sharply produced signals (pulses) may be at later times recorded as sharp ones; such a "sharpness conservation" is precisely what leads us to the linear superposition above.

Next, let us see how the same scenario takes place in (2+1) dimensions. Let us start by considering the "hypothetical" configuration: $\rho(\vec{y}, t') = q\delta^2(\vec{y})\delta(t')$. Now, its potential reads:

$$\Phi_{pulse}(\vec{x}, t) = \frac{q}{2\pi}\frac{\Theta(t - |\vec{x}|)}{\sqrt{t^2 - |\vec{x}|^2}}. \qquad (8)$$

Clearly, although such a signal has been sharply sent (at $t = 0$ it was just at $|\vec{x}| = 0$) it cannot later be recorded as a sharp one: the pulse develops a "tail" (its spreading in time) and so it reverberates. Therefore, we now need a *very long time to record* a sharp signal sent at an earlier time. Next, we obtain the superposed case, which is got from $\rho(\vec{y}, t') = q\delta^2(\vec{y})\Theta(t')$, and reads:

$$\Phi_{sup}(\vec{x},t) = +\frac{q}{2\pi}\ln\left(\frac{t+\sqrt{t^2-|\vec{x}|^2}}{|\vec{x}|}\right)\Theta(t-|\vec{x}|)\,. \tag{9}$$

Furthermore, we may see another peculiar characteristic of the signals: they superpose in a logarithmic (and not linear) way. Such a sort of superposition leads us to an interesting point if we compare with previous results (eqs. (8) and (9)), when $t \gtrsim |\vec{x}|$ (t equal or slightly greater than $|\vec{x}|$): while the single pulse' potential, eq. (8), appears to be very strong, the contrary happens to the superposed case, which is very weak there! A somewhat "paradoxal" superposition of signals. However, as time goes by, things straighten up: while single pulses fall off, their superposition appears to broaden the potential. [The expressions for the electric field are also easily obtained and exhibit similar phenomenon concerning reverberation, while the superposition is "better-behaved" than the Φ-potential]. Moreover, notice that as (and only as) $t \to \infty$, we recover the static potential, eq. (6):

$$\Phi(\vec{x}) = -\frac{q}{2\pi}\ln|\vec{x}| + \frac{q}{2\pi}\lim_{t\to\infty}\ln|t+\sqrt{t^2-|\vec{x}|^2}|.$$

Thus, the results discussed above bring an additional complication to the (classical, at least) electrodynamics of a system of interacting charges, since even single pulses (of potentials or fields) emitted by an electric charge will demand a very long time to be completely '*felt*' by another one. In other words, even the static (for concreteness) feature of the potentials and fields *will be no longer* determined only by the (static) configuration of the charges; it rather demands a very long time to actually happen, since at finite times the electromagnetic quantities are time-dependent.

Indeed, in (2+1)D, we may regard the classical propagation of a signal as if the wave front travels with velocity c, and decreasing in a such a way that the back point of the signal has null-velocity (this is exactly what eq. (8) says). These effects appear to be consequences of the failure of the Huyghens' principle (the reverberation itself) as well as of the dimensionality of the space-time (logarithmic superposition). Physically, such a propagation may be compared, to some extent, to the propagation of disturbances in an infinite planar physical medium, like an infinitely extended planar membrane.

Actually, similar conclusions concerning the reverberation of signals were already discussed by other authors [11, 15]. For instance, Courant and Hilbert in their classical book[12] analyse such a propagation and, by virtue of the failure of the Huyghens principle, they conclude that D'Alembertian' waves (in general), even if sharply produced, cannot be later recorded with the same sharpness.

Furthermore, we would like here to raise a question in view of what we have understood about the spreading that unavoidably affects the classical propagation of sharp signals in (2+1)D. By facing an electromagnetic signal rather as a wave, reverberation affects its propagation and we can no longer speak of sharp pulses; on the other hand, if we are to give the electromagnetic signal the status of a particle, we wonder whether the concept of photon as a localised energy packet should not be reassessed in the framework of planar Electromagnetism. An analogous question is pertinent in the MCS-case (next section). There, however, by virtue of the mass gap, reverberation is more expected to happen, since massive (Klein-Gordon or Proca-like) fields exhibit such a phenomenum even in (3+1) dimensions [18, 19].

ii) Maxwell-Chern-Simons model

Let us write the Lagrangian for the Maxwell-Chern-Simons Electrodynamics (MCS):

$$\mathcal{L}_{MCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\kappa}A_\mu\partial_\nu A_\kappa + j_\mu A^\mu \, , \tag{10}$$

where $\frac{m}{2}\epsilon^{\mu\nu\kappa}A_\mu\partial_\nu A_\kappa = \frac{m}{2}A_\mu\tilde{F}^\mu$ is the (Abelian) Chern-Simons term, which provides a mass for the boson, A_μ, without breaking the original local gauge symmetry of the action [2], $S_{MCS} = \int d^{2+1}x\, \mathcal{L}_{MCS}(x)$. Moreover, the mass parameter, m, may be taken to be positive or negative. Depending on the choice of its signal, the 'massive photon' will carry polarisation equal to +1 ($m > 0$) or -1 ($m < 0$).[6] Notice, however, that in both cases, massless or massive,

[6]Talking about *spin* in (2+1) dimensions, we should be careful, since its meaning is rather different from its (3+1)D-counterpart. In fact, for a massive particle, its "spin" in (2+1)D has some similarities with the helicity of its massless correspondent in (3+1)D: only the positive, +1, or negative, -1, polarisations may take place, while no component of zero-polarisation appears. See, for example, Binegar's paper in Ref.[4]. See also

the "photon" carries *only one physical degree-of-freedom*, which highlights its 'scalar nature'. Actually, since its mass is given by means of a topological mass term, we do not expect to have any additional degree-of-freedom.

The field-strength obeys $\partial_\mu F^{\mu\nu} = j^\nu + m\tilde{F}^\nu$ and also the Bianchi identity, $\partial_\mu \tilde{F}^\mu = 0$, whence there follow:

$$\nabla B = \partial_t \vec{E}^* + \vec{j}^* + m\vec{E}\,, \quad \nabla \cdot \vec{E} = \rho + mB \quad \text{and} \quad \nabla \cdot \vec{E}^* = \partial_t B\,.$$

Now, the A_μ-potential satisfies (where we have introduced a gauge-fixing term whose parameter is α):

$$\left(\Box\eta^{\mu\nu} - m\epsilon^{\mu\nu\kappa}\partial_\kappa - \frac{\alpha+1}{\alpha}\partial^\mu\partial^\nu\right) A_\nu(x) \equiv \mathcal{O}^{\mu\nu}_x A_\nu(x) = j^\mu(x).$$

After having obtained $\mathcal{O}^{-1}$, we remove the gauge-fixing and longitudinal terms. Next, integrating over the k-variable, the solutions may be written in terms of Green's functions:

$$A^\mu(x) = \int d^{2+1}y \left[G^{2+1}(x-y)\eta^{\mu\nu} + \frac{m}{m^2}\left(G^{mass}(x-y) - G^{2+1}(x-y)\right)\epsilon^{\mu\nu\kappa}\partial_\kappa\right] j_\nu(y)\,, \quad (11)$$

where the massive Green' function is given by:

$$G^{mass}_{\substack{ret \\ adv}}(x-y) = -\frac{1}{2\pi}\frac{\Theta[t^2-r^2]\cos\left(m\sqrt{t^2-r^2}\right)}{\sqrt{t^2-r^2}}\Theta[\pm\, t].$$

with $t = x^0 - y^0$ and $r = |\vec{x} - \vec{y}|$. We clearly see that, as $m \to 0$, then $G^{mass} \to G^{2+1}$. Similarly to its massless counterpart, G^{mass} does not satisfy the Huyghens' principle: again, the support spreads throughout the whole region $(x-y)^2 \geq 0$.

Next, the general expression for A_μ, as produced by a single point-like charge, takes the form:

$$\begin{aligned} A^\mu(x) = \quad & +\frac{q}{2\pi} - \int_{-\infty}^{+\infty} ds\, \Theta(x^0 - z^0(s))\Theta[(x-z)^2] \left\{ \frac{\cos(m\sqrt{(x-z)^2})}{\sqrt{(x-z)^2}}\dot{z}^\mu + \right. \\ & + \frac{m}{m^2}\epsilon^{\mu\nu\kappa}\left[\dot{z}_\nu (x-z)_\kappa \left(\frac{m\sin(m\sqrt{(x-z)^2})}{(\sqrt{(x-z)^2})^2} + \frac{\cos(m\sqrt{(x-z)^2}) - 1}{(\sqrt{(x-z)^2})^3}\right) + \right. \\ & \left.\left. + \ddot{z}_\nu \dot{z}_\kappa \left(\frac{\cos(m\sqrt{(x-z)^2}) - 1}{\sqrt{(x-z)^2}}\right)\right]\right\} \,, \end{aligned} \quad (12)$$

Ref. [20].

from which we may notice the difficulties which arise in trying to solve it for arbitrary motions of the charge (indeed, the general solution to such an expression deeply depends on the massless one). There is also a new sort of term, not present in the massless case, which is explicitly acceleration-dependent (a radiation-like term, the last one in the eq. above).[7] Such a term, in turn, will lead to another one that explicitly depends on d^3z/ds^3 in the expression for $F_{\mu\nu}$: a back-reaction-like term. By virtue of its length, we shall not give the explicit form for this field here. We refer the reader to Ref.[17], where a detailed derivation of the results above will be presented. We only anticipate that the possibility that the radiation increases like a $\ln|\vec{r}|$ also takes place here.

Even though a general solution for A_μ (and $F_{\mu\nu}$) for arbitrary motions appears to be far off our possibilities, it is instructive to work out static quantities which already exhibit some of the new properties brought about by the Chern-Simons term. They read as follows:

$$A^\mu(x) = \begin{cases} \Phi(\vec{x}) = +\frac{q}{2\pi}K_0(m|\vec{x}|) \\ A^i(\vec{x}) = -\frac{q}{2\pi}\frac{m}{m^2}\frac{\epsilon^{ij}x^j}{|\vec{x}|}\left(\frac{1}{|\vec{x}|} - m\,K_1(m|\vec{x}|)\right) \end{cases}, \tag{13}$$

$$F_{\mu\nu}(x) = \begin{cases} E^i(\vec{x}) = -\frac{q}{2\pi}\frac{mx^i}{|\vec{x}|}K_1(m|\vec{x}|) \\ B(\vec{x}) = +\frac{q}{2\pi}mK_0(m|\vec{x}|) = m\Phi(\vec{x}) \end{cases}. \tag{14}$$

Now, we see that A_μ acquires a better asymptotic behaviour: $A_\mu \to 0$ as $|\vec{x}| \to \infty$ (at large distances, K_0 and K_1 roughly behave as $e^{-|m\vec{x}|}/\sqrt{|m\vec{x}|}$). Indeed, even the long-range sector of $\vec{A}$ now decreases as $|\vec{x}|^{-1}$ (such a sector is related to the well-known non-dynamical massless pole and also to the possibility of topological objects such as vortex-like magnetic field). In addition, due to the Chern-Simons term, the charge now produces a non- vanishing static magnetic field. Nevertheless, this does not lead to radiation at all. Indeed, it is easily to show that $\nabla\cdot\vec{S}^* = \nabla\cdot(\vec{E}^*B) = 0$, with $\vec{S}^*$ being the Poynting vector. We should now comment on the short-distance behaviour of these quantities. By recalling that, for $z \ll 1$ ($z > 0$), the modified Bessel functions behave as $K_0(z) \approx -\ln(z/2)$ and $K_1(z) \approx z^{-1}$, we see that, near the charge, Φ and B diverge as $\ln|m\vec{x}|$ while $\vec{E}$ blows up as $|\vec{x}|^{-1}$. The vector potential, on

[7] Although in a different context, we would like to point out the works of Ref.[21], where the action for a relativistic charged particle minimally coupled to an Abelian Chern-Simons is shown to be equivalent to a higher-derivative action whenever the U(1)-field is integrated out.

the other hand, exhibits a very peculiar behaviour: it vanishes as $|\vec{x}| \to 0$! Such a result is actually in accordance with eq. (11): the A^i components should vanish as $\sqrt{t^2 - |\vec{x}|^2} \to 0$, what indeed happens, in the static case, when we get very close to the charge. Hence, we see that classical MCS model recover the massless one when we get very close to the charge.

Next, we shall treat the propagation of signals in the Maxwell-Chern-Simons framework. We shall start by obtaining and analysing the single pulse case, which is produced by $\rho(\vec{y}, t') = q\delta^2(\vec{y})\delta(t')$. The quantities read (we have omitted $\Theta(t - |\vec{x}|)$ in all expressions below):

$$\Phi_{pulse}(\vec{x}, t) = +\frac{q}{2\pi}\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2})}{\sqrt{t^2 - |\vec{x}|^2}}\,, \tag{15}$$

$$A^i_{pulse}(\vec{x}, t) = -\frac{q}{2\pi}\frac{m}{m^2}\epsilon^{ij}\partial_j\left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2}) - 1}{\sqrt{t^2 - |\vec{x}|^2}}\right)\,,$$

for the potentials, while the fields are:

$$E^i_{pulse}(\vec{x}, t) = +\frac{q}{2\pi}\partial_i\left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2})}{\sqrt{t^2 - |\vec{x}|^2}}\right) + \frac{q}{2\pi}\frac{m}{m^2}\epsilon^{ij}\partial_t\partial_j\left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2}) - 1}{\sqrt{t^2 - |\vec{x}|^2}}\right) \tag{16}$$

$$B_{pulse}(\vec{x}, t) = -\frac{q}{2\pi}\frac{m}{m^2}\nabla^2_x\left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2}) - 1}{\sqrt{t^2 - |\vec{x}|^2}}\right)\,.$$

The reverberation of the pulse is evident: it is very strong when $t \gtrsim |\vec{x}|$ and decreases as time goes by, vanishing as $t \to \infty$. The superposed case may also be readily obtained (essentially, by integrating expressions above from $|\vec{x}|$ to t). For example, the scalar potential superposes as:

$$\Phi_{sup}(\vec{x}, t) = \int_{|\vec{x}|}^{t} \Phi(\vec{x}, \tau)d\tau = +\frac{q}{2\pi}\int_{|\vec{x}|}^{t}\frac{\cos(m\sqrt{\tau^2 - |\vec{x}|^2})}{\sqrt{\tau^2 - |\vec{x}|^2}}d\tau\,, \tag{17}$$

Here, a new result takes place in the MCS framework: we cannot exactly evaluate how electromagnetic signals superpose for an arbitrary case, since the integral above is not available, in closed form, unless $t \to \infty$ (the other quantities also depend on the same integral). At this limit, we get (see, for example, Ref. [13], page 419, eq. 3.754-2):

$$\lim_{t\to\infty}\int_{|\vec{x}|}^{t}\frac{\cos(m\sqrt{\tau^2 - |\vec{x}|^2})}{\sqrt{\tau^2 - |\vec{x}|^2}}d\tau = K_0(m|\vec{x}|),$$

which, in turn, leads us to the static potential, eq. (13), as $t \to \infty$. A similar scenario holds for the other quantities, such as the vector potential and the field-strengths. Thus, we see that, in the case of the $\vec{E}$-field, only its longitudinal component survives asymptotically.

iii)Dirac-like monopole and its tangential electric field

Now, let us draw the attention to the introduction of a Dirac-like object into the previously studied models and to discuss some characteristics and consequences of the fields produced by this sort of monopole.

As it is well-known, such an (point-like) object shows up by breaking the Bianchi' identity[22]:[8] $\partial_\mu \tilde{F}^\mu = g$, which in terms of the potentials gets the form:

$$\int_t dt \int_{xy} d^2x \; \left(\epsilon_{ij} \, [\partial_i, \partial_t] A_j(\vec{x}, t) - [\partial_x, \partial_y] \Phi(\vec{x}, t)\right) = g \, ; \tag{18}$$

in the static limit, it reduces to:

$$[\partial_x, \partial_y] \Phi(\vec{x}) = -g\delta^2(\vec{x}) \, . \tag{19}$$

Now, the above equation may be satisfied only if Φ carries a "singular structure". Indeed, by recalling that

$$[\partial_x, \partial_y] \arctan\left(\frac{y}{x}\right) = \partial_x \left(\frac{x}{x^2+y^2}\right) + \partial_y \left(\frac{y}{x^2+y^2}\right)$$

exactly coincides with

$$\nabla^2 \ln\sqrt{x^2+y^2} = +2\pi\delta(x)\delta(y),$$

we identically solve eq. (19) by taking (as usual $r = \sqrt{x^2+y^2}$ and $\varphi = \arctan(y/x)$)

$$\Phi(\vec{x}) = -\frac{g}{2\pi} \arctan\left(\frac{y}{x}\right) \; \Rightarrow \; \Phi(r, \varphi) = -\frac{g}{2\pi}\varphi \, , \tag{20}$$

It is worthy noticing the remarkable feature of such a potential: it has an angular, instead of a radial dependence.

[8] In the Maxwell-Chern-Simons case, the naïve breaking of such an identity yields the breaking of gauge invariance. Thus, one should take into account that the monopole induces an extra electric current in order to balance $\partial_\mu j^\mu = 0$, and so restores gauge invariance (see Ref.[24, 25] for details. See also Ref.[26] for an alternative approach to a similar problem in (3+1) dimensions).

This leads to a very interesting (static) electric field ($\vec{\mathcal{E}} = -(\nabla\Phi + \partial_t \vec{A})$, as usual):

$$\vec{\mathcal{E}}(x,y) = +\frac{g}{2\pi}\frac{x\hat{y} - y\hat{x}}{x^2+y^2} \Rightarrow \vec{\mathcal{E}}(r,\varphi) = +\frac{g}{2\pi}\frac{\hat{e}_\varphi}{r}\,. \tag{21}$$

Whence, we clearly see the announced property of the g-monopole: it yields a (static) tangential electric field[9]. [As far as we have seen, such a peculiarity takes place only in (2+1)D Electrodynamics. Furthermore, we do expect that such a property survives at time-dependent regimes]. Moreover, it is worth noticing that a point-like magnetic vortex is characterised by a vector potential identical in structure to the tangential electric field above[23]. Thus, we may identify a "duality" between both objects: the vortex is obtained from the monopole (more precisely, from its "string" -see below) by taking the electric field and the charge of the first to be respectively the vector potential and the magnetic flux associated to the latter.

As it is well-known, the "angle-function" has a semi-rect of 'ill-definition', say, it is globally defined only on the $x-y$-plane without, for instance, the branch $x \in \{[0,+\infty)\}$[10]. Therefore, the scalar potential suffers from such a 'singularity', which in turn should not be identified with the string of the (2+1)D monopole. Actually, in this space-time, the "string" of a Dirac-like monopole shrinks to a (spatial) point while the monopole itself appears to be a sort of 'instanton' (see, for example, Ref.[24, 25]). Thus, the modified Bianchi equation, $\partial_\mu \tilde{F}^\mu = g\delta^2(\vec{x})$, have to be rather viewed as an equation for the "string", which appears to be localised at static limit (it is clear that the "string" may be chosen to be at any point on the plane. By means of the gauge-transformation approach this involves a time-dependent function relating different spatial points). Although such a localisation seems to state us that g should be rather faced as a peculiar electric charge, we stress that this is not so. Indeed, what happens is that, at static limit, the vanishing of radiation, $\nabla\cdot\vec{S}^* = \nabla\cdot(\vec{\mathcal{E}}^*\mathcal{B}) = 0$, de-

[9]Strictly speaking, such a field does not produce a genuine Newton's force on another charge (usual or peculiar one), since the force between them would not lie on the line that links both particles, as may be readily seen.

[10]Thought as an one-form, such a "function" it is the simpler (lower-dimensional) example of a closed but not exact one-form ($d^2\varphi \neq 0$), so it is not properly a function in the $x-y$-plane (recall its multi-valuedness character). However, as thought as the imaginary part of the ln-function in the context of Riemann surfaces it becomes a well-defined (say, single-valued) function.

mands that the monopole' magnetic field must also vanish. [Notice that such a requiriment, $\mathcal{B}=0$, is intimately related to the tangential feature of $\vec{\mathcal{E}}$, once that $\vec{\mathcal{E}}^*$ becomes radial, and so $\nabla\cdot\vec{\mathcal{E}}^*\neq 0$]. Hence, what we may state is that such an object yields only non-vashing (tangential) electric field at the static limit.

Next, we analyse the (classical) dynamics of a usual electric charge, q, with mass m, moving under the action of such a tangential field. Its equations of motion are readily obtained and read as follows:

$$\frac{2\pi m}{gq}\ddot{x}=-\frac{y}{x^2+y^2}\quad\text{and}\quad\frac{2\pi m}{gq}\ddot{y}=+\frac{x}{x^2+y^2}\,,\tag{22}$$

or in (r,φ)-coordinates:

$$\frac{2\pi m}{gq}(\ddot{r}-r\dot{\varphi}^2)=0\quad\text{and}\quad\frac{2\pi m}{gq}\frac{d}{dt}(r^2\dot{\varphi})=1\,.\tag{23}$$

Now, due to the angle-dependent feature of the potential, we notice that the particle' "angular momentum" is clearly not conserved; also the eqs. of motion are rather coupled and cannot be analitically solved. Indeed, the following relation

$$\ddot{x}x+\ddot{y}y=0\quad\Longrightarrow\quad\ddot{\xi}=-\frac{1}{\xi}$$

readily follows from (22), with $\xi(t)=\frac{x(t)}{y(t)}$, whose solution reads:

$$\int\frac{d\xi}{\sqrt{-2\ln\xi\,+c_1}}=t+c_2\quad\Longrightarrow\quad e^{-c_1/2}\int e^{-u^2/2}du=t+c_2\,,$$

where we have defined $\xi(t)=e^{(c_1-u^2)/2}$. Now, since the Gaussian integration is not available for arbitrary limits, an explicit solution for our problem cannot be achieved and numerical resolution is demanded. In this line, a typical plot of the motion ($x-y$-coordinates) of the charged particle is shown in Fig. 1.a. By virtue of the tangentially repulsive nature of the electric field, the particle is quickly drifted away, despite the signals of the charges.

A further system which deserves more attention is that in which we also have the presence of an external (constant, for simplicity) magnetic field. A realistic planar system may be

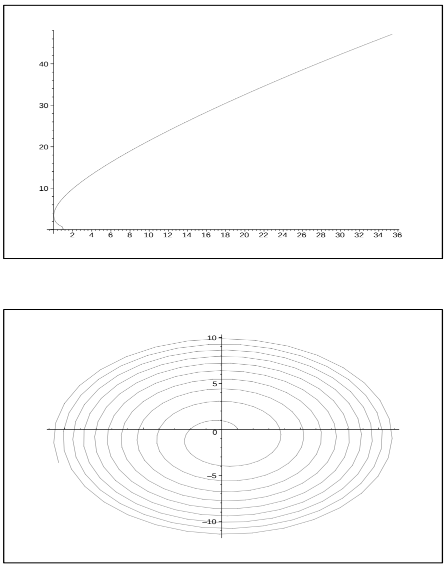

Figure 1: Typical plot (x-y-coordinates) of the trajectory of a charged particle (initially at rest in $x = 1$, $y = 0$); a) under the action of the tangential electric field alone (with $2\pi m/qg = 1$); and b) also with an external magnetic field, B_0 (with $m = qB_0 = \frac{qg}{2\pi} = 1$).

obtained at very low temperatures (around or less than 1K) and suficiently strong magnetic field (at least 10T) perpendicular to a very thin plate [11]. Such a perpendicular field is got by taking a vector potential entirely confined to the 2D-spatial plane, say

$$\vec{A} = \vec{A}_1 = B_0 x\hat{\jmath}\,, \quad \vec{A} = \vec{A}_2 = -B_0 y\hat{\imath}\,, \tag{24}$$

(Landau gauges) or still (symmetric gauge)

$$\vec{A} = \frac{\vec{A}_1 + \vec{A}_2}{2} = \frac{B_0}{2}\left(x\hat{\jmath} - y\hat{\imath}\right)\,. \tag{25}$$

Now, our present system is composed by the electric charge subject to the external magnetic and to the tangential electric field as well. Again, the classical eqs. of motion are easy to be obtained and read (eqs. of motion in r, φ imediately follow):

$$\frac{m}{q}\ddot{x} = -\frac{q}{2\pi}\frac{y}{x^2 + y^2} + B_0\dot{y} \quad \text{and} \quad \frac{m}{q}\ddot{y} = +\frac{g}{2\pi}\frac{x}{x^2 + y^2} - B_0\dot{x}\,, \tag{26}$$

[11]Such systems may be realised, for instance, in the interface between two semi-conductors. Furthermore, since the motion of the charges (electrons, for concreteness) takes place as if the third dimension (perpendicular to the plane of motion) were frosen, the generally employed 2D (spatial) treatment is justified, and has been shown to gives us a very good explanation of the physical phenomena which occur whithin such systems, e.g., the Quantum Hall Effect.

Or, by defining complex dynamical variables as $\eta = x + iy$ and $\eta^* = x - iy$, we get:

$$2m(\ddot{\eta}\eta^* + \eta\ddot{\eta}^*) + iqB_0(\dot{\eta}\eta^* - \eta\dot{\eta}^*) = 0 \quad \text{and} \quad 4\pi m(\ddot{\eta}\dot{\eta}^* + \dot{\eta}\ddot{\eta}^*) + iqg\frac{(\dot{\eta}\eta^* - \eta\dot{\eta}^*)}{\eta\eta^*} = 0.$$

Despite their symmetric appearance, the resolution of the eqs. above is not too easy. Indeed, we claim that they may be even more difficult to be solved than those in the absence of magnetic field (former case). On the other hand, numerical resolution shows us that the magnetic field tends to compensate the repulsive effect of the electric one so that the (classical) motion of the particle appears to drift in a more slower way, describing an almost regular spiral-like pattern (see Fig. 1.b). Notice also that the distance between two neighbour arms of such a pattern decreases as the radial distance increases: the particle asymptotically 'approaches' to perform a closed trajectory (in the next section, we shall see that, the quantum dynamics of the charged particle asymptoticaly, $r \to \infty$, reduces to that of one central harmonic oscillator). Indeed, such a pattern is strongly dependent on the signal of the charges and the direction of the magnetic field, say, some choices of these signals may lead us to other types of trajectory of the q-charge (yet presenting some regularities; more details will appear in Ref.[17]).

There is, however, at least one important information which may be analytically obtained: in both cases, $B_0 = 0$ and $B_0 \neq 0$,the velocity of the charged particle is bounded by the angle, as below:

$$(\vec{v})^2 = \frac{qg}{m\pi}\varphi + (\vec{v}_0)^2 . \tag{27}$$

It is worthy noticing that the number of windings of the charge around the origin must be taken into account, i.e., the kinectic energy is determined by the total angle descrided by the charge. [As a sort of quantum counterpart, we shall see that as $r \to \infty$ the (angular) energy eigenvalues have to be shifted as $\varphi \to \varphi + 2\pi$ (see next section for details)].

iv) Preliminary analysis of the quantum charge-monopole system

Next, we shall present a preliminary quantum (non-relativistic) analysis of the system above: one electric charge, q, moving under the action of the monopole scalar potential, $V \propto$

$\arctan(y/x)$, and of an external constant magnetic field, B_0. The Hamiltonian (the pure gq-system is readily got by setting $\vec{A}=0$),

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + qV$$

for this system is obtained by taking $\vec{A}$ in a particular gauge (Landau or symmetric, eqs. (24-25), as well as $V(x,y) = -\frac{g}{2\pi}\arctan(y/x) = -\frac{g}{2\pi}\arg(\vec{r})$. [Notice that the potential remains invariant under general scale transformation, say: $x \to f(x,y)\,x$ and $y \to f(x,y)\,y$, but, the same symmetry is not present in the full Hamiltonian, even for $f(x,y) = a = constant$.

For the analysis to be presented here, concerning the non-conservation of the angular-momentum and some of its consequences, as well as asymptotic bahaviours of the present system, it will be more convenient to write the Hamiltonian above in polar coordinates, r, φ, and $\vec{A}$ in the symmetric gauge, like below:

$$H = \frac{1}{2m}\left[{p_r}^2 + \frac{p_r}{r} + (qB_0)^2 r^2\right] + \frac{1}{2m}\frac{{p_\varphi}^2}{r^2} + \frac{qB_0}{2m}p_\varphi - \frac{gq}{2\pi}\varphi\,, \tag{28}$$

with r and φ defined as before and $\vec{p} = p_r\hat{e}_r + \frac{p_\varphi}{r}\hat{e}_\varphi$, whence there follows that $p_r \leftrightarrow -i\hbar\frac{\partial}{\partial r}$ and $p_\varphi \leftrightarrow -i\hbar\frac{\partial}{\partial\varphi}$.

Now, we notice the first remarkable feature of this Hamiltonian: H is explicitly angle-dependent and so non-invariant under rotations; conversely, the angular momentum operator, $J = p_\varphi = -i\hbar\frac{\partial}{\partial\varphi}$, is not conserved, $[J, H] = +i\hbar gq/2\pi \neq 0$.

Although other angle-dependent Hamiltonians have been studied and shown to be relevant in Physics (see for example [27]), a remarkable difference between them and the one presented here is that the latter is not separable. Indeed, as far as we have seen, the system appears to present an intricate coupling between its degrees-of-freedom, despite of the coordinates chosen. [Perhaps, some non-standard tranformation could lead us to such a separation, but could also lead us, on the other hand, to results which were of hard physical interpretation. Such an issue remains to be investigated].

It is clear, from the Hamiltonian (28) and also from the fundamental commutation relations, $[r, \varphi] = [p_r, p_\varphi] = 0$ and $[r, p_r] = [\varphi, p_\varphi] = +i\hbar$, that the non-separability arises from the non-conservation of the angular momentum, $[J, H] \neq 0$. Indeed, as it may be easily checked, such an angular sector would be separable if it had the general form $\frac{1}{r^2}(J^2 + aJ + b\varphi)$. So, it is the lack of a $1/r^2$-factor in J and in φ-terms what prevents us from a split of variables.

On the other hand, by facing H as being non-separable, the analytical resolution of the eigenvalue problem, $H|\psi >= E|\psi >$, appears to be of very hard achievement. [Actually, the presence of the terms proportional to φ and r -or powers of r- in H prevents us from solving this eigenvalue problem by means of, for example, hypergeometric functions (see, for example Ref. [28])].

Therefore, a numerical resolution appears to be a more suitable (and direct) attempt towards solving the problem (results will be communicated as soon as they were obtained). Here, however, we shall deal with some analytical results at asymptotic limits, even though some of them appear to be quite qualitative. We shall mainly discuss the limits $r \to \infty$ and $r \to 0$:

i)$r \to \infty$: supposing that the canonical momenta remain finite in this limit, we get:

$$H(r, \varphi)_{r\to\infty} \approx \frac{1}{2m}(p_r^2 + q^2 B_0^2 r^2) + \frac{qB_0}{2m} p_\varphi - \frac{gq}{2\pi}\varphi\,, \tag{29}$$

in which the variables appear explicitly split, say, $H_{r\to\infty} = H^r_{r\to\infty} + H^\varphi_{r\to\infty}$. Thus, at this limit, we have that (the limit $r \to \infty$ is implicit hereafter)

$$(H|\psi(r, \varphi) >) = (E_n|\psi(r, \varphi) >) \quad \Longrightarrow \quad (H^r + H^\varphi)|R(r)\Phi(\varphi)) >= ((E^r + E^\varphi)|R\Phi >)\,, \tag{30}$$

which leads us to:

$$H^r R_k(r) = E^r_k R_k(r) \quad \text{and} \quad H^\varphi \Phi_l(\varphi) = E^\varphi_l \Phi_l(\varphi)\,. \tag{31}$$

Therefore, as $r \to \infty$, we get the following set of differential eqs.:

$$\hbar^2 \frac{d^2}{dr^2} R + (2mE^r - q^2 B_0^2 r^2) R = 0\,, \tag{32}$$

$$i\hbar \frac{d}{d\varphi}\Phi + (\epsilon^\varphi + \beta\varphi)\Phi = 0\,, \tag{33}$$

with $\beta = +mg/\pi B_0$ and $\epsilon^\varphi = 2mE^\varphi/qB_0$.

We notice that, as $r \to \infty$, the radial part of the Hamiltonian reduces to that of one central harmonic oscillator, whose solutions may be written in terms of Hermite polynomials, H_n:

$$R_k(u) = R_0 e^{-u^2/2} H_k(u),$$

with $u = qB_0 r$ and k any non-negative integer. This implies the well-known eigenvalues $E_k^r = \hbar B_0(k + 1/2)$.
On the other hand, the angular sector appears to be quite unusual. Indeed, by solving the differential equation in φ,we readily obtain

$$\Phi_l(\varphi) = \Phi_0\, exp\left[\frac{i}{\hbar}\left(\frac{\beta\varphi}{2} + \epsilon_l^\varphi\right)\varphi\right]. \tag{34}$$

It is worth noticing the new φ^2-like phase factor, along with the usual linear one. As a first remark, we should stress that it cannot be removed by any suitable gauge tranformation; indeed, it must rather be faced as a consequence of the φ-like scalar potential. Although quite unusual, it leads us to new and interesting results, as we shall see in what follows.

First, notice that $\Phi(\varphi)$ has periodicity $2\pi(\beta\pi + \epsilon_l^\varphi)$. Thus, the requirement that Φ be single-valued, i.e., continuous, is equivalent to set

$$2\pi(\beta\pi + \epsilon_l^\varphi) = 2\pi l\hbar. \tag{35}$$

The lowest value, $\epsilon_0^\varphi = -\beta\pi\hbar$, gives us

$$E_0^\varphi = -gq\hbar/2\,, \tag{36}$$

which may be viewed as a quantisation condition on the gq-product (notice that the product is uniquely determined by the lowest angular energy eigenvalue, E_0^φ). The parameter l, in turn, is to be identified with the number of windings the q-particle gives around the origin (so, it measures the full angle described by the q-charge). Whence, l has to be taken as a non-negative integer. Therefore, the eigenvalues associated to the angular variable feel whether it is running between 0 and 2π, 2π and 4π, and so forth. In other words, whenever φ is shifted, say, by 2π, its associated eigenvalues respond to this change by shifting up their values. the latter result may be understood as the quantum analogue of the classical one, as expressed by eq. (27).

Moreover, we could be tempted to naively apply J-operator on $|\psi_{kl}>$ above, to get

$$J|\psi_{kl}>= (\beta\varphi + \epsilon_l^\varphi)|\psi_{kl}>= (\beta(\varphi+\pi)+l)|\psi_{kl}>,$$

and hence, guess that $|\psi_{kl}>$ carry continuous angular momentum. However, this is not a legitimate procedure, because $|\psi_{kl}>$ are not eigenvectors of J (recall that $[J,H]\neq 0$). Actually, as far as we have seen, the only two quantities which may be simoutaneously diagonalised in $|psi_{kl}>$-basis are H_r and H_φ (the components of the full asymptotic Hamiltonian, eq. (29)).

Clearly, the results and remarks above are strictly valid only at the asymptotic limits specified previously. Whether similar scenario does happen at arbitrary distances (as the classical result (27) does), remains to be studied and will be strongly dependent on the separation of variables in the original Hamiltonian, eq. (28). Now, let us discuss the $r\to 0$-limit.

ii) $r\to 0$: we have seen that near the origin (where the "string" is localised), the charged particle experiences a very strong tangentially repulsive electric field (see previous section for details). Since as $r\to 0$ this field blows up, it is expected that q can never reach the origin, say, its wave-function must vanish there: $|\psi(r=0,\varphi)>\equiv 0$. Such a requirement may be viewed as the counterpart of the Dirac-veto in (3+1)D: a single charge moving under the action of the magnetic monopole field could not cross the string of its associated vector potential[22].

Thus, what remains to be determined is how quickly $|\psi>$ do vanishes as $r\to 0$. Nevertheless, contrary to the $r\to\infty$-limit, in which the Hamiltonian gets separable, here the variables are no longer naïvely separated. This arises because p_φ^2/r^2 is now one of the leading terms, similar to the original problem, described by the Hamiltonian (28). Thus, we claim that, even for $r\to 0$, we demand numerical techniques in order to get some information about the gq-system; say, for instance, the form of its wave-function and eigenvalues.

A naive analysis of the limits discussed above would lead us to conclude that, since the charged particle is repelled from the origin by the φ-potential and since as $r \to \infty$ its dynamics reduces to that of one central harmonic oscillator (whose wave-functions fall off exponentially), it is expected that the system yields physical bound states. Therefore, even though the pure gq-system does not admit bound states (once that the confining r^2-type potential is absent, for this case, in eq.(29), we get indeed a radially free particle), when it is supplemented by an external magnetic field (constant , for concreteness and strong enough), the possibility for such states may be raised.

Nevertheless, when charged particles (say, electrons) are moving on the plane subject only to a perpendicular magnetic field, then the choice of Landau gauge immediately reduce the quantum problem to that of one harmonic oscillator in one dimension, along with a free particle motion in the other. In this case, we cannot have bound states. However, when the system is supplemented by an extra, say, scalar potential (as in the present case), it is also well-known that bound states show up, even in the case of repulsive potential[29]. Here, we have just raised such a question, and a precise answer demands further investigation. These and other quantum aspects of the present system are been studied [30], and we would like to communicate eventual results about them in a forthcoming paper.

v) Concluding remarks

We have shown that classical (2+1)D Maxwell and Maxwell-Chern-Simons Electrodynamics present some interesting novelties as compared to Maxwell theory in (3+1)D, namely, the reverberation of signals and the far-from-trivial question of a Larmor-like formula. As we have seen, such phenomena are intimately related to the failure of the Huyghens' principle. Namely, the latter is very difficult to be obtained even for constant accelerated motions (parabolic and hyperbolic ones). The integrals involved are highly non-trivial and appear to diverge, so demanding some suitable regularisation scheme. On the other hand, we hope that some hints about such a Larmor' formula could be obtained with the help of numerical calculations.

Concerning the Dirac-like monopole, it also presents some new properties whenever compared to its (3+1)D-counterpart; for instance, its static tangential electric field. Furthermore, whenever acting on a single charged particle, it leads us to interesting classical and quantum results. For example, the gq-system (with B_0) has been shown to give rise, at least asymptoticaly and at non-relativistic regimes, to one central harmonic oscillator, with an interesting angular sector which contribute to the energy-eigenvalues. As future perspectives, it remains to be studied whether these results survive at finite distances and, still completely open, the issue concerning possible effects of this peculiar potential on spin particles, for instance, planar Dirac fermions. Moreover, by virtue of its peculiar scalar potential (and unusual consequences), such a monopole could be relevant to Condensed Matter problems. For instance, by facing this object as a sort of impurity (scatter) within a sample, could its presence modify the Hall conductivity? And possibly, how would actually look like such a modification?

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