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The Statistical Turbulence Suppress by a Strong Magnetic Field a Feynman Path Integral Argument

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Abstract

We show the suppress of the randomness of the magnetohydrodynamic flux (modeled by the Navier-Stones equation) in the presence of a random stirring; by a Feynman path integral one-loop and low viscosity argument. Additionaly, we show exactly the above supressing turbulence phenomenon in the context of Beltrami fluxes in appendix A.

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One of the long-standing unsolved problems in turbulent magnetohydrodynamics going back to L. Landau (ref. [1] – pags. 238) is to produce quantitative arguments for the charged fluid motion turbulence suppressing by the presence of a strong magnetic field. In this letter, we propose to present such an argument by using the Wild-Rosen Feynman path integral formulation at a low viscosity (turbulent) regime in the context of a background one-loop (Gaussian) approximation. Additionally, we show in the appendix, the above cited suppressing phenomena in the path integral context of Beltrami fluxes.

Let us start our analysis by considering the Navier-Stokes equation for a charged fluid fluxe in the presence of a uniform magnetic field $\vec{B}$ in the z-direction and a uniform electric field $\vec{E}$ in the plane (x, y)

$$\frac{\partial \vec{V}(\vec{r},t)}{\partial t}+(\vec{V}\cdot\vec{\nabla})\vec{V}(\vec{r},t) = e(\vec{E}+\frac{1}{c}(\vec{V}(\vec{r},t)\times\vec{B})+\nu\Delta\vec{V}(\vec{r},t)-(\overrightarrow{gradP})(\vec{r},t)+\vec{F}^{ext}(\vec{r},t) \quad (1)$$

Here, the random stirring Force (responsible by the fluid fluxe turbulent regime in the statistical-Langevin approach ([2])) is such that its satisfies the white-noise Gaussian statistics

$$\langle (F^{ext})_i(\vec{r},t)(F^{ext})_j(\vec{r}\,',t')\rangle = D\delta_{ij}\delta^{(3)}(\vec{r},\vec{r}\,')\delta(t-t') \quad (2)$$

It is worth remark that the presence of such external randon fluctuations are considered in this statistical framework with the same conceptual role as used in the usual Langevin-Einstein approach for study the Brownian motion. It is still missing a turbulence theory for first principles leading to the master equation (1).

Now a simple functional integral shift ([2]), leads to the exactly characteristic functional path integral expression for the random (turbulence) process defined by the magneto hidrodynamic (incompressible) fluid fluxe eq. (1) after taking into account the fluid incompressibility condition on the fluid motion $d_{iv}\vec{V}(\vec{r},t) = 0$ and disregarding the transverse part piece of the term $((\vec{V}\cdot\nabla)\vec{V})^{tr}$ coming from the above mentioned incompressibility condition (see ref. [2]). We obtain, thus, the following Burger-Magneto hydrodynamical turbulent path integral as the main object of our study.

$$\begin{aligned} Z[\vec{J}(\vec{r},t)] \quad &= \quad \frac{1}{Z[0]}\int D^F[\vec{V}(\vec{r},t)] \\ &\exp \quad \left\{-\frac{1}{2D}\int_{-\infty}^{+\infty} d^3\vec{r}\int_{-\infty}^{\infty} dt \left[\frac{\partial}{\partial t}\,\vec{V} - \nu\Delta\vec{V}\right.\right. \end{aligned}$$

$$+ \quad \left((\vec{V}\cdot\vec{\nabla})\vec{V}\right) - e\left(\vec{E} + \frac{1}{c}\,(\vec{V}(\vec{r},t)X\vec{B})\right)\Bigg]^2 (\vec{r},t)\Bigg\}$$

$$\times \quad \exp\left\{ i\int_{-\infty}^{+\infty} d^3\vec{r}\int_{-\infty}^{\infty} dt\vec{J}(\vec{r},t)\cdot\vec{V}(\vec{r},t)\right\} \tag{3}$$

In order to show the randomness suppressing of the fluid flux motion in the plane (x, y) normal to the magnetic field B at its strong field limit regime, we consider the back-ground flux decomposition

$$\vec{V}(\vec{r},t) = \vec{\phi} + \sqrt{D}\ \vec{v}(\vec{r},t) \tag{4}$$

where the back-ground flux $\vec{\phi}$ satisfies the steady-isotropic Ohm's law fluxe condition $(i = 1, 2, 3)$

$$eE_i = -\frac{1}{c}\ \varepsilon_{ijk}\phi_j B\delta_{k3} \tag{5}$$

As usual pertubative arguments in Quantum Field theories ([3] – pags. 29), we consider the usual one-loop (Gaussian) approximation for the path integral eq. (3) ([2]) with the following result:

$$S_{one-loop}[\vec{v}(\vec{r},t)] = (v_1(\vec{r},t),\ v_2(\vec{r},t),\ v_3(\vec{r},t)$$

$$x\begin{bmatrix} \left(-\left(\frac{\partial}{\partial t}+\phi_i\,\frac{\partial}{\partial x_i}\right)^2+\frac{e^2B^2}{c^2}\right)+\nu^2|\vec{k}|^2 & \frac{2eB}{c}\left(\frac{\partial}{\partial t}+\phi_i\frac{\partial}{\partial x_i}\right)+\frac{2\nu_e}{c}\ B\Delta_{\vec{r}} & 0 \\ -\frac{2eB}{c}\left(\frac{\partial}{\partial t}+\phi^i\frac{\partial}{\partial x_i}\right)-\frac{2e\nu}{c}\ B\Delta_{\vec{r}} & \left(-\left(\frac{\partial}{\partial t}+\phi_i\frac{\partial}{\partial x_i}\right)^2+\frac{e^2B^2}{c^2}\right)+\nu^2|\vec{k}|^2 & 0 \\ 0 & 0 & -\left(\frac{\partial}{\partial t}+\phi_i\,\frac{\partial}{\partial x_i}\right)^2+\nu^2|\vec{k}|^2 \end{bmatrix}$$

$$\begin{pmatrix} v_1(\vec{r},t) \\ v_2(\vec{r},t) \\ v_3(\vec{r},t) \end{pmatrix} \tag{6}$$

Let us show the vanishing of the two-point correlation fluxe at the limit of very strong magnetic field $B \to \infty$. In the plane (x, y) within the very small viscosity limit which constraint us to neglect the term $\nu^2(\Delta\vec{v})^2$ in relation to other linear terms in eq. (6). We get, thus:

$$S^{(\nu-small)}_{Gaussian}[\vec{v}(\vec{r},t)] = (v_1(\vec{r},t),\ v_2(\vec{r},t),\ v_3(\vec{r},t)$$

$$\begin{bmatrix} \left(-\left(\frac{\partial}{\partial t}+\phi_i\,\frac{\partial}{\partial x_i}\right)^2+\frac{e^2B^2}{c^2}\right) & \frac{2eB}{c}\left(\frac{\partial}{\partial t}+\phi_i\frac{\partial}{\partial x_i}\right)+\frac{2\nu B}{c}\,\Delta_{\vec r} & 0 \\ -\frac{2eB}{c}\left(\frac{\partial}{\partial t}+\phi^i\frac{\partial}{\partial x_i}\right)-\frac{2e\nu B}{c}\Delta_r & \left(-\left(\frac{\partial}{\partial t}+\phi_i\frac{\partial}{\partial x_i}\right)^2+\frac{e^2B^2}{c^2}\right) & 0 \\ 0 & 0 & -\left(\frac{\partial}{\partial t}+\phi_i\,\frac{\partial}{\partial x_i}\right)^2 \end{bmatrix}$$

$$\begin{pmatrix} v_1(\vec r,t) \\ v_2(\vec r,t) \\ v_3(\vec r,t) \end{pmatrix} \tag{7}$$

A simple reading of eq. (7) gives us the following result in Fourier transformed form for the x-two point correlation turbulent function

$$\begin{aligned} &\langle v_1(\vec r,t)v_1(\vec r\,',t')\rangle \\ =\;& \int d^3\vec k\cdot e^{i\left[\vec k\left((\vec r-\vec r\ )-\vec\phi(t-t')\right)\right]} \\ \times\;& \frac{1}{2}\int_{-\infty}^{+\infty} d\bar w e^{-i\bar w(t-t')}\left\{\left(\frac{1}{\bar w^2+\frac{2B}{c}\ \bar w+\left(\frac{H^2}{c^2}+\frac{2i\nu}{c}\ B|\vec k|^2\right)}\right)\right. \\ +\;& \left.\left(\frac{1}{\bar w^2-\frac{2B}{c}\bar w+\left(\frac{H^2}{c^2}-\frac{2iv}{c}\ B|\vec k|^2\right)}\right)\right\} \end{aligned} \tag{8}$$

The first frequency $\bar w$-integral eq. (8) above obtained may easily be evaluated by means of the residue theorem applied in the (*causal*) region lower-half-plane $I_m(\bar w)<0$. Namely, for $t>t'$

$$\begin{aligned} &\int_{-\infty}^{+\infty} d\bar w e^{-i\bar w(t-t')}\left(\frac{1}{\bar w^2+\frac{2B}{c}\ \bar w+\left(\frac{H^2}{c^2}+\frac{2iv}{c}\ B|\vec k|^2\right)}\right) \\ =\;& 2\pi i\left\{-\frac{e^{-i\left[-\frac{B}{c}-\left(\frac{\sqrt{2Bc\nu}}{\nu}\right)|\vec k|\left(-\frac{\sqrt2}{2}+i\ \frac{\sqrt2}{2}\right)\right](t-t')}}{2\left[-|\vec k|\cdot\sqrt{\frac{2Bc}{\nu}}\ \frac{\sqrt2}{2}+i|\vec k|\cdot\sqrt{\frac{2Bc}{\nu}}\ \frac{\sqrt2}{2}\right]}\theta(t-t')\right\} \end{aligned} \tag{9}$$

Note that we have considered only the residue at the pole $w_+=-\frac{B}{c}+|\vec k|\sqrt{\frac{2Bc}{\nu}}-i\ \sqrt{\frac{2}{2}}\times|\vec k|\sqrt{\frac{2Bc}{\nu}}$ in the lower-half plane $I_m(w)<0$, which by its turn, allows the application of the residue theorem and, thus, leads directly to the causal Fourier transform *vanishing* at the limit of $t\to+\infty$.

Now it is easy to see that at large $B\to\infty$ limit with ν very small *but fixed*, the integral eq. (9) vanishes. Similar result holds true for the second term of the $\bar w$-integral eq. (8)

and producing, thus, the suppressing of the randomness at the limit of $B \to \infty$ within our one-loop leading small viscosity approximation, namelly

$$\lim_{B\to\infty} \langle v_1(\vec{r},t) v_1(\vec{r}\,',t)\rangle \to 0 \tag{10}$$

Analogously:

$$\lim_{B\to+\infty} \langle v_1(\vec{r}\,',t) v_2(\vec{r}\,',t')\rangle = \lim_{B\to\infty} \langle v_2(\vec{r},t) v_2(\vec{r}\,',t)\rangle = 0 \tag{11}$$

However, the fluxe randomness in the B-direction in this context is given exactly by the non-vanishing result.

$$\langle v_3(\vec{r},t), v_3(\vec{r},t)\rangle = \delta^{(3)}\left((\vec{r}-\vec{r}\,') - \vec{\phi}(t-t')\right) \frac{e^{-\nu(t-t')}}{\nu}\,\theta(t-t') \tag{12}$$

At this point, it is very important to remark that the zero viscosity (full tubulent) regime is a *singular* limit never reached in our approach, result opposite to those of instantons calculatations of ref. [4]. Work on the next loops corrections will be reported elsewhere. (See appendix A for an exactly solubler reduced model solved in our path integral framework).

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Appendix A – A soluble model an turbulence suppresing in turbulent Magneto Hydrodynamics

The celebrated Landau conjecture on the suppressing of turbulence (or randomness in the statistical approach) by a strong magnetic field analyzed, in the text within, the context of the path magnetic integral approximations stated there, can be displayed exactly in the following reduced degrees of freedom model of Beltrami fluxes in a turbulent regime defined analitically by $rot\ \vec{v}(\vec{r},t) = \lambda\vec{v}(\vec{r},t)$.

Let us, thus, start with the Magneto-hydrodynamical Navier-Stokes eq. (1) in the following suitable form for Beltrami fluxes

$$\begin{aligned} & \frac{\partial\vec{v}(\vec{r},t)}{\partial t} + \left\{\frac{1}{2}\ grad\left[(\vec{v}\cdot\vec{v})(\vec{r},t)\right]\right. \\ - & \left.(\vec{v}\times\ rot\ \cdot\vec{v})(\vec{r},t)\right\} \\ = & -gradP(\vec{r},t) + \nu\Delta\vec{v}(\vec{r},t) + \vec{F}^{ext}(\vec{r},t) \\ + & \frac{1}{c}\left(\vec{v}(\vec{r},t)\times\vec{B}\right) \end{aligned} \tag{13}$$

It is important to remark that the wave vectors of the Beltrami hydrodynamical motions eq. (1) have eddies of a fixed scale $|\vec{k}| = \lambda$, as consequence of a Fourier (Wave vector) transformation of the Beltrami condition.

Let us, thus, suppose that the random stirring force in eq. (1) satisfies the *spatially non-local* Gaussina statistics

$$\begin{aligned} & \langle (F^{ext})_i(\vec{r},t);(F^{ext})_j(\vec{r}\ ',t')\rangle_{|\vec{k}|=\lambda} \\ = & \lambda^2\delta_{ij}\left((\Delta^{-1})\delta(\vec{r}-\vec{r}\ ')\right)\delta(t-t') \end{aligned} \tag{14}$$

At this point, we remark that we assume implicity the same vector constraint $|\vec{k}| = \lambda$ in our random stirring eq. (2), namelly the exactly expression for the right-hand side eq. (14) is given below

$$\langle (F^{ext})_i(\vec{r},t);(F^{ext})_j(\vec{r}\ ',t)\rangle = \frac{\delta_{ij}\cdot\lambda\cdot Sen(\lambda|\vec{r}-\vec{r}\ '|)}{2\pi|\vec{r}-\vec{r}\ '|}\cdot\delta(t-t') \tag{15}$$

At this point of our study, we consider the already mentioned Beltrami flux condition and its direct consequences, namelly:

$$\lambda^2\vec{v}(\vec{r},t) = rot(rot\ \vec{v}(\vec{r},t) = -\Delta\vec{v}(\vec{r},t) \tag{16}$$

$$(\vec{v} \times \ rot \ \vec{v})(\vec{r},t) = (\vec{v} \times \lambda\vec{v})(\vec{r},t) = 0 \tag{17}$$

in order to replace the Navier-Stokes equation (13) by the exactly soluble Langevin like fluid flux stirred by the external force $\vec{\Omega}^{\ ext}(\vec{r},t) = rot\ (\vec{F}^{ext})(\vec{r},t)$.

$$\frac{\partial(\vec{r},t)}{\partial t} = (-\nu\lambda^2)\vec{v}(\vec{r},t) + \frac{1}{\lambda}\ \vec{\Omega}^{ext}(\vec{r},t) + \frac{1}{c\lambda}\ (\vec{B}\cdot\vec{\Delta})\vec{v}(\vec{r},t) \tag{18}$$

Here the new external stirring $\vec{\Omega}^{ext}(\vec{r},t)$ satisfies a Gaussian process with the following two-point correlation function (again with $|\vec{k}| = \lambda!$)

$$\begin{aligned}
&\langle \Omega_\ell^{ext}(\vec{t},t)\ ;\ \Omega_{\ell'}^{ext}(\vec{t}\,',t)\rangle \\
= &\ \lambda^2\delta^{\ell\ell'}\delta^{(3)}(\vec{r}-\vec{r})\delta(t-t') \\
- &\ \lambda^2\partial_\ell^{(r)}\partial_{\ell'}^{(r')}\left(\Delta_r^{-1}\delta(\vec{r}-\vec{r}\ ')\right)\delta(t-t')
\end{aligned} \tag{19}$$

Now a simple functional integral shift leads to the following exactly soluble Feynman path integral for our Beltrami magneto-hydrodynamic reduced model

$$\begin{aligned}
Z[\vec{j}(\vec{r},t) \ &= \ \frac{1}{Z(0)} \times \int D^F[\vec{v}(\vec{r},t)] \\
&\times \ \exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} dr^3 \int_0^\infty dt \left[\frac{\partial\vec{v}}{\partial t} - \frac{1}{\lambda c}(\vec{B}\cdot\vec{\nabla})\vec{v} + \nu\lambda^2\vec{v}\right]^2 (\vec{r},t)\right\} \\
&\quad \exp\left\{i\int_{-\infty}^{+\infty} d^3\vec{r}\int_0^\infty dt(\vec{j}\cdot\vec{v})(\vec{r},t)\right\}
\end{aligned} \tag{20}$$

It is work remark that we have used incompressible constraint $\partial_i^{(r)}v^i(\vec{r},t) = 0$ to obtain that the spatially non-local piece of eq. (19) does not contribute to the final path integral weight in eq. (20).

Now the Fourier transformed expression for the correlations functions associated to the Beltrami-Magneto-hydrodynamical model eq. (8-A) are given exactly by (note again the wave vector constraint $|\vec{k}| = \lambda$ in the model formulae!).

$$\begin{aligned}
&\langle v_i(\vec{r},t)v_j(\vec{r}\ ',t')\rangle = \\
&\delta_{ij} \times \frac{e^{-\nu\lambda^2|t-t'|}}{\nu\lambda^2} \times \left\{\int_{|\vec{k}|=\lambda} d^3\vec{k}e^{i\vec{k}\left[(\vec{r}-\vec{r}\ ')-\frac{\vec{B}}{c\lambda}\ (t-t')|\right]}\right\} \times \\
= \ &2\delta_{ij}\left(\frac{e^{\nu\lambda^2|t-t'|}}{\nu\lambda}\right)\ \frac{sen\left[\lambda\left(|\vec{r}-\vec{r}\ ')-\frac{\vec{B}}{c\lambda}(t-t')|\right)\right]}{|(\vec{r}-\vec{r}\ ')-\frac{\vec{B}}{c\lambda}(t-t')|}
\end{aligned} \tag{21}$$

Its is a direct consequenc of the exact expression of eq. (21) that of the limit of strong magnetic field $|\vec{B}| \rightarrow \infty$, we get the vanishing of the correlations functions eq. (21) and, thus, the exactly randomness suppressing in our Beltrami-Magneto-hydrodynamical model eq. (18) and corroborate the Landau conjecture ref. [1] and the correctness of our general (1-loop) calculations presented in the bulk of this letter [see equations ((10) and (11))].