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The Nucleon-Air Nuclei Interaction Probability Law With Rising Cross-Section

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Abstract

The diffusion equation of cosmic-ray nucleons is exactly integrated using the successive approximation method for a general distribution of the primary component, and taking into account the rising with energy of the nucleon-air nuclei cross-section.

The interaction probability law for the nucleon in the atmosphere is obtained as a consequence of the respective diffusion equation. If the nucleon-air nuclei cross-section rises logarithmically, this probability law assumes a binomial form, and for the constant cross-section it is purely poissonian.

We also compared our calculated nucleon intensity with the measured at sea-level: a good consistency is obtained with an average inelasticity coefficient around 0.60.

Key-words: Nucleon; Binomial distribution; Successive approximation.

1 Introduction

It's a well know fact that if we suppose that the inelasticity coefficient, k, and the nucleon-air nuclei interaction mean-free path, λ, are constants, then a nucleon that makes "n" interactions in traversing a depth x of the atmosphere, will have its energy reduced from $E_0 = \dfrac{E}{(1-k)^n}$ to E. So, the elementary energy contribution of the primary spectrum to the x level differential energy intensity is given by $G(E_0)dE_0 = G\left(\dfrac{E}{(1-k)^n}\right)\dfrac{dE}{(1-k)^n}$. Besides, G. Brooke et al[1] assuming, a priori, that the probability of a nucleon to make n interactions is given by the Poisson distribution law, $P_n(x) = e^{-x/\lambda}\dfrac{(x/\lambda)^n}{n!}$, obtained for the total flux at depth x the following expression

$$F(x,E) = \sum_{n=0}^{\infty} P_n(x)G(E_0)/(1-k)^n$$

F.M. Castro[2] derived the same expression integrating the diffusion equation of nucleons using the successive approximation method without any hypothesis on the collision's probability law, which resulted so to be poissonian.

In a similar way and using the same successive approximation method we solved exactly the nucleon diffusion equation supposing that the nucleon-air nuclei interaction mean-free path decreases with energy but the nucleon elasticity is still constant. In this case the interaction probability law is non-poissonian. Assuming, a logarithmic parametrization for the nucleon-air nuclei cross-section in the form $\sigma = \sigma_0(1+a\ln E/TeV)$ we obtained a binomial distribution.

We compared also our calculated nucleon flux with the measured at sea-level for different values of the inelasticity coefficient.

We estimated the effect of the rise of the nucleon-air nuclei cross-section of the type $\sigma(E) = \sigma_0(1 + a\ln E/TeV)$ on the exponent of the energy nucleon spectrum with the increase of the atmospheric depth. Here this power index rises non linearly with increase of the atmospheric depth.

2 Nucleon Diffusion Equation

The diffusion equation for nucleons in the atmosphere can be written as

$$\frac{\partial F(E,x)}{\partial x} = -\frac{F(E,x)}{\lambda(E)} + \frac{(E/\eta,x)}{\lambda(E/\eta)\eta} \tag{2.1}$$

where the nucleon elasticity, η is still constant but the interaction mean free path, $\lambda(E)$ decreasing with energy.

The equation (2.1) must be integrated with the boundary condition, $F(O,E) = G(E)$, where $G(E)dE$ stands for the differential energy spectrum of nucleons at the top atmosphere, $(x = 0)$.

Let us put

$$F(x,E) = e^{-x/\lambda(E)} y(x,E) \tag{2.2}$$

with this substitution, equation (2.1), and its initial condition become

$$\frac{\partial y(x,E)}{\partial x} = \frac{e^{-\delta(E,\eta)x}}{\eta\lambda(E/\eta)}\, y(x,E/\eta) \tag{2.3}$$

and

$$y(0,E) = G(E) \tag{2.4}$$

where

$$\delta(E,\eta) = \left(\frac{1}{\lambda(E/\eta)} - \frac{1}{\lambda(E)}\right) \tag{2.5}$$

The differential equation (2.3) with (2.4) is equivalent to the following single integral equation

$$y(E,x) = G(E) + \frac{1}{\eta\lambda(E/\eta)} \int_0^x e^{-\delta(E,\eta)t} y(E/\eta,t)dt \tag{2.6}$$

In order to solve equation (2.6), we use the following successive approximations:

$$y_0(E,x) = G(E) \tag{2.7}$$

and

$$y_n(E,x) = G(E) + \frac{1}{\eta\lambda(E/\eta)} \int_0^x e^{-\delta(E,\eta)t} y_{n-1}(E/\eta,t)dt \tag{2.8}$$

The approximation of order n order n is

$$y_n(E,x) = G(E) + \sum_{j=1}^{n} \frac{G(E/\eta^j)}{\eta^j\lambda(E/\eta)\lambda(E/\eta^2)\cdots\lambda(E/\eta^j)} \cdot$$
$$\cdot \int_0^x dt_n e^{-t_n\delta(\eta,E)} \cdot \int_0^{tn} dt_{n-1} e^{-t_{n-1}\delta(\eta,E/\eta)} \cdots$$
$$\cdots = \int_0^{t_2} dt_1 e^{-t_1\delta(\eta,E/\eta^{n-1})} \tag{2.9}$$

which can be compactly rewritten as

$$y_n(E,x) = G(E) + \sum_{j=1}^{n} \frac{\varepsilon_j(E)G(E/\eta^j)}{\eta^j}\ \phi_j\left(x, \delta(E,\eta)\right) \tag{2.10}$$

where

$$\xi_j(E) = \prod_{j=1}^{n} \frac{1}{\lambda(E/\eta^j)} \tag{2.11}$$

and $\phi_j(x, \delta(E,\eta))$ represents the multiple integral

$$\phi_j(x,\delta(E,\eta)) = \int_0^x dt_n e^{-t_n\cdot\left(\frac{1}{\lambda(E/\eta)}-\frac{1}{\lambda(E)}\right)} \cdot \int_0^{t_n} dt_{n-1} e^{-t_{n-1}\cdot\left(\frac{1}{\lambda(E/\eta^2)}-\frac{1}{\lambda(E/\eta)}\right)}$$

$$\cdots \int_0^{t_2} dt e^{-t_1\cdot\left(\frac{1}{\lambda(E/\eta^n)}-\frac{1}{\lambda(E/\eta^{n-1})}\right)} \tag{2.12}$$

and the approximation of order n for the function $F(x,E)$ is

$$F_n(x,E) = \left[G(E) + \sum_{j=1}^{n} \xi_j(E)\phi_j(x,\delta(E,\eta))\ \frac{G(E/\eta^j)}{\eta^j}\right] e^{-x/\lambda(E)} \tag{2.13}$$

3 Nucleon-air Nuclei Interaction Probability Law

The total nucleon flux, $F(x,E)$, can also be expressed in the following way,

$$F(x,E) = \sum_{n=0}^{\infty} P_n(x,E_0)\frac{G(E_0)}{\eta^n} \tag{3.1}$$

where $P_n(x,E_0)$ is the probability that a nucleon with energy E_0/η^n at $x=0$ interacts n times down to the depth x and it assumes different forms for different dependences of the interaction length with energy.

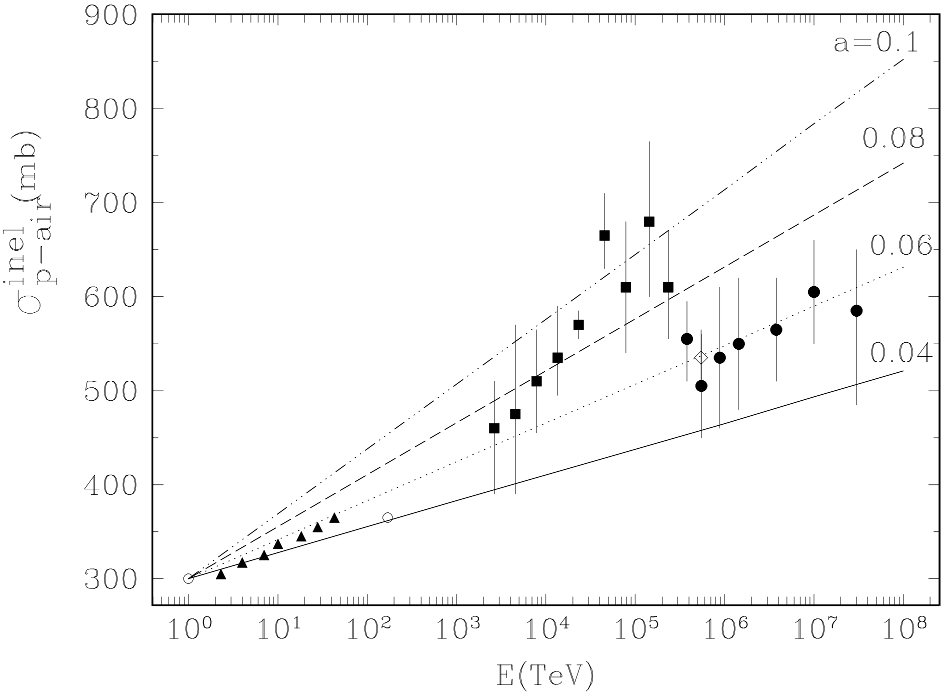

Figure 1: Inelastic cross-sections of p-air. Data are from (3) for ▲, (4) for ■, (5) for ◊, (6) for • and (7) for ◦. Four lines are drawn for a =0.04, 0.06, 0.08 and 0.10 in the formula $\sigma = 300(1 + a \ln E/TeV)$.

In Figure 1, the cross-sections of inelastic interactions between protons and air are plotted against energy. Data are from air shower experiments [3–6] and from accelerator experiments [7]. For the latter, $\sigma(pp)$ or $\sigma(\bar{p}p)$ were converted to σ (p-air) by the empirical formula of Hillas [8].

Several functional forms have been proposed to fit the behaviour of rising cross-section, among which we adopt the following one in our calculation

$$\sigma = \sigma_0(1 + a \ln E/\varepsilon) \tag{3.2}$$

In figure 1, three cases of the logarithmically energy-dependent cross-section are shown by full lines for a guide.

In this case, the expressions (2.10) and (2.11) take the forms

$$\xi_j(E) = \delta(\eta)^j \ \frac{\Gamma(Z(E)+j+1)}{\Gamma(Z(E)+1)} \ . \tag{3.3}$$

and

$$\phi_j(x, \delta(\eta)) = \frac{1}{j!} \ \left(\frac{1 - e^{-\delta(\eta)x}}{\delta(\eta)} \right)^j$$

where

$$\begin{aligned} Z(E) &= \frac{1 + a\ln(E/\varepsilon)}{\delta(\eta)\lambda_0} \ , \\ \delta(\eta) &= \frac{a}{\lambda_0} \ \ln/\eta \end{aligned}$$

and

$$\frac{1}{\lambda(E)} = \frac{1}{\lambda_0} \ (1 + a\ln E/\varepsilon) \ , \ \lambda_0 = 80g/cm^2$$

Thus, the approximation of order n (2.12) becomes

$$y_n(x,E) = e^{-x/\lambda(E)} \left\{ G(E) + \sum_{j=1}^{n} \frac{(1 - e^{-\delta(\eta)x})^j}{j!} \ \frac{\Gamma(Z+j+1)}{\Gamma(Z+1)} \ \frac{G(\frac{E}{\eta^j})}{\eta^j} \right\} \tag{3.4}$$

As described in the appendix, this partial sum converges absolutely and uniformly to the solution of the integral equation (2.6) as

$$y(E,x) = \lim_{n\to\infty} \ y_n(E,x) \tag{3.5}$$

The differential nucleon flux at a depth x with energy between E and $E + dE$ is

$$F(x,E) = \sum_{n=0}^{\infty} \ e^{-x/\lambda(E_0\eta^n)} \ \frac{\Gamma(Z+n+1)}{n!\Gamma(Z+1)} \ \left(1 - e^{-\delta(\eta)x}\right)^n \tag{3.6}$$

Thus, we get immediately the differential intensity at the atmospheric depth, x, without any hypotheses on the collision's probability distribution, which results to be the binomial,

$$P_n(E_0,x) = e^{-x/\lambda(E_0\eta^n)} \ \frac{\Gamma(Z+1+n)}{n!\Gamma(Z+1)} \ \left(1 - e^{\delta(\eta)x}\right)^n \tag{3.7}$$

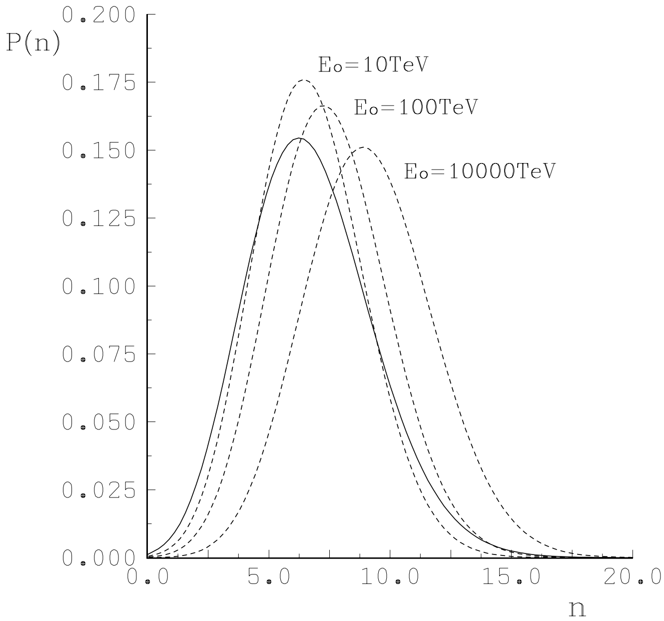

Figure 2: Nucleon-air nuclei interaction probability law at $x = 540g/cm^2$. Solid curve represents the Poisson distribution law for $\lambda = 80g/cm^2$ (constant) and the broken curves the binomial distribution for $\lambda(E) = 80(1 + 0.06 \ln E/TeV)$.

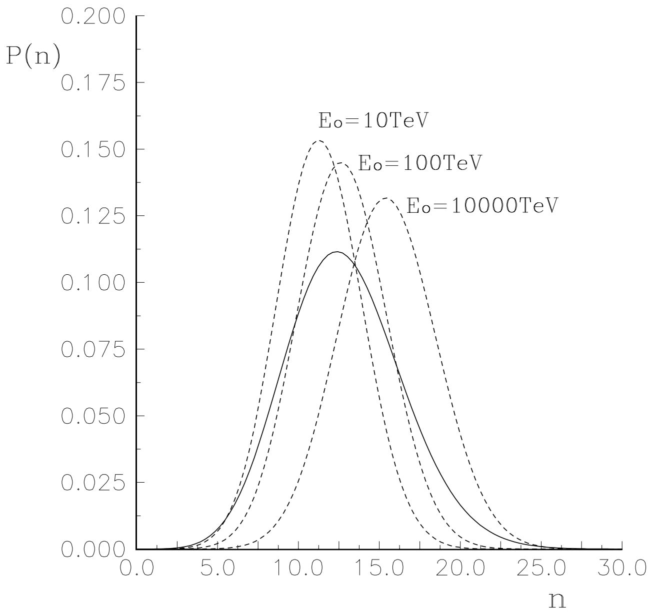

Figure 3: Nucleon-air nuclei interaction probability law at $x = 1030g/cm^2$. Solid curve represents the Poisson distribution law for $\lambda = 80g/cm^2$ and the broken curves the binomial distribution for $\lambda(E) = 80(1 + 0.06 \ln E/TeV)$.

Figure 2 and Figure 3 shows the nucleon-air nuclei interaction probability law plotted against the number of interactions (n) for two different atmospheric depth, $x = 540 g/cm^2$ $x = 1030 g/cm^2$. These figures show a comparison of our solution (expression 3.8) with the Poisson Distribution obtained for the case of the nucleon interaction length constant.

We notice from the figures that as the primary energy increases the mean value of the nucleon interactions in the atmosphere increases. For the case of the Poisson distribution this value is constant, independent of energy. Both distributions are normalized

$$\sum_{n=0}^{\infty} P_n(E_0, x) = 1 \ , \text{ for a fixed } E_0 \ . \tag{3.8}$$

In the usual Grigorov approximate solution, $P_n(E_0, x)$ can be written as

$$P_n(E_0, x) = \frac{e^{-x/\lambda(E_0 \eta^n)}}{n!} \left(\frac{x}{\lambda(E_0 \eta^n)} \right)^n \tag{3.9}$$

assuming constant elasticity. This expression is a Poissonian Distribution, where the mean-free path of a nucleon at depth x is dependent of the energy. The sum $P_n(E_0, x)$ at a given value of E_0 is smaller than unity, which results in lower interaction probability and higher nucleon number ours.

4 Comparison With Experimental Data

In order to make a comparison with the nucleon fluxes measured at sea level(10) we need to take into account the primary cosmic-ray flux in addition to the cross-section. At the top of the atmosphere, protons share the majority of incoming cosmic-ray particles, but the number of nuclei cannot be negligible in order to study the nucleon flux. Bhattacharyya(11) analysed experimental data of balloon-borne experiments and reported the nucleon flux at the top the atmosphere to be 2.237 $E^{-2,7}$ $(cm^2.st.GeV/nucleon)^{-1}$.

Thus, for the primary nucleon flux of the form

$$G(E) = N_0 E^{-(\gamma+1)} \tag{4.1}$$

our solution (3.7) becomes

$$F(X, E) = N_0 E^{-(\gamma+1)} e^{-x/\lambda(E)} (1 - \alpha)^{-(Z+1)} \tag{4.2}$$

with

$$\alpha = \eta^\gamma \left(1 - e^{-\delta(\eta)x}\right) \tag{4.3}$$

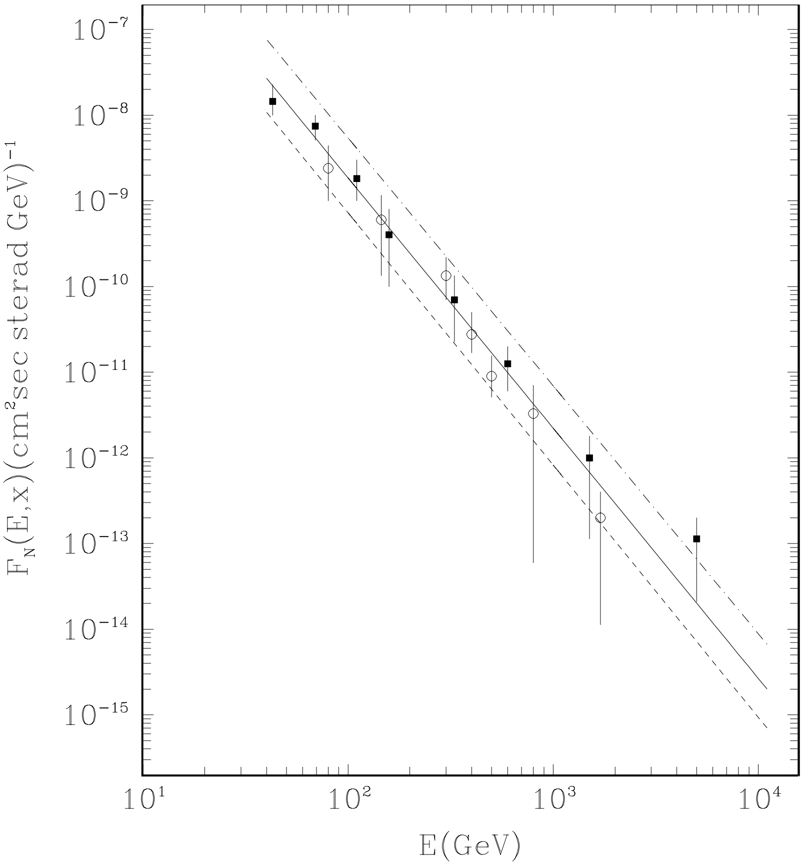

Figure 4: Differential nucleonic intensity sea level. Data are from (10) ($\bullet, 0$). Three lines are drawn for differents values of the inelasticity coefficient: ($- \cdot - \cdot -$) for $< K >= 0.5$, (—) for $< K >= 0.6$ and ($- - - -$) for $< K >= 0.7$.

Figure 4 shows the comparison of our solution (4.2) with the nucleon fluxes measured at sea-level[10]. Three curves of the mean value of nucleon inelasticity coeficient, $< K >=$ 0.5, 0.6 and 0.7 are also drawn in the figure. We see that the curve $< K >= 0.6$ gives a good agreement with experimental data.

Using the approximation $\ln(1-\alpha) \simeq -\alpha$, we derived from our solution (4.2) the power index γ' at the atmospheric depth x, as

$$\gamma' = \gamma + \frac{ax}{\lambda_0} + \frac{\eta^{\gamma}(1 - \eta^{ax/\lambda_0})}{\ln \eta} \tag{4.4}$$

From the usual and compact Grigorov approximate solution[9] which has been widely accepted and used in the field;

$$F(X, E) = N_0 E^{-(\gamma+1)} e^{-x/L(E)} \tag{4.5}$$

with

$$L(E) = \frac{\lambda(E)}{1 - <\eta^{\gamma}>} \tag{4.6}$$

the power index γ' is obtained as

$$\gamma' = \gamma + \frac{ax}{\lambda_0} - \frac{ax}{\lambda_0}\,\eta^{\gamma} \tag{4.7}$$

If the nucleon interaction mean free path is constant, the exponent γ' is also constant and independent of the atmospheric depth. We have

$$\gamma' = \gamma \tag{4.8}$$

for the case $\lambda(E) = \lambda_0$

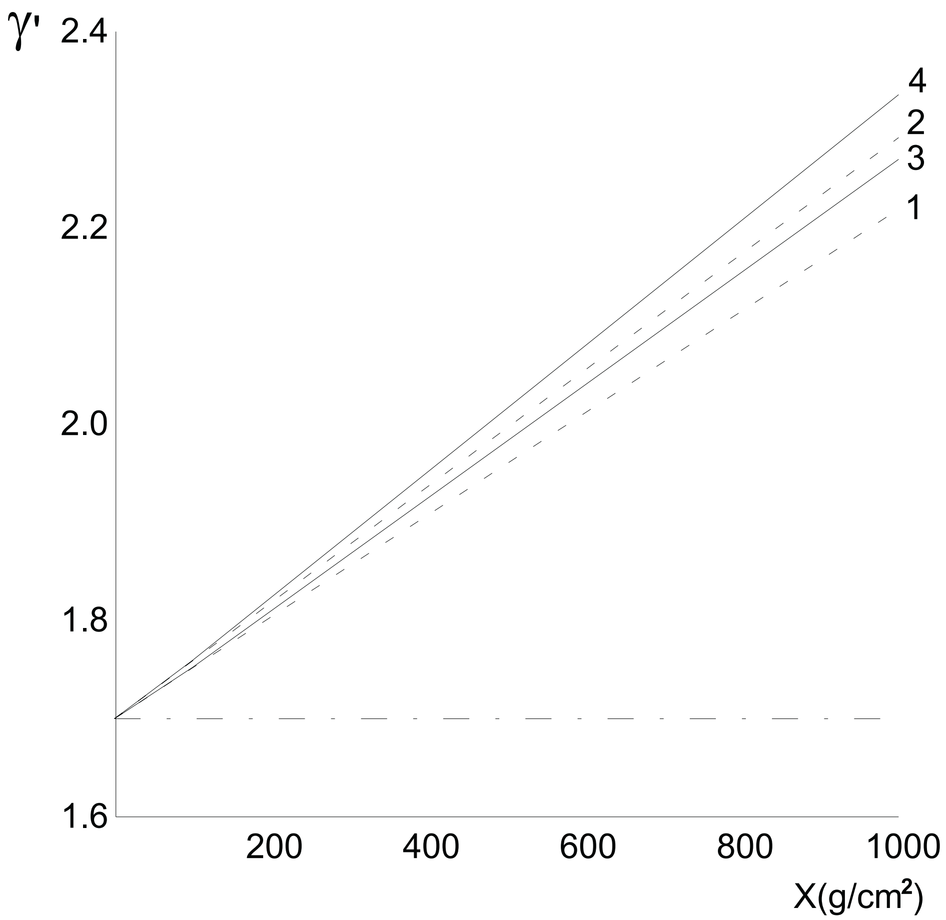

Figure 5: Power index of the nucleon spectrum against atmospheric depths. Five lines are drawn: The dashed-dotted line represents the expresssion (4.8), the dashed lines 1 and 2 represent the expression (4.7) for $< K >= 0.5$ and 0.6 respectively and the full lines 3 and 4 represent the expression (4.4) for $< K >= 0.5$ and 0.6 respectively.

Figure 5 shows the variation of the power index γ' with atmospheric depth in case of $\gamma = 1.7$, $a = 0.06$, and $\lambda_0 = 80 g/cm^2$. The dashed-dotted line represent the expression (4.8), the dashed lines 1 and 2 represents the expression (4.7) for $< K >= 0.5$ and $< K >= 0.6$ respectively and full lines 3 and 4 represents the expression (4.4) for $< K >= 0.5$ and $< K >= 0.6$ respectively. We notice from the figure that the power index derived from our solution becomes larger at deeper atmospheric depths than the others two.

5 Discussions and Conclusions

We have solved the diffusion equation of cosmic-ray nucleons analytically and exactly taking into account the rising cross-section of the form $\sigma_0(1 + a \ln E/TeV)$. We obtained further the binomial distribution law for the nucleon-air nuclei interactions as a consequence of the respective diffusion equation. In the case of constant cross-section, the function $\delta(\eta)$ is zero and the $P_n(E_0, x)$ is purely poissonian and independent of the primary energy, E_0.

Through a comparison of our solution with differential nucleon fluxes measured at sea level, we have found that the mean value of inelasticity, $0,60$, gives a good agreement. But, is important to observe that a change of the dependence with energy of the nucleon interaction lengths and the inclusion of an elasticity distribution largely affects the nucleon flux.

We show also that the rise of the inelastic cross-section of the type $\sigma(E) = \sigma(1 + a \ln E/TeV)$ up to energies $10^7 GeV$ lead to an increase of the exponent of the nucleon spectrum with increase of the atmospheric depth and to a softening of the nucleon intensity. The power index rises non linearly with the increase of the atmospheric depth and rises quicker than the derived from the Grigorov approximate solution.

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Appendix

Convergence of $y_n(E,n)$

The function $G(E)$ is supposed to be continuous, positive and bounded ($G(E) < M$) for $E > 0$. Thus we have

$$|y_n(E,x)| \leq M \sum_{j=0}^{n} \frac{\Gamma(Z+1+j)}{\Gamma(Z+1)j!}\, \beta^j$$

where M is a positive constant and $\beta = \dfrac{(1-e^{-\delta x})}{\eta}$. The right-hand side of this expression is the partial sum of order n of the expansion $M(1-\beta)^{-(Z+1)}$ with convergence radius $\rho = 1$ when $|\beta| < 1$. In this manner the partial sum $y_n(E,x)$ represented in (3.5) converges absolutely and uniformly to the solution (3.6). The function $y(X,E)$ is continuous, in the closed set T whose points satisfy the conditions $E > 0$ and $e^{-\delta x} > 1 - \eta$.

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