

Zero-point energy of massless scalar fields in the presence of soft and semihard boundaries in D dimensions

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ABSTRACT

The renormalized energy density of a massless scalar field defined in a D -dimensional flat spacetime is computed in the presence of “soft” and “semihard” boundaries, modeled by some smoothly increasing potential functions. The sign of the renormalized energy densities for these different confining situations is investigated. The dependence of this energy on D for the cases of “hard” and “soft/semihard” boundaries are compared.

Key-words: Vacuum energy; Soft boundaries.

PACS numbers: 03.70.+k, 12.20.Ds, 04.62.+v.

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1 Introduction

In many situations in Quantum Field Theory it is assumed that the fields are defined in a region limited by some finite “classical” cavity and submitted to some particular boundary condition. “Classical” means here that the boundary which confines the fields has a very precise spatial location and a well defined geometrical shape (“hard” boundary). In the majority of the papers discussing the Casimir effect these “hard” boundaries are assumed, although they are unquestionably an idealization. In this scenario, one can argue in what extension does the accumulated experience in the Casimir effect depend on the assumption of this kind of boundaries. In some papers these conditions have been relaxed — the most recent ones quoted in Refs. [1-5].

In this paper we will deepen into the investigation on how the Casimir energy of a massless scalar field, defined in a D-dimensional flat spacetime, depends on the boundary conditions and on the dimensionality. For this purpose, three different kinds of confining boundaries are considered: “hard”, “soft” and also “semihard” ones. The meaning of this terminology will be clarified latter.

The problem of determining the expectation value of a physical observable is related to the question: how to implement a renormalization scheme in a given situation? In 1948 Casimir presented a scheme to obtain a finite result from the divergent zero-point energy of the electromagnetic field [6]. Although formally divergent, the difference between the vacuum energy of different physical configurations can be finite. If one of these configurations is assumed to have a zero vacuum energy, then the difference of the vacuum energy of both configurations is the renormalized one. Therefore the formal definition of the Casimir energy is

$$E_{ren}(\partial\Omega) = E_0(\partial\Omega) - E_0(0), \quad (1)$$

where $E_0(\partial\Omega)$ and $E_0(0)$ are, respectively, the zero-point energies in the presence and in the absence of boundaries. In the case of scalar fields, Casimir’s approach can be summarized in the following steps: a complete set of mode solutions of the Klein-Gordon equation satisfying an appropriate boundary condition and the respective eigenfrequencies are found; the divergent zero-point energy is regularized by the introduction of an ultraviolet cut-off and, finally, the polar part of the regularized energy is removed using a renormalization procedure.

It is well known that there are two quantities which might be expected to correspond to the total renormalized energy of quantum fields [7]. The first is called the mode sum energy $\langle E \rangle_{ren}^{mode}$,

$$\langle E \rangle_{ren}^{mode} = \int_0^\infty d\omega \frac{1}{2}\omega(N(\omega) - N_0(\omega)), \quad (2)$$

where $\frac{1}{2}\omega$ is the zero-point energy for each mode, $N(\omega)d\omega$ is the number of modes with frequencies between ω and $\omega + d\omega$ in the presence of boundaries and $N_0(\omega)d\omega$ is the corresponding quantity evaluated in empty space. Eq. (2) gives the renormalized sum of the zero-point energy of each mode. The second one is the volume integral of the renormalized energy density, $\langle E \rangle_{ren}^{vol}$, obtained by the Green’s function method [8]. In the latter method, in order to calculate the renormalized energy for any field, a certain second order

differential operator is applied to the renormalized Green's function $G_{ren}(x, x')$, *i.e.*,

$$\langle T_{\mu\nu} \rangle_{ren} = \lim_{x \rightarrow x'} D_{\mu\nu} [G(x, x') - G_0(x, x')], \quad (3)$$

where $G(x, x')$ is the Green's function in the presence of the boundary ($\partial\Omega$) and $G_0(x, x')$ is the Green's function in the absence of boundaries. Deutsch and Candelas [7] refer to the quantity between the brackets as the renormalized Green's function, since both Green's functions give rise to the same ultraviolet singularity structure (as $x \rightarrow x'$). If x belongs to the boundary ($\partial\Omega$) the renormalized stress-tensor $\langle T_{\mu\nu} \rangle_{ren}$ can diverge as one gets close to this surface. However, as was stressed by these authors, the above argument is not a proof that the renormalized stress-tensor $\langle T_{\mu\nu} \rangle_{ren}$ will diverge as we get closer to ($\partial\Omega$), but it suggests that if the renormalized stress tensor is bounded near ($\partial\Omega$) it means that a delicate cancellation must occur. In the case of a perfectly conducting spherical shell in the presence of an electromagnetic field both inside and outside the cavity there is a cancellation between the TE and TM modes, giving rise to a finite energy density even on the boundary [9].

It is important to point out that, for the minimally coupled scalar field, such cancellation does not occur, which renders the concept of the renormalized vacuum energy density $\langle T_{00} \rangle_{ren}$ ambiguous. However it is well known that the total renormalized vacuum energy associated with a minimally coupled scalar field obtained by the sum of modes method, $\langle E \rangle_{ren}^{mode}$, must be equal to that of the conformally coupled case, since both fields satisfy the same wave equation and have the same density of states. Nevertheless, the total renormalized vacuum energies obtained from the Green's function method, $\langle E \rangle_{ren}^{vol}$, for the minimal and conformal scalar fields, are different. Actually, $\langle E \rangle_{ren}^{mode}$ is found to be divergent. Which of these quantities, $\langle E \rangle_{ren}^{mode}$ or $\langle E \rangle_{ren}^{vol}$, therefore, is the "physical" renormalized energy of a minimally coupled scalar field? In the bag model this problem is also present [10].

Using the Green's function method, Bender and Hays [11] obtained a quadratic divergence for minimal scalar fields confined in the interior of the bag. Also Milton, investigating the zero-point energy of vector fields (gluons) confined in a spherical bag [12], obtained the same kind of quadratic divergence.

It has often been suggested by many authors that a full quantum mechanical treatment of boundary conditions can solve the above mentioned problem. Recently, Ford and Svaiter have confirmed these especulations [5] assuming fluctuating boundaries. By considering confining plates as quantum objects with a position probability distribution $|\psi(q)|^2$, it was shown that this approach is able to remove the discrepancy between $\langle E \rangle_{ren}^{mode}$ and $\langle E \rangle_{ren}^{vol}$ for the minimally coupled scalar field, solving a long standing paradox concerning the renormalized energy of the minimally and conformally coupled scalar fields.

There are many other different approaches in order to relax the classical boundary conditions. A long time ago, investigating the bag model, some authors discussed quantum corrections to this model by quantizing fluctuations around the classical bag solution [13]. Working in the same direction, Creutz [14] studied the effects of considering different bag configurations using the path integral approach in a theory with different massive scalar fields inside and outside the bag. Golestanian and Kardar [4] also use a path integral approach to investigate the problem of perfectly reflecting cavities that undergo an

arbitrary dynamical deformation. These authors were able to calculate the behavior of the mechanical response function (*i.e.*, the ratio between the induced force and the deformation field in the linear regime). Some authors, on the other hand, employed a simpler alternative approach, which allows one to deal with more general physical situations than the “hard” classical boundary conditions currently used in the literature. They imagine a confining “soft” boundary as modeled by a given smoothly increasing potential function representing some distribution of matter which interacts with the quantum field [1, 2]. Using this approach, it is possible to recover “hard” boundaries in some limit. This point will be clarified latter.

The aim of this paper is to discuss the Casimir effect for massless scalar fields subjected not only to “hard” boundaries in four dimensional spacetime but also to “soft” and “semihard” ones in a general D -dimensional flat spacetime. A classic question to be analysed is what actually determines the attractive or repulsive nature of the Casimir force. As it is well known, the sign of the Casimir energy may depend on the type of boundary conditions, on the ratio of the finite characteristic lengths of the cavity and on many other geometrical and topological features [15, 16]. Recently, the sign of the Casimir energy was discussed in Ref. [17] where some results obtained in Ref. [18] were generalized, but still assuming only Dirichlet b.c. It is our purpose here to address the question of the sign of the Casimir energy in these “new” confining situations in a D -dimensional flat spacetime.

The paper is organized in the following way. In Section II we review the most simple example of the Casimir energy dependence on the ratio between the characteristic lengths of a “hard” cavity. In Section III we analyse the Casimir effect for a minimally coupled scalar field in a D -dimensional spacetime in the presence of “hard” boundaries. In Section IV we investigate the Casimir effect in the presence of “soft” and “semihard” boundaries. In Section V we consider the situation of more than one confining potential being imposed onto the field. Conclusions are given in Section VI. Throughout this paper we use $\hbar = c = k_B = 1$.

2 The Casimir energy in a two-dimensional classical box

In order to get some insight on the problem of renormalized quantities confined in compact domains, we review, in this Section, a well known example. The most simple situation that we can imagine in which the vacuum energy is dependent on the ratio of characteristic lengths of a cavity is that of a minimally coupled scalar field satisfying classical boundary conditions in a $3D$ spacetime.

Let us consider a free massless scalar field confined in a $2D$ rectangular box satisfying Dirichlet boundary conditions. Although the presence of the corners are unphysical features, the model was also used by Peterson, Hansson and Johnson [19] in the study of loop diagrams of a confined scalar field in boxes, and it is suitable for our purposes.

A free real massless scalar field $\varphi(x)$ defined in a flat spacetime must satisfy the homogeneous Klein-Gordon equation. If we restrict the field to the interior of the box, the field modes are denumerable and the positive and negative frequency parts form

a complete orthonormal set. The renormalized energy can be obtained after suitable regularization and renormalization procedure of the infinite sum of the zero-point energy of each field mode. Because there is no difference between the density of modes of the minimal and of the conformal scalar fields, the example below covers both situations. In this Section we will make use of a standard procedure, following, for instance, Svaiter and Svaiter [20].

In the Fock representation, there must exist a particular vector $|0\rangle$, called the vacuum or the no-particle state. In a 3D spacetime the eigenfrequencies of the field are given by

$$\omega_{n_1 n_2} = \left(\left(\frac{n_1 \pi}{L_1} \right)^2 + \left(\frac{n_2 \pi}{L_2} \right)^2 \right)^{\frac{1}{2}} \quad n_1, n_2 = 1, 2, \dots, \quad (4)$$

where L_1 and L_2 are the lengths of the sides of the box. The zero-point energy is

$$E(L_1, L_2) = \frac{1}{2} \sum_{n_1, n_2=1}^{\infty} \omega_{n_1 n_2}, \quad (5)$$

where $\omega_{n_1 n_2}$ is given by Eq. (4). This expression is divergent and can be written as:

$$E_{\zeta}(L_1, L_2, s) = \frac{1}{2} \sum_{n_1, n_2=1}^{\infty} \omega_{n_1 n_2}^{-2s}, \quad (6)$$

for $s = -\frac{1}{2}$. Eq. (6) is analytic for $Re(s) > 1$. An analytic regularization method consists in evaluating the analytic continuation of the zeta function at the point $s = -\frac{1}{2}$. Algebraic manipulations of Eq. (6), using Eq. (5), give

$$E_{\zeta}(L_1, L_2, s) = \frac{1}{8} A\left(\left(\frac{\pi}{L_1}\right)^2, \left(\frac{\pi}{L_2}\right)^2; s\right) - \frac{1}{4} \left(\left(\frac{L_1}{\pi}\right)^{2s} + \left(\frac{L_2}{\pi}\right)^{2s} \right) \zeta(2s), \quad (7)$$

where $\zeta(2s)$ is the Riemann zeta function and $A(a, b, s)$ is the Epstein zeta function defined as:

$$A(a_1, a_2, s) = \sum'_{n_1, n_2=-\infty}^{\infty} (a_1 n_1^2 + a_2 n_2^2)^{-s}.$$

The prime sign in the summation means that the term $n_1 = n_2 = 0$ is to be excluded. Therefore $E_{\zeta}(L_1, L_2; s)$ is analytic in the complex s-plane for $s \in C/\{\frac{1}{2}, 1\}$, and the evaluation of $E_{\zeta}(L_1, L_2, -\frac{1}{2})$ gives the Casimir energy $U(L_1, L_2)$:

$$U(L_1, L_2) = \frac{\pi}{48} \left(\frac{1}{L_1} + \frac{1}{L_2} \right) - \frac{L_1 L_2}{32\pi} \sum'_{p, q=-\infty}^{\infty} (p^2 L_1^2 + q^2 L_2^2)^{-\frac{3}{2}}. \quad (8)$$

Instead of analytically regularizing the zero-point energy we can obtain the Casimir energy by introducing a suitable cut-off. For the details of these calculations see Ref. [21], and for a general discussion about analytic regularization methods used to obtain the renormalized vacuum energy of free fields in an arbitrary ultrastatic spacetime see Ref. [22]. A simple inspection of Eq. (8) shows that the sign of the Casimir energy depends on the ratio between L_1 and L_2 , and its behavior is shown in Fig. (1) and Fig. (2). In the next two Sections we will extend these calculations to a D-dimensional spacetime assuming not only Dirichlet boundary conditions but also other categories of boundaries.

3 The Casimir energy of a massless scalar field in the presence of “hard” boundaries in D dimensions

Let us consider a free massless scalar field $\varphi(t, \vec{x})$ defined in a $D = d + 1$ dimensional Minkowski spacetime. If we assume Dirichlet b.c. in a $D-1$ dimensional box with lengths L_1, L_2, \dots, L_{D-1} , the eigenfrequencies are given by:

$$\omega_{n_1, n_2, \dots, n_{D-1}} = \left[\left(\frac{n_1 \pi}{L_1} \right)^2 + \left(\frac{n_2 \pi}{L_2} \right)^2 + \dots + \left(\frac{n_{D-1} \pi}{L_{D-1}} \right)^2 \right]^{1/2}. \quad (9)$$

Using the condition $a = L_{D-1} \ll L_i, i = 1, 2, \dots, D-2$, the energy of the vacuum state is

$$E_D(L_i, a) = \frac{1}{2} \left(\prod_{i=1}^{D-2} L_i \right) \frac{1}{(2\pi)^{D-2}} \int_0^\infty dk_1 \dots \int_0^\infty dk_{D-2} \sum_{n=1}^\infty \left[(k_1)^2 + \dots + (k_{D-2})^2 + \left(\frac{n\pi}{a} \right)^2 \right]^{\frac{1}{2}}. \quad (10)$$

Note that the summation starts at $n = 1$ because for the scalar field one should not include the modes for which all integers n_1, n_2, \dots, n_{D-1} vanish. As was stressed in the previous Section there are two different ways to obtain the Casimir energy using the sum of modes method. The first one is to use dimensional regularization in the continuous variable in Eq. (10) and analytically extend the Epstein zeta function that will appear after dimensional regularization. A different approach is to use dimensional regularization in the continuous variables and to introduce a cut-off in the discret one. Let us use this second approach in this case. For “soft” and other types of boundaries it may be useful to consider both dimensional and zeta function analytic regularizations (see Section IV).

The angular part of the integral over the $D - 2$ dimensional k space can be calculated straightforwardly and if we define the energy per unit area by $\epsilon_D = \frac{E_D}{\prod L_i}$, for $i = 1, 2, \dots, D - 2$, we have

$$\epsilon_D(a) = F(D) \sum_{n=1}^\infty \int_0^\infty r^{D-3} \left(r^2 + \left(\frac{n\pi}{a} \right)^2 \right)^{\frac{1}{2}}, \quad (11)$$

where

$$F(D) = \frac{(2\sqrt{\pi})^{2-D}}{\Gamma(\frac{D-2}{2})}. \quad (12)$$

The energy per unit area is divergent and should be regularized. Let us introduce in Eq. (11) a convergence factor, *i.e.*, an ultraviolet regulator

$$\exp \left[-\lambda \left(r^2 + \left(\frac{n\pi}{a} \right)^2 \right)^{\frac{1}{2}} \right], \quad (13)$$

valid for $Re(\lambda) > 0$. The regularized energy per unit area is finite provided $Re(\lambda) > 0$, and is given by

$$\epsilon_D(\lambda, a) = F(D) \sum_{n=1}^\infty \int_0^\infty r^{D-3} \left(r^2 + \left(\frac{n\pi}{a} \right)^2 \right)^{\frac{1}{2}} \exp \left[-\lambda \left(r^2 + \left(\frac{n\pi}{a} \right)^2 \right)^{\frac{1}{2}} \right]. \quad (14)$$

The Casimir energy per unit area (the renormalized vacuum energy per unit area) is defined by

$$U_D(a) = \lim_{\lambda \rightarrow 0, R \rightarrow \infty} [\epsilon_D(\lambda, a) + \epsilon_D(\lambda, R - a) - \epsilon_D(\lambda, \eta R) - \epsilon_D(\lambda, (1 - \eta)R)], \quad (15)$$

where η is a real number between zero and one. A straightforward calculation gives (see [21]):

$$U_D(a) = -\frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2})\zeta(D)}{a^{D-1}}. \quad (16)$$

Thus the Casimir energy is negative for any D in this particular configuration. This result is in agreement with Ambjorn and Wolfram [16]. After a schematic review of this well known case, we are now in position to investigate two different kinds of boundaries: the “soft” and “semihard” ones. It is important to stress that since we are using the sum of modes method to find the Casimir energy, both the cases of minimally and conformally coupled scalar fields are covered. As we discussed before, this comes from the fact that there is no difference between the density of modes of the minimal and of conformal scalar fields.

4 The effect of “soft” and “semihard” boundaries in the Casimir energy

In this Section we will investigate the Casimir effect of a massless scalar field in the presence of “soft” and “semihard” boundaries in a general D -dimensional spacetime. The idea is to replace the “hard” Dirichlet walls by some confining potential $V(\vec{x})$ in the x_{D-1} direction [1] (the Dirichlet condition corresponds to the particular case where $V(\vec{x})$ vanishes inside the cavity and $V(\vec{x})$ becomes infinite on the boundary). In this case, the spatial modes of the scalar quantum field satisfy a Schrödinger-like equation and its spectrum will be denoted by ν_n^2 . This confining potential may be interpreted as representing some distribution of matter with which the quantum field interacts. Since the potential acts as effective plates, it is relevant to state that the Casimir forces will act upon the matter distribution modeled by $V(\vec{x})$. When all modes are completely suppressed only for $x \rightarrow \infty$, the effective boundary is called “soft”. We can also imagine an intermediate situation between “hard” and “soft” — that may be called “semihard” — where the complete suppression happens for a given finite x value. In this case, the potential $V(\vec{x})$ decreases smoothly from an infinite value on the boundary surface $\partial\Omega$ to $V = 0$ far from $\partial\Omega$. In this sense, Actor and Bender attribute a sort of “texture” to the boundary effective surface [1], where the case of the harmonic oscillator potential in a particular direction, say x_{D-1} , was investigated for $D = 4$ (see also [2]).

Assuming that the boundary conditions in the direction $x_{D-1} = x$ are dictated by a generic potential $V(x, a)$, (where a is a characteristic length of the system), the vacuum energy per unit area can be written as

$$\epsilon_D(a) = -\frac{1}{2(4\pi)^{\frac{D-1}{2}}} \Gamma\left(\frac{1-D}{2}\right) \sum_n^{\infty} (\nu_n^2)^{\frac{D-1}{2}}. \quad (17)$$

The first situation that we would like to discuss is that of a potential which is “semi-hard” near the origin and “soft” for large x . An example of such situation can be given by the following potential, plotted in Fig. (3):

$$V(x, a, b) = V_0^{1/2} \left(\frac{a}{x} - \frac{x}{b} \right)^2, \quad (18)$$

where a and b have dimension of $[length]$ and V_0 has the dimension of $[length]^{-2}$. The solution of the Schrödinger's equation in the case $b = a$ is well known [23]. However, the above potential is more suitable to work out some limits and indeed only slight changes are needed to get the solution for $b \neq a$. It is straightforward to show that the energy levels of the Schrödinger's equation for $b \neq a$ are given by

$$\nu_n^2 = \left(\frac{32V_0}{b^2} \right)^{\frac{1}{2}} \left[n + \frac{1}{2} + \frac{1}{4} \left(\sqrt{8V_0a^2 + 1} - \sqrt{8V_0a^2} \right) \right], \quad n = 0, 1, 2, \dots \quad (19)$$

Substituting Eq. (19) in the vacuum energy density given by Eq. (17) we obtain:

$$\epsilon_D(a, b) = -\frac{1}{2(4\pi)^{\frac{D-1}{2}}} \left(\frac{32V_0}{b^2} \right)^{\frac{D-1}{4}} \Gamma \left(\frac{1-D}{2} \right) \zeta \left(\frac{1-D}{2}; q \right). \quad (20)$$

After doing the analytic extension of the Hurwitz zeta-function

$$\zeta(z; q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z},$$

which is analytic at the beginning of an open connected set of points of the complex plane, *i.e.*, $Re(z) > 1$, we obtain for the vacuum energy density:

$$\epsilon_D(a, b) = -\frac{1}{2} \frac{1}{(4\pi)^{\frac{D-1}{2}}} \left(\frac{32V_0}{b^2} \right)^{\frac{D-1}{4}} \left[\int_1^{\infty} dt t^{-\frac{(1+D)}{2}} \frac{e^{t(1-q)}}{e^t - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n B_n(q)}{n!} \frac{1}{n - \frac{D+1}{2}} \right]. \quad (21)$$

In the above equation $B_n(q)$ are the Bernoulli polynomials [24] and q is given by

$$q = \frac{1}{2} + \frac{1}{4} \left(\sqrt{8V_0a^2 + 1} - \sqrt{8V_0a^2} \right). \quad (22)$$

To obtain the Casimir energy we have to subtract the polar part of the above equation, which is easily seen to be a single term in the summation, since the integral is finite. Two comments are in order: the first is that a renormalization procedure is necessary only for odd-dimensional ($D = 2m - 1, m = 1, 2, \dots$) spacetimes, because Eq. (20) is already finite for even D ; second is that the “soft” boundaries change the structure of the poles of the model, *i.e.*, the residues of the polar part are given by (D is an odd-integer number):

$$Res(\epsilon_D(a, b)) = \frac{(-1)^{\frac{D+1}{2}}}{\left(\frac{D+1}{2}\right)!} B_{\frac{D+1}{2}}(q).$$

Two limits are of interest (we keep b fixed from now on): (i) $a^2 \ll (V_0)^{-1}$ (hence $q \rightarrow 3/4$) in which case the potential behaves just as a “hard” impenetrable (Dirichlet) wall for $x \rightarrow 0$, while it behaves as a harmonic oscillator potential restricted to $x > 0$, and (ii) $a^2 \gg (V_0)^{-1}$ ($q \rightarrow 1/2$). In the formalism of Actor and Bender [1], it is also possible to obtain limit (i) from the harmonic oscillator (HO) result by just discarding some of the eigenvalues in the zeta-function, and this can be called the $\frac{1}{2}HO$ limit.

First let us investigate the limit $a^2 \ll (V_0)^{-1}$. In order to compare the Casimir energy with the value obtained in Ref. [1], where the $D = 4$ case was treated, we need to use the particular value of the Hurwitz zeta-function $\zeta(-3/2; 3/4) = 0.02093$. In addition, one can still define $\lambda = (V_0/b^2)^{-1/4}$, with dimension of $[length]$, and interpret λ as the “characteristic separation distance” between the “hard wall” at $x = 0$ and the “soft” one at $x \approx \lambda$. In this way, from Eq. (20) we readily obtain:

$$\epsilon_4(a, b) = -\frac{2^{7/4}}{3\pi}\lambda^{-3}\zeta(-3/2; 3/4), \quad (23)$$

and ϵ_4 is half the value found in Ref. [1]. It should be noted, however, that the value found in Ref. [1] for the well known Casimir energy in $D = 4$ between two Dirichlet plates, as the limit of their result for the harmonic oscillator potential, is also twice the value found in Refs. [16, 18]. In the limit of large separation between the “walls” ($\lambda \rightarrow \infty$), one recovers the free half-space result: $\epsilon \rightarrow 0^-$.

Let us examine now the values of the vacuum energy density given by Eq. (21), for $D = 2, 3, 4$, in both limiting cases $a^2 \gg (V_0)^{-1}$ and $a^2 \ll (V_0)^{-1}$ ($q \rightarrow 1/2$ and $q \rightarrow 3/4$ respectively). An interesting feature for low dimensional spacetimes, *i.e.*, for $D = 2$ and $D = 3$ is found: indeed, when $a^2 \gg (V_0)^{-1}$ the vacuum fluctuations give rise to a repulsive force corresponding to energy densities

$$\lim_{a^2 \gg (V_0)^{-1}} \epsilon_2(a, b) = [+0.0724]\lambda^{-1} \quad (24)$$

and

$$\lim_{a^2 \gg (V_0)^{-1}} \epsilon_3(a, b) = [+0.0160]\lambda^{-2}, \quad (25)$$

respectively, while in the limit $a^2 \ll (V_0)^{-1}$ the force becomes attractive, corresponding to energy densities

$$\lim_{a^2 \ll (V_0)^{-1}} \epsilon_2(a, b) = [-0.0562]\lambda^{-1} \quad (26)$$

and

$$\lim_{a^2 \ll (V_0)^{-1}} \epsilon_3(a, b) = [-0.0108]\lambda^{-2}. \quad (27)$$

Thus, for $D = 2, 3$, there must exist some finite a for which the Casimir energy vanishes. The same behavior was found in the case of the two-dimensional classical box considered in Section II, where the sign of the Casimir energy was shown to depend on the ratio between the lengths of the box [17]. It is important to emphasize that single poles appear in both limits only for $D = 3$. In the four-dimensional case, for both limits, the Casimir force is found to be always attractive, with corresponding values for energy densities

$$\lim_{a^2 \gg (V_0)^{-1}} \epsilon_4(a, b) = [-0.0059]\lambda^{-3} \quad (28)$$

and

$$\lim_{a^2 \ll (V_0)^{-1}} \epsilon_4(a, b) = [-0.0075]\lambda^{-3}, \quad (29)$$

and the latter is exactly the value of Eq. (23). Only in the limit $a^2 \ll (V_0)^{-1}$ the Casimir energy has the same (negative) sign as in the lowest spacetime dimensions, while in the limit $a^2 \gg (V_0)^{-1}$ the sign of the Casimir energy for $D = 4$ is opposite to those of $D = 2$ and $D = 3$.

In Table 1 we present the values of the Casimir energy for a massless scalar field in the presence of the asymmetric potential $V(x, a, b)$ in both limits $a^2 \ll (V_0)^{-1}$ and $a^2 \gg (V_0)^{-1}$, for spacetime dimensionality varying in the range $2 \leq D \leq 20$. Although we went up to $D = 100$, we report on the table, for simplicity, only the values for $D \leq 20$. From these numerical results we can discuss the sign of the Casimir energy, which is not straightforward from the analytical expression Eq. (21). The first point we would like to stress is that, independently of a , the Casimir energy ϵ_D has the sign $(-1)^{m+1}$ for $D = 4(m + 1)$, $m = 0, 1, 2, 3$. For $m \geq 4$, the sign becomes $(-1)^m$; indeed, for $D \geq 18$ this pattern of signs is broken. In any case, we have checked that even for the wide range $2 \leq D \leq 100$ the modulus of ϵ_D always decrease (we have assumed always $\lambda = 1$). Secondly, we would like to point out that, for this potential, it is possible to conceive a *gedanken experiment* to investigate the possibility of higher dimensions. Whenever it is found, varying a , that the Casimir energy vanishes (and hence changes sign), it is safe to assert that $D = 4l + 2$ or $D = 4l + 3$ (l being a natural number). Alternatively, as for $D = 4$, if the sign never changes, spacetime dimensionality should be $D = 4l$ or $D = 4l + 1$. This is the maximum we can conclude from the experience in this case. But once again we stress that the sign pattern is altered for $D \geq 18$: while for $D \leq 18$ ϵ_{4l} and ϵ_{4l+1} have the same sign, for $D \geq 18$ they have opposite signs.

The second situation that we would like to discuss is the case of an increasing potential in the x_{D-1} direction which becomes infinite for $x = 0$ and $x = a$. To represent this situation let us assume that the potential is given by

$$V(x) = V_0^{1/2} \cot^2\left(\frac{\pi x}{a}\right). \quad (30)$$

Using the Actor and Bender terminology [1], one can say that this situation is equivalent to two “semihard walls”, one at $x = 0$, the other at $x = a$. Note that, in this case, the field is confined to the interior of the region $0 < x < a$ (see Fig. (4)). It is straightforward to show that the values of ν_n^2 are given by

$$\nu_n^2 = \frac{\pi^2}{a^2}(n^2 + 4n\beta - 2\beta), \quad n = 1, 2, \dots \quad (31)$$

where β is defined by

$$\beta = \frac{1}{4} \left(\sqrt{\frac{8}{\pi^2} V_0 a^2 + 1} - 1 \right). \quad (32)$$

It follows that the vacuum energy density is now given by

$$\epsilon_D(a) = -\frac{\pi^{\frac{D-1}{2}}}{2^D} \frac{1}{a^{D-1}} \Gamma\left(\frac{1-D}{2}\right) \sum_{n=1}^{\infty} (n^2 + 4n\beta - 2\beta)^{\frac{D-1}{2}}. \quad (33)$$

Here it is necessary to analytically extend a modified Epstein zeta-function. This was done by Ford and also by Birrell and Ford [25]. In this case the analytic expression for the vacuum energy density is very complicated and it is not reported here, but an important difference between the first potential and this second one is that in some limits it is possible to recover exactly the Dirichlet walls, *i.e.*, perfectly conducting plates separated by a distance a . These limits can be obtained by expanding the potential around the point $x = \frac{a}{2}$ and neglecting terms higher than the second order. Let us first consider the limiting case $V_0 \rightarrow 0$. We see that, in this case, we come back to the problem of classical parallel plates (Dirichlet b.c.) placed in the x_{D-1} direction. When $V_0 \rightarrow 0$ also $\beta \rightarrow 0$ and the Casimir energy reduces again to (see Eq. (16)):

$$\epsilon_D(a) = -\frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2})\zeta(D)}{a^{D-1}}, \quad (34)$$

after making use of the reflection formula for $\zeta(1-D)$,

$$\Gamma(D/2)\pi^{-D/2}\zeta(D) = \Gamma\left(\frac{1-D}{2}\right)\pi^{\frac{D-1}{2}}\zeta(1-D).$$

For large values of the eigenvalue β , and for $n \ll \beta$, *i.e.*, for the lower levels, we get:

$$\nu_n^2 = \frac{\pi}{a}(n + 1/2). \quad (35)$$

This limit is completely analogous to the case of the harmonic oscillator potential studied in Refs. [1, 2]. In the next Section we compute the vacuum energy of a scalar field in some configurations where the field is constrained by different potentials in different directions.

5 The Casimir energy of a massless scalar field in a hyperbox with different boundary conditions

In Section IV the scalar field was supposed to be constrained by a hyperbox where only in one direction the “hard” Dirichlet plates were replaced by a confining potential. In this Section we analyse different situations in which, out of the D dimensions of spacetime, in p of them the field should satisfy Dirichlet boundary conditions, and in each of the remainder $D-p-1$ directions it is subjected to confining potentials. Moreover, as we are working in Cartesian coordinates, our formalism allows us to choose different potentials acting upon the field in each one of the remainder $D-p-1$ directions.

In the directions $x^i, i = 1, 2, \dots, p$, we impose the vanishing of the field at parallel plates located at $x^i = 0, L_i, i = 1, 2, \dots, p$ (Dirichlet boundary conditions); in the directions $x^j, j = p+1, p+2, \dots, D-1$, we choose $D-p-1$ potentials $V_j(x_j, a_j)$ (all of them may be *a priori* different), each one depending upon different characteristic sizes a_j . The vacuum energy is easily written as:

$$E_D = \frac{1}{2} \sum_{n_1, \dots, n_{D-1}} \left[\left(\frac{n_1 \pi}{L_1} \right)^2 + \dots + \left(\frac{n_p \pi}{L_p} \right)^2 + \sum_{j=p+1}^{D-1} \nu_{n_j}^2(a_j) \right]^{1/2}, \quad (36)$$

where the functions $\nu_{n_j}^2(a_j)$ represent the spectra of eigenvalues of the corresponding Schrödinger's equation.

Now, if all L_i are made much greater than all a_j , then we can replace the first p summations by integrals:

$$E_D = \frac{1}{2} \left(\prod_{i=1}^p L_i \right) \sum_{n_{p+1}, \dots, n_{D-1}} \int \frac{d^p k}{(2\pi)^p} \left[k^2 + \sum_{j=p+1}^{D-1} \nu_{n_j}^2(a_j) \right]^{1/2}, \quad (37)$$

where k is the length of the vector (k_1, \dots, k_p) . The integral above is in a well-suited form to apply dimensional regularization, with the result:

$$E_D = -\frac{1}{2(4\pi)^{\frac{p+1}{2}}} \left(\prod_{i=1}^p L_i \right) \Gamma\left(-\frac{1+p}{2}\right) \sum_{n_{p+1}, \dots, n_{D-1}} \left[\nu_{n_{p+1}}^2(a_{p+1}) + \dots + \nu_{n_{D-1}}^2(a_{D-1}) \right]^{\frac{1+p}{2}}. \quad (38)$$

Instead of discussing the general case, let us work out two specific cases, considering a $4D$ spacetime. In one spatial direction (for example z) let us impose Dirichlet boundary conditions (with plates separated by a distance L); hence $p = 1$. Besides, we will subject the field to the same potential in the x and y directions:

$$V(x, a, b) = V_0^{1/2} \left(\frac{a}{x} - \frac{x}{b} \right)^2 \quad (39)$$

and

$$V(y, a, b) = V_0^{1/2} \left(\frac{a}{y} - \frac{y}{b} \right)^2. \quad (40)$$

The respective spectra of the Schrödinger's equation are already known to us:

$$\nu_n^2 = \left(\frac{32V_0}{b^2} \right)^{\frac{1}{2}} (n + q) \quad (41)$$

and

$$\nu_m^2 = \left(\frac{32V_0}{b^2} \right)^{\frac{1}{2}} (m + q), \quad (42)$$

where the value of q is given by Eq. (22). Exploiting the symmetry between x and y directions, the summation which then appears from Eq. (38),

$$\sum_{n,m=0}^{\infty} [n + m + 2q], \quad (43)$$

can be put in a more tractable form by using (see [26]):

$$\sum_{n,m=0}^{\infty} [n + m + c]^{-s} = \zeta(s-1; c) + (1-c)\zeta(s; c). \quad (44)$$

From the general result of Eq. (38), we worked out the limits $a^2 \gg (V_0)^{-1}$ and $a^2 \ll (V_0)^{-1}$ (respectively $q \rightarrow 1/2$ and $q \rightarrow 3/4$). In the limit $a^2 \gg (V_0)^{-1}$ the vacuum energy is given by:

$$\lim_{a^2 \gg (V_0)^{-1}} E_4(a, b) = L \left(\frac{V_0}{b^2} \right)^{\frac{1}{2}} \frac{\sqrt{2}}{\pi} \left[\int_1^\infty dt \frac{t^{-3}}{e^t - 1} + \sum_{n=0}^\infty \frac{B_n}{n!} \frac{1}{n-3} \right]. \quad (45)$$

It is now easy to obtain the energy density, *i.e.*, energy per unit area, from the expression above. First one divides the expression by L . Although the plates in the x and y directions are replaced by confining potentials, one can still assign to these directions a “characteristic distance” between the “walls”, as stressed in the previous section, given by $\lambda = \left(\frac{V_0}{b^2}\right)^{-1/4}$. Therefore, in order to obtain the energy density ϵ , one further divides Eq. (45) by λ , which yields:

$$\lim_{a^2 \gg (V_0)^{-1}} \epsilon_4(a, b) = \lambda^{-3} \frac{\sqrt{2}}{\pi} \left[\int_1^\infty dt \frac{t^{-3}}{e^t - 1} + \sum_{n=0}^\infty \frac{B_n}{n!} \frac{1}{n-3} \right]. \quad (46)$$

where the integral is finite and the polar part, given by the fourth term in the summation, is identically zero because $B_3 = 0$ (otherwise it would be discarded as usual). In this way, the Casimir energy in this limit reads:

$$\lim_{a^2 \gg (V_0)^{-1}} \epsilon_4(a, b) = [-0.0069] \lambda^{-3}. \quad (47)$$

which is quite the same value of the limit obtained in Section IV, Eq. (28) (confining potential only in one direction).

In the other limit $a^2 \ll (V_0)^{-1}$ the vacuum energy density is given by (see Eq. (44)):

$$\begin{aligned} \lim_{a^2 \ll (V_0)^{-1}} \epsilon_4(a, b) &= \lambda^{-3} \frac{\sqrt{2}}{\pi} \left[\left(\int_1^\infty dt \frac{t^{-3} e^{-t/2}}{e^t - 1} + \sum_{n=0}^\infty \frac{(-1)^n B_n(3/2)}{n!} \frac{1}{n-3} \right) \right. \\ &\quad \left. + \frac{1}{4} \left(\int_1^\infty dx \frac{x^{-2} e^{-x/2}}{e^x - 1} + \sum_{m=0}^\infty \frac{(-1)^m B_m(3/2)}{m!} \frac{1}{m-2} \right) \right], \end{aligned} \quad (48)$$

where each summation contains a pole, with corresponding residues $\frac{-B_3(3/2)}{6} = \frac{-1}{8}$ and $\frac{B_2(3/2)}{2} = \frac{11}{24}$. The regularized vacuum energy density or, simply, the Casimir energy in this case is evaluated to give

$$\lim_{a^2 \ll (V_0)^{-1}} \epsilon_4(a, b) = [-0.0310] \lambda^{-3}, \quad (49)$$

also negative, giving rise to an attractive Casimir force. This value is to be compared to that of Eq. (29). Thus, in any case, the replacement of two parallel Dirichlet plates in one further direction, y , in comparison with the case of Section IV, makes the absolute value of the Casimir energy to increase.

For completeness, let us calculate the Casimir energy when this “soft” potential acts in all three spatial directions, again for $D = 4$; so $p = 0$ in this situation. We can

obtain, in this case, the vacuum energy density from Eq. (38), by simply dividing it by $\lambda^2 = \left(\frac{V_0}{b^2}\right)^{-1/2}$; it reads:

$$\epsilon_4(a, b) = +2^{1/4}\lambda^{-3} \sum_{n,m,l=0}^{\infty} [n + m + l + 3q]^{1/2}. \quad (50)$$

Again this summation can be simplified [26] if use is made of the relation:

$$\sum_{n,m,l=0}^{\infty} [n + m + l + c]^{-s} = \frac{1}{2}\zeta(s - 2; c) + \left(\frac{3}{2} - c\right)\zeta(s - 1; c) + \frac{1}{2}(c - 1)(c - 2)\zeta(s; c). \quad (51)$$

In the limit $a^2 \gg (V_0)^{-1}$ the Casimir energy is given by:

$$\lim_{a^2 \gg (V_0)^{-1}} \epsilon_4(a, b) = [-0.0132]\lambda^{-3}, \quad (52)$$

and in the limit $a^2 \ll (V_0)^{-1}$ it reads:

$$\lim_{a^2 \ll (V_0)^{-1}} \epsilon_4(a, b) = [+0.0255]\lambda^{-3}. \quad (53)$$

For this new configuration the result is qualitatively different from the situation with confining potentials only in two directions (previous case). In the limit $a^2 \gg (V_0)^{-1}$, ϵ_4 is still negative and twice the value of Eq. (47); in the other limit, $a^2 \ll (V_0)^{-1}$, ϵ_4 changes sign but its absolute value is smaller than the value of Eq. (49).

6 Final remarks

We have examined how the Casimir energy of a massless scalar field confined to the interior of a $D - 1$ dimensional hyperbox depends on different kinds of boundary conditions and on the dimensionality of spacetime. “Classical” Dirichlet boundary conditions in one, two and three directions were relaxed; in these directions the constraints on the field were assumed to be given by some smoothly increasing potential that represents some distribution of matter which interacts with the quantum field. The new contribution of the present work regards the study of the Casimir effect generated by two different types of boundary conditions, namely, the “soft” and “semihard” ones. In particular, we have discussed in details the case of a confining asymmetric potential $V(x) = V_0^{1/2}(a/x - x/b)^2$, which presents the feature of being “soft” for $x \rightarrow \infty$ and “semihard” for $x \rightarrow 0$. Although the choice of the potential is in general dictated by the solvability of the Schrödinger’s equation and by the manageable structure of the sum of proper modes, the study of how the Casimir effect depends on the boundary conditions opens new perspectives, which could lead to a deeper understanding of the interaction of real (not perfectly conducting) boundaries with the field.

Let us stress now some remarkable differences between the case studied here and the case of the Casimir energy in a D -dimensional “hard” hyperbox considered in Ref. [18]. The first one regards the repulsive or attractive nature of the Casimir force. In Ref. [18]

it was shown that the force is attractive if the number p of finite and equal edges of a rectangular box is *odd* or for very large even values of p , irrespective of D . However, it was also shown in that paper that for each small *even* p there exists a critical spacetime dimension $D_c(p)$ such that the force is repulsive if $D < D_c(p)$ and attractive otherwise. Our calculations have shown that, for the asymmetric potential considered here, there is no critical dimension D_c such that for $D > D_c$ the Casimir energy have always the same sign. What we have demonstrated is that, independently of the parameter a , there is a regular pattern for the sign of the Casimir energy for $D < 18$; for other values of D this sign pattern is broken (see Table 1). The computation of the vacuum energy up to $D = 100$ showed that $|\epsilon_D|$ always decrease with increasing D .

A second comment concerning the sign of the Casimir energy is related to how it changes when p' pairs of “hard” Dirichlet walls are substituted by “soft” and/or “semi-hard” potentials. It is well known that the “hard” wall Casimir energies in $D = 4$ are negative for $p = 1, 3$ and positive for $p = 2$. A significant difference between the “hard” and the “soft” wall Casimir effects for long waveguides was first pointed out in Ref. [1] and occurs only in the case where two pairs of Dirichlet plates were replaced with harmonic oscillator potential; when one or three pairs were substituted by the same potential, there is a qualitative similarity between the two cases. Our result, based on a different smoothly increasing potential, corroborates this tendency, but attention should be drawn to the following point when $p' = 3$. Our result in the limit $a^2 \ll (V_0)^{-1}$ should be compared to what is called the $\frac{1}{2}HO$ limit of Ref. [1], which is positive and equal to our result (up to a systematic factor of 2, as stressed in the text). However, in the other asymptotic limit $a^2 \gg (V_0)^{-1}$, the sign changes. Similarly, regarding “hard” walls for $D = 4$ and $p = 3$, it is known that in the symmetric configuration with equal edges ($L_1 = L_2 = L_3$), the Casimir energy is negative, while in Ref. [17] it was demonstrated that allowing $L_1 \neq L_2 \neq L_3$ the sign can also change.

As a last comment let us revisit the question of the instability of the semiclassical Abraham-Lorentz-Casimir model for the electron. The original Casimir’s idea was that the electrostatic repulsion due to the electron’s distribution of charge could be balanced by the zero-point fluctuations of the electromagnetic field inside and outside the conducting shell, assumed at that time to be “hard”. Unfortunately, in $D = 4$, the Casimir force is found to be repulsive [27]. How this fact depends on D was analysed in Ref. [18]. There, it was shown that the stability of a Casimir electron model in higher-dimensional spacetimes would be possible only for a number of dimensions $D \geq 8$. Therefore, one can imagine a toy model of a stable semiclassical electron where the Poincaré stress has quantum electromagnetic origin only if one lived in a higher-dimensional flat spacetime. All these results take into account that the boundaries of the electron are “hard” ones. In Ref. [17] it was shown that a negative zero-point energy can be obtained for such b.c. only for a very unexpected particular (and antisymmetric) shape and size. Nevertheless, in the light of the new results for confining “soft” boundaries, the Casimir idea of how the semiclassical electron could be stabilized may be revived. Indeed, there are two results that could be interpreted as an indication in this direction. First, it was demonstrated in Ref. [1] that the Casimir energy of a spherical “soft” cavity is negative. Second, Eqs. (52) and (53) show that it is possible to find a particular value of a which compensates the electrostatic repulsion in $D = 4$. Thus, these results suggest how the hypothesis

of a perfect conducting shell confining the electron was overwhelming. On the other hand, these results give rise to an important general question which, to the best of our knowledge, has no general answer yet, namely, how the sign of the Casimir energy changes when one changes the physical parameters and boundary conditions. It seems to us that the only way to discuss the attractive or repulsive character of the Casimir effect is, up to now, by direct computation case by case.

Finally, it should be pointed out that the mode sum energy defined by Deutsch and Candelas [7] stresses the fact that a classical “perfect conductor” boundary condition is unphysical and there is a sufficiently high frequency ν_0 for which the modes are not confined by the plates (in the case of dielectric materials this is called the plasma frequency). In view of this, the modes in the continuum will cancel out, leading us to assert that the only relevant modes to consider in the Casimir effect are the discrete ones. Consequently, a natural extension of this paper is to consider a partially transparent boundary, which can be modeled, for example, by the modified Pöschl-Teller potential given by $V(x) = -\frac{1}{2}\alpha \frac{\lambda(\lambda-1)}{\cosh^2 \alpha x}$ (a partially transparent boundary is the one which is transparent only for frequencies greater than ν_0). Another direction to look on is to investigate these confining potentials in different geometries, as for example a spherical one, trying to generalize the results obtained by Bender and Milton [28]. Interesting physical situations are those of a partially transparent sphere and spheres with “soft” and “semihard” boundaries. It is clear that this problem is of great interest in the framework of the bag model and may shed additional light on the Abraham-Lorentz-Casimir model for the electron. This subject is under investigation by the authors.

7 Acknowledgement

This paper was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico do Brazil (CNPq).

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D	$a^2 \gg (V_0)^{-1}$	$a^2 \ll (V_0)^{-1}$ ($\frac{1}{2}HO$ limit)
2	$+0.724 \times 10^{-1} \lambda^{-1}$	$-0.561 \times 10^{-1} \lambda^{-1}$
3	$+0.160 \times 10^{-1} \lambda^{-2}$	$-0.108 \times 10^{-1} \lambda^{-2}$
4	$-0.590 \times 10^{-2} \lambda^{-3}$	$-0.750 \times 10^{-2} \lambda^{-3}$
5	$-0.116 \times 10^{-2} \lambda^{-4}$	$-0.157 \times 10^{-2} \lambda^{-4}$
6	$-0.451 \times 10^{-3} \lambda^{-5}$	$+0.437 \times 10^{-3} \lambda^{-5}$
7	$-0.999 \times 10^{-4} \lambda^{-6}$	$+0.823 \times 10^{-4} \lambda^{-6}$
8	$+0.335 \times 10^{-4} \lambda^{-7}$	$+0.361 \times 10^{-4} \lambda^{-7}$
9	$+0.640 \times 10^{-5} \lambda^{-8}$	$+0.786 \times 10^{-5} \lambda^{-8}$
10	$+0.244 \times 10^{-5} \lambda^{-9}$	$-0.243 \times 10^{-5} \lambda^{-9}$
11	$+0.541 \times 10^{-6} \lambda^{-10}$	$-0.460 \times 10^{-6} \lambda^{-10}$
12	$-0.177 \times 10^{-6} \lambda^{-11}$	$-0.180 \times 10^{-6} \lambda^{-11}$
13	$-0.336 \times 10^{-7} \lambda^{-12}$	$-0.398 \times 10^{-7} \lambda^{-12}$
14	$-0.127 \times 10^{-7} \lambda^{-13}$	$+0.127 \times 10^{-7} \lambda^{-13}$
15	$-0.282 \times 10^{-8} \lambda^{-14}$	$+0.241 \times 10^{-8} \lambda^{-14}$
16	$+0.914 \times 10^{-9} \lambda^{-15}$	$+0.919 \times 10^{-9} \lambda^{-15}$
17	$+0.173 \times 10^{-9} \lambda^{-16}$	$+0.203 \times 10^{-9} \lambda^{-16}$
18	$-0.656 \times 10^{-10} \lambda^{-17}$	$+0.656 \times 10^{-10} \lambda^{-17}$
19	$+0.145 \times 10^{-10} \lambda^{-18}$	$-0.124 \times 10^{-10} \lambda^{-18}$
20	$+0.470 \times 10^{-11} \lambda^{-19}$	$+0.471 \times 10^{-11} \lambda^{-19}$

Table 1: Some limits of the Casimir energy for confining potential $V(x, a, b) = V_0^{1/2} \left(\frac{a}{x} - \frac{x}{b} \right)^2$ in one direction and Dirichlet b.c. in the other $D - 2$ directions for various D , where $\lambda = (V_0/b^2)^{-1/4}$.

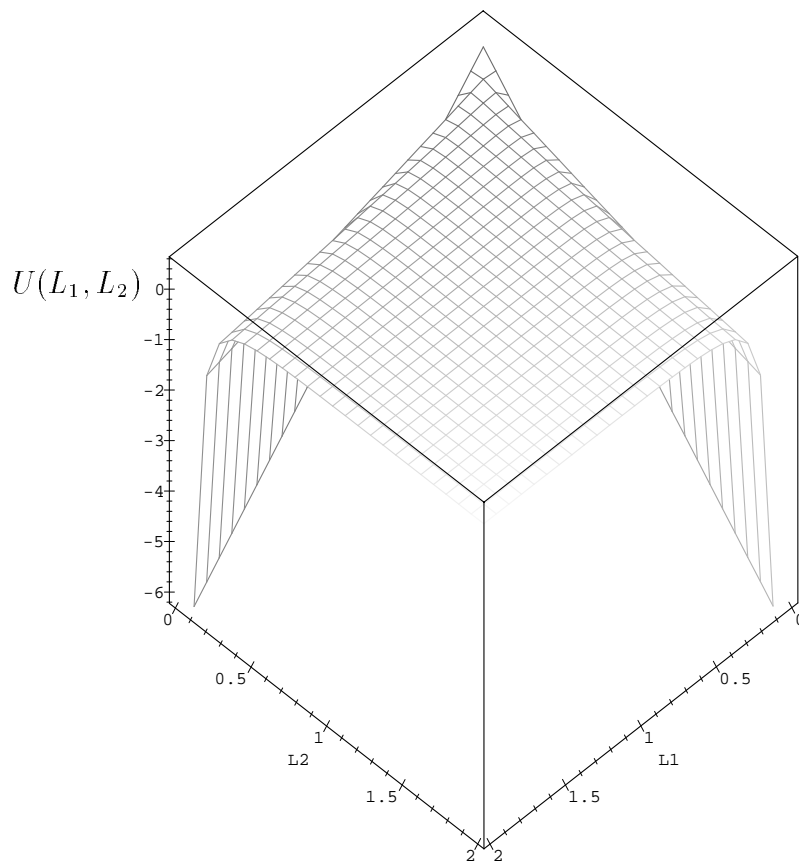


Figure 1: The Casimir energy of a two-dimensional classical box $U(L_1, L_2)$ as a function of its lengths.

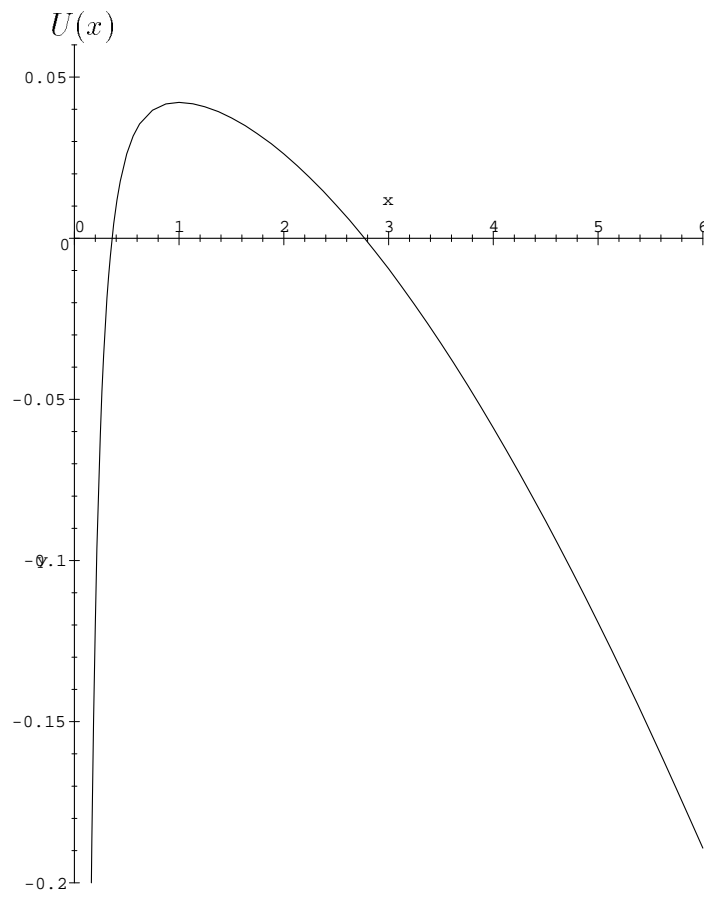


Figure 2: The Casimir energy of a two-dimensional classical box $U(L_1, L_2)$ as a function of $x = L_1/L_2$.

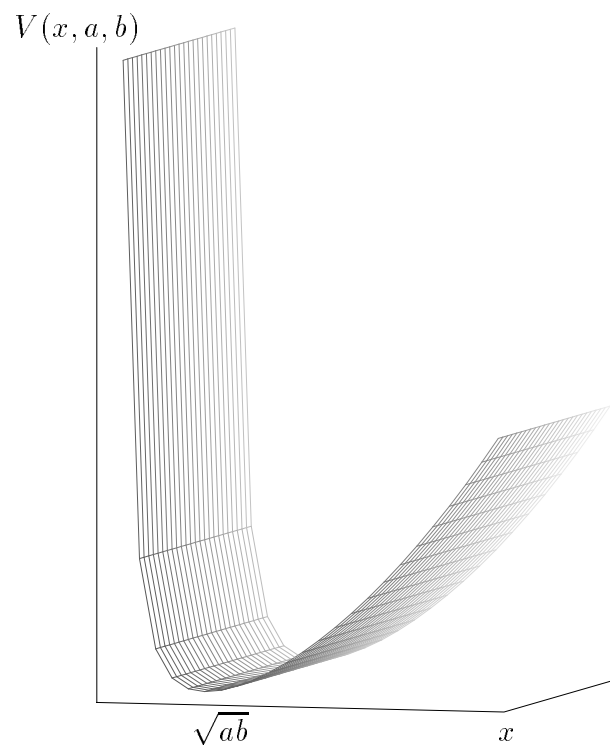


Figure 3: The asymmetric potential $V(x, a) = V_0^{1/2} \left(\frac{a}{x} - \frac{x}{b} \right)^2$, which is “semihard” for $x \rightarrow 0$ and “soft” for $x \rightarrow \infty$.

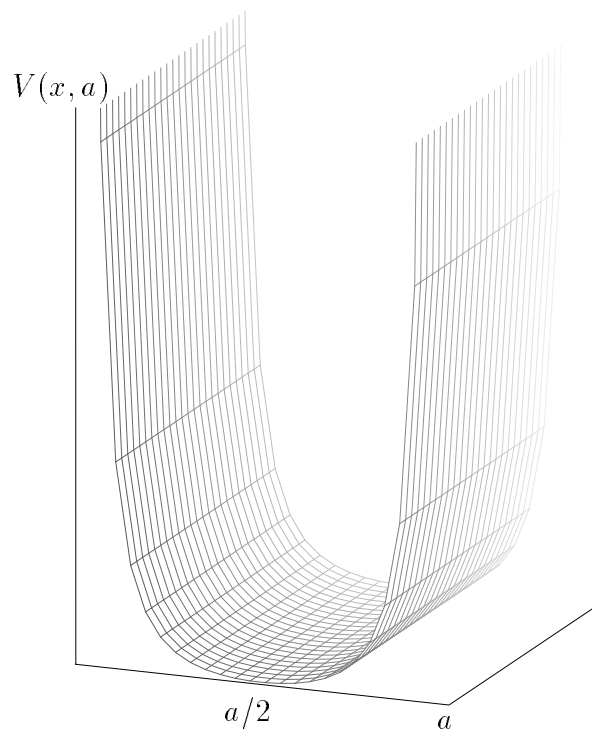


Figure 4: The semihard potential $V(x) = V_0^{1/2} \cot^2(\frac{\pi}{a}x)$, which reduces to Dirichlet walls separated by a in the limit $V_0 \rightarrow 0$.