

“A Deformed Model for the Unstable Phonon Spectrum in ${}^4\text{He}$ ”

M.Rego-Monteiro^{+a}, L.M.C.S. Rodrigues^{*b} and S. Wulck^{**c}*

^{*}Centro Brasileiro de Pesquisas Físicas - CBPF
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

⁺INFN – Sezione di Torino
Dipartimento di Fisica dell’Università di Torino
Via Pietro Giuria, 1
10125 Torino, Italy

^{**}Instituto de Física
Universidade Federal do Rio de Janeiro
Cidade Universitária - Ilha do Fundão
21945-970 – Rio de Janeiro, RJ – Brazil

ABSTRACT

The theoretical derivation of the phonon anomalous dispersion relation, $w^{ph} = c_0 p(1 - \gamma p^2)$, predicts that γ is positive and therefore that the four-phonon process is the dominant scattering, thus contradicting experimental results of phonon lifetime in neutron scattering and of specific heat measurements. We show that this discrepancy between theory and experiment concerning the stability of phonon spectrum is overcome when we treat the gas of phonon excitations in ${}^4\text{He}$ as an ideal deformed bosonic gas. When this model is compared with data from ${}^4\text{He}$ specific heat measurements for $T < 1\text{ K}$, it reproduces within 5% of accuracy the curves resulting from least-squares fits of those data.

Key-words: Liquid Helium; q -Oscillators; Superfluidity.

PACS: 67.40.Kh, 67.90.+z, 05.30.Jp .

^(a)MONTEIRO@to.infn.it

^(b)ligia@cbpfsu1.cat.cbpf.br

^(c)steniow@if.ufrj.br

1 Introduction

The phenomenon of superfluidity, which is the ability of a liquid to flow through narrow capillaries without any friction, was discovered by Kapitza [1]. It appears in liquid Helium [1, 2] below the λ -point, which is the temperature where it undergoes a second order phase transition ($T_\lambda = 2.18\text{ K}$) [3].

The superfluid properties of ${}^4\text{He}$ are well described by Landau theory [4]. In this theory, which is based on a quantum liquid concept, the superfluidity in ${}^4\text{He}$ follows from phonon and roton excitations [5]. Even for very low temperatures (below 1 K), the phonons present an anomalous dispersion given by

$$\omega^{ph} = c_0 p (1 - \gamma p^2) ,$$

where c_0 is the sound velocity. This dispersion relation was theoretically derived from a hydrodynamic Hamiltonian [6]: through a self-energy calculation using the lowest order perturbation theory, the constant γ was estimated to be positive. The positivity of γ implies that the three-phonon process is not allowed and therefore the four-phonon process would be the dominant scattering. This is in disagreement with experimental results. On one side, a negative γ is obtained (for most values of the pressure) in ${}^4\text{He}$ specific heat measurements [7, 8]. Besides, experimental data of phonon lifetime in scattering of neutrons show that the phonon spectrum is unstable, that is, the three-phonon process does occur [9].

In ${}^4\text{He}$, phonon excitations are dominant in the lowest part of the dispersion curve [4]. As the momentum approaches $p_0/\hbar = 1.9\text{ \AA}^{-1}$, the rotons contribution becomes more and more relevant. Their dispersion is given by $\omega^{rot}(p) = \Delta + \frac{(p - p_0)^2}{2\mu}$, where $\frac{\Delta}{k_B} = 8.6\text{ K}$ and the effective mass of the roton $m = 0.15\text{ }m_{\text{He}}$. For temperatures not too close to the λ -point, the gas of elementary excitations is treated as an ideal bosonic gas. The phonon gas obeys Bose-statistics and the minimum roton energy is high enough so that the rotons follow Boltzmann distribution.

The purpose of this paper is to present a model where the discrepancy between theory and experiment concerning the stability of the phonon spectrum in ${}^4\text{He}$ is overcome [10]. We show that treating the gas of phonon excitations as an ideal deformed bosonic gas, we find positive values for γ , but due to the influence of the deformation parameters, the phonon spectrum is unstable (in accordance with the experimental data). These results are obtained by comparing our model with data from ${}^4\text{He}$ specific heat measurements [8] and we reproduce, within 5% of accuracy, the curves resulting from least squares fits of those data.

A deformed ideal bosonic q -gas is a system that generalizes the ideal boson gas using a set of independent q -oscillators [11, 12], which are objects that satisfy deformed Heisenberg algebra, instead of the standard bosonic oscillator. Deformed Heisenberg algebras are non-trivial generalizations of the Heisenberg algebra through the introduction of one (or more) [13] deformation parameters such that the non deformed case is recovered for a special limit of the parameter (s). It is worth to note that q -oscillators can be interpreted as usual quantum oscillators presenting a particular type of non-linearity [14]. Moreover, a special class of deformed algebra, called “quasi-triangular Hopf algebras” in the mathematical

literature [15], presents an interesting connection with q -oscillators: the q -analogue of classical Lie Algebras [16, 17], which are examples of quasi-triangular Hopf algebras, can be realized in terms of q -oscillators [11, 18]. Finally, interesting systems possessing deformed symmetry algebras have been found in several areas of physics [19–21].

Most of the literature on the ideal boson q -gas refers to a representation of a q -oscillator algebra that we call “fundamental representation” [22]. Different representations characterized by a parameter ν_0 have been obtained [23]. Recently, the consequences of using such inequivalent representations on an ideal q -gas were discussed and ν_0 was interpreted as a “background effect” [22]. This parameter is crucial to construct our ideal q -gas model for the phonon spectrum in ${}^4\text{He}$ [10].

This paper is organized as follows: in section 2 we review the ideal q -gas in its fundamental representation ($\nu_0 = 0$); in section 3 we extend to the ($\nu_0 \neq 0$) inequivalent representations; in section 4 we present our model for the phonon spectrum in ${}^4\text{He}$ and finally, in section 5 we compare it with data on C_V measurements of ${}^4\text{He}$ and present our conclusions.

2 Ideal q -gas in the “fundamental representation”

One calls bosonic q -oscillators (or deformed Heisenberg algebra) [11, 12] the associative algebra generated by the elements a, a^+ and N satisfying the relations

$$\begin{aligned} [N, a^+] &= a^+ \quad , \quad [N, a] = -a \\ aa^+ - qa^+a &= q^{-N} \quad , \end{aligned} \tag{2.1}$$

where the deformation parameter $q \in \mathbb{R}$. Note that a and a^+ are generalizations of the usual annihilation and creation operators and that the standard algebra is recovered in the limit $q \rightarrow 1$.

It is possible to construct the representation of relation (2.1) in the Fock space \mathcal{F} generated by the normalized eigenstates $|n\rangle$ of the number operator N as

$$\begin{aligned} a|0\rangle &= 0 \quad , \quad N|n\rangle = n|n\rangle \quad ; \quad , \quad n = 0, 1, 2 \dots \\ |n\rangle &= \frac{1}{\sqrt{[n]!}} (a^+)^n |0\rangle \quad ; \end{aligned} \tag{2.2}$$

where $[n]! \equiv [n] \cdots [1]$, $[n] = (q^n - q^{-n})/(q - q^{-1})$ and $[n] \rightarrow n$ as $q \rightarrow 1$.

In \mathcal{F} one can express the deformed oscillators in terms of the standard bosonic ones b, b^+ , which obey the usual Heisenberg algebra,

$$bb^+ - b^+b = 1 \quad , \quad N = b^+b \quad , \tag{2.3}$$

as [24]

$$a = \left(\frac{[N+1]}{N+1} \right)^{1/2} b \quad , \quad a^+ = b^+ \left(\frac{[N+1]}{N+1} \right)^{1/2} \quad ; \tag{2.4}$$

and it can easily be shown, using (2.3), that

$$aa^+ = [N + 1] \quad , \quad a^+a = [N] . \quad (2.5)$$

It is well known that the $SU(2)$ algebra

$$[J_0, J_{\pm}] = \pm J_{\pm} \quad , \quad [J_+, J_-] = 2J_0 \quad (2.6)$$

can be realized à la Jordan Schwinger through the use of a pair of creation and annihilation operators as

$$J_+ = b_1^+ b_2 \quad , \quad J_- = b_2^+ b_1 \quad , \quad J_0 = \frac{1}{2}(N_1 - N_2) = \frac{1}{2}(b_1^+ b_1 - b_2^+ b_2) . \quad (2.7)$$

It was shown by Biedernarn and Macfarlane that the appropriate objects to realize $SU_q(2)$ (the deformed $SU(2)$ algebra),

$$[J_0, J_{\pm}] = \pm J_{\pm} \quad , \quad [J_+, J_-] = 2[J_0] \quad (2.8)$$

are the q -oscillators. In fact, considering a pair of independent q -oscillators a_1 and a_2 , one realizes $SU_q(2)$ à la Jordan Schwinger as [11]

$$\begin{aligned} J_+ &= a_1^+ a_2 \quad , \quad J_- = a_2^+ a_1 \\ J_0 &= \frac{1}{2}(N_1 - N_2) \neq \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2) . \end{aligned} \quad (2.9)$$

Further, with

$$n_1 = j + m \quad , \quad n_2 = j - m \quad , \quad (2.10)$$

one can realize the $|j, m\rangle$ basis of $SU_q(2)$ by means of the above q -oscillators as:

$$|j, m\rangle = |n_1\rangle |n_2\rangle = \frac{(a_1^+)^{j+m}}{\sqrt{[j+m]!}} \frac{(a_2^+)^{j-m}}{\sqrt{[j-m]!}} |0\rangle . \quad (2.11)$$

As expected, in the $q \rightarrow 1$ limit the non-deformed algebras are recovered.

We are interested in real $q > 1$ deformed q -oscillators. We note that in the $q \rightarrow \infty$ limit for $n \geq 2$, $[n] \rightarrow \infty$ and as a result when $q = \infty$ Fock space

$$|n\rangle = \frac{1}{\sqrt{[n]!}} (a^+)^n |0\rangle ,$$

is reduced to a fermionic one since $|n\rangle$ vanishes for $n \geq 2$ [25].

Let us consider an ideal deformed system described by the Hamiltonian [27, 25, 26]:

$$H = \sum_i \omega_i a_i^+ a_i = \sum_i \omega_i [N_i] , \quad (2.12)$$

where a_i, a_i^+ and N_i are interpreted respectively as annihilation, creation and occupation number operators of particles in level i , with energy ω_i , which satisfy the algebra (2.1) and commute for different levels.

The grand canonical partition function is:

$$Z = Tr \exp[-\beta(H - \mu N)] = \exp(-\beta\Omega), \quad (2.13)$$

where T is the temperature, k_B is the Boltzmann constant, $\beta = (k_B T)^{-1}$ and $N, N = \sum_i N_i$, is the total number operator; μ is the chemical potential and Ω is the grand canonical potential. For the above system Z factorizes and the grand canonical potential is given by a sum over single level partition functions [26]

$$\Omega = -\frac{1}{\beta} \sum_i \ln Z_i^0(\omega_i, \beta, \mu), \quad (2.14)$$

where

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i[n] - \mu n)}. \quad (2.15)$$

The system is enclosed in a large ν -dimensional volume V and as in the usual procedure, the sum over levels is replaced by an integral over the \vec{p} -space. With the energy spectrum of each q -particle obeying the dispersion law

$$\omega_i \rightarrow \gamma p^\alpha, \quad (2.16)$$

the general expression for the grand canonical potential (2.14) is

$$\Omega = \frac{-V}{h^\nu \beta} \int d^\nu p \ln \sum_{n=0}^{\infty} e^{-\beta(\gamma p^\alpha [n] - \mu n)}; \quad (2.17)$$

for $\alpha = 1(2)$ one recovers the ultrarelativistic (non-relativistic) case, with $\gamma = 1(1/2m)$.

Integrating over the angular variables, defining the new variable $\eta = \beta \gamma p^\alpha$ and integrating (2.17) by parts, we finally have

$$\Omega = -\frac{-\Lambda^{-\nu}}{\Gamma(\frac{\nu}{\alpha} + 1)\beta} \int_0^\infty d\eta \eta^{\nu/\alpha} \frac{\sum_{n=0}^{\infty} [n] z^n e^{-[n]\eta}}{\sum_{n=0}^{\infty} z^n e^{-[n]\eta}}, \quad (2.18)$$

where $z = e^{\beta\mu}$ is the fugacity. The thermal wavelength Λ , $\Lambda^{-\nu} = \frac{\pi^{\nu/2} \Gamma(\frac{\nu}{\alpha} + 1)}{\Gamma(\frac{\nu}{2} + 1) h^\nu (\beta \gamma)^{\nu/\alpha}}$, is the relevant expansion parameter in the thermodynamic functions.

The pressure $P = -\frac{\Omega}{V}$ and the density $n = \frac{\partial P}{\partial \mu}|_{T,V}$ for the q -oscillator in ν -spatial dimensions and energy spectrum given by (2.16) are then

$$P(T, z) = kT \Lambda^{-\nu} Y_q(z) \quad (2.19a)$$

$$n(T, z) = \Lambda^{-\nu} y_q(z), \quad (2.19b)$$

where

$$Y_q(z) = \frac{1}{\Gamma(\frac{\nu}{\alpha} + 1)} \int_0^\infty d\eta \eta^{\nu/\alpha} \frac{\sum_{n=0}^{\infty} [n] z^n e^{-[n]\eta}}{\sum_{n=0}^{\infty} z^n e^{-[n]\eta}} \quad (2.20a)$$

$$\begin{aligned} y_q(z) &= z \partial_z Y_q(z) \\ &= \frac{1}{\Gamma(\frac{\nu}{\alpha} + 1)} \int_0^\infty d\eta \eta^{\nu/\alpha} \left[\frac{\sum_{n=0}^{\infty} [n] n z^n e^{-[n]\eta}}{\sum_{n=0}^{\infty} z^n e^{-[n]\eta}} \right. \\ &\quad \left. - \frac{\left(\sum_{n=0}^{\infty} [n] z^n e^{-[n]\eta} \right) \left(\sum_{n=0}^{\infty} n z^n e^{-[n]\eta} \right)}{\left(\sum_{n=0}^{\infty} z^n e^{-[n]\eta} \right)^2} \right]. \end{aligned} \quad (2.20b)$$

We can see numerically that for high q (which, means $q > 3$) [25, 26, 28], the series in functions $Y_q(z)$ and $y_q(z)$ can be approximated by their first three terms ($n = 0, 1, 2$):

$$Y_q(z) \cong \frac{1}{\Gamma(\frac{\nu}{\alpha} + 1)} \int_0^\infty d\eta \eta^{\nu/\alpha} \left[\frac{z e^{-\eta} + (q + q^{-1}) z^2 e^{-(q+q^{-1})\eta} + \dots}{1 + z e^{-\eta} + z^2 e^{-(q+q^{-1})\eta} + \dots} \right], \quad (2.21a)$$

$$\begin{aligned} y_q(z) &\cong \frac{1}{\Gamma(\frac{\nu}{\alpha} + 1)} \int_0^\infty d\eta \eta^{\nu/\alpha} \left[\frac{z e^{-\eta} + 2(q + q^{-1}) e^{-(q+q^{-1})\eta} + \dots}{1 + z e^{-\eta} + z^2 e^{-(q+q^{-1})\eta} + \dots} \right. \\ &\quad \left. - \frac{(z e^{-\eta} + (q + q^{-1}) z^2 e^{-(q+q^{-1})\eta} + \dots)(z e^{-\eta} + 2z^2 e^{-(q+q^{-1})\eta} + \dots)}{(1 + z e^{-\eta} + z^2 e^{-(q+q^{-1})\eta} + \dots)^2} \right]. \end{aligned} \quad (2.21b)$$

Similarly, to the ideal gas, in the high temperature limit, the fugacity z is small compared to one; performing the integrations and keeping terms up to the third order in z , we obtain

$$P = \frac{\Lambda^{-\nu}}{\beta} z [1 + F_1 z + F_2 z^2 + O(z^3)], \quad (2.22)$$

and

$$n = \Lambda^{-\nu} z [1 + 2z F_1 + 3z^2 F_2 + O(z^3)], \quad (2.23)$$

where the coefficients F are given by:

$$F_1 = \frac{-1}{2^{\frac{\nu}{\alpha}+1}} + \frac{1}{(q + q^{-1})^{\nu/\alpha}}, \quad (2.24a)$$

$$F_2 = \frac{1}{3^{\frac{\nu}{\alpha}+1}} - \frac{1}{(1 + q + q^{-1})^{\nu/\alpha}}. \quad (2.24b)$$

Inverting (2.23) and substituting into (2.22), we find the virial expansion of the deformed gas [25, 26, 29]:

$$P = \frac{n}{\beta} \left[1 - \left(\frac{-1}{2^{\frac{\nu}{\alpha}+1}} + \frac{1}{(q+q^{-1})^{\nu/\alpha}} \right) (n\Lambda^\nu) + 2 \left(\frac{1}{2^{\frac{2\nu}{\alpha}+1}} + \frac{1}{3^{\frac{\nu}{\alpha}+1}} - \frac{2}{(2+2q+2q^{-1})^{\nu/\alpha}} + \frac{1}{(1+q+q^{-1})^{\nu/\alpha}} \right) (n\Lambda^\nu)^2 + \dots \right]. \quad (2.25)$$

For lower temperatures, if n is kept constant, $n\Lambda^\nu$ increases and so does z . When z reaches 1, the temperature T attains its critical value T_c^q , defined by

$$n\Lambda_c^\nu = y_q(1), \quad (2.26)$$

which can be expressed as:

$$T_c^q = \frac{\gamma \Gamma^{\alpha/\nu} (\frac{\nu}{2} + 1) h^\alpha n^{\alpha/\nu}}{k \pi^{\alpha/2} \Gamma^{\alpha/\nu} (\frac{\nu}{\alpha} + 1) y_q^{\alpha/\nu}(1)}. \quad (2.27)$$

From (2.15) we see that the ground state ($\omega_i = 0$) is not affected by the deformation. Therefore, the series in (2.20) will only converge if $z < 1$ and μ has to be non-negative as in the usual ideal gas. Also, when $T \rightarrow 0$, there will be an accumulation of particles in this state: Bose-Einstein condensation is present in this deformed system [28]. Comparing T_c to the critical temperature for non-deformed gases of the same density n , one finds

$$\frac{T_c^q}{T_c} = \left(\frac{2.61}{y_q(1)} \right)^{\alpha/\nu}. \quad (2.28)$$

In the vicinity of T_c^q we have to take into account the zero-point energy and single out its contribution in (2.20). Therefore the expressions for P and n become:

$$P(T, z) = \beta^{-1} \Lambda^{-\nu} Y_q(z), \quad (2.29a)$$

$$n(T, z) = \frac{1}{V} \frac{z}{1-z} + \Lambda^{-\nu} y_q(z), \quad (2.29b)$$

where the first term on the right-hand side of (2.29.b), which is due to the contribution of the zero energy, is relevant only for $T \leq T_c^q$. In this region z remains equal to one, as in the usual case.

The specific heat per mole C_V is defined as

$$C_V = V \left. \frac{\partial \tilde{e}}{\partial T} \right|_n, \quad (2.30)$$

where \tilde{e} is the energy density (internal energy per volume) and V the molar volume. From the thermodynamic definitions of \tilde{e} and of the entropy density one has

$$\tilde{e} = n\mu - \beta \left. \frac{\partial P}{\partial \beta} \right|_\mu - P, \quad (2.31)$$

with P given by (2.29.a). It is easy to find that

$$\tilde{\epsilon} = \frac{\nu}{\alpha} P, \quad (2.32)$$

a generalization of the standard result for $q = 1$.

To obtain C_V one has to compute $\partial_T \tilde{\epsilon}|_n \equiv (\partial \tilde{\epsilon} / \partial T)|_n$ which from (2.32) can be obtained in the two regimes from

$$\begin{aligned} P(T) &= \beta^{-1} \Lambda^{-\nu} Y_q[z(T)] & T > T_c \\ P(T) &= \beta^{-1} \Lambda^{-\nu} Y_q(1) & T \leq T_c. \end{aligned} \quad (2.33)$$

Now, for $T > T_c^q$,

$$\partial_T \tilde{\epsilon}|_n = \frac{\nu}{\alpha} \left(\frac{\nu}{\alpha} + 1 \right) k \Lambda^{-\nu} Y_q(z) + \frac{\nu}{\alpha} k T \Lambda^{\nu} Y_q'(z) \partial_T z|_n, \quad (2.34)$$

where $Y_q'(z) = \partial_z Y_q(z)$. From $y_q(z) = \Lambda^{\nu} n$,

$$\partial_T z|_n = -\frac{\nu}{\alpha} \frac{\Lambda^{\nu} T^{-1} n}{y_q'(z)}. \quad (2.35)$$

Substituting (3.35) in (3.34) one gets

$$\frac{C_V}{k} = \frac{\nu}{\alpha} \left(\frac{\nu}{\alpha} + 1 \right) (\Lambda^{\nu} n)^{-1} Y_q(z) - \left(\frac{\nu}{\alpha} \right)^2 \frac{y_q(z)}{z y_q'(z)}. \quad (2.36)$$

Also, for $T < T_c^q$,

$$\frac{C_V}{k} = \frac{\nu}{\alpha} \left(\frac{\nu}{\alpha} + 1 \right) (\Lambda^{\nu} n)^{-1} Y_q(1). \quad (2.37)$$

Clearly equations (2.36) and (2.37) also generalize the usual $q = 1$ ideal bosonic gas specific heat.

It is seen from above that for $\frac{\nu}{\alpha} > 1$ a deformed q -gas has the Bose-Einstein condensation phenomenon with the specific heat exhibiting a sort of λ -point discontinuity. The critical temperature being higher than for the bosonic ideal gas, we see that the presence of deformation favours the Bose-Einstein condensation phenomenon.

3 Ideal q -gas in the inequivalent ($\nu_0 \neq 0$) representations

We show now the consequences of choosing different representations of a q -oscillator algebra on a system which generalizes the ideal quantum boson gas [22, 23, 30].

Let us consider the deformed algebra (2.1). Assuming that a, a^+ are mutually adjoint, $N = N^+$ and the spectrum is non-degenerate, representations of algebra (2.1) in a Hilbert space \mathcal{H} were built [23].

Denoting the normalized basis vectors by $|n\rangle$, for $q > 1$ (which is the case we shall be interested) the following representations were obtained [23]:

$$\begin{aligned} a^\dagger |n\rangle &= q^{-\nu_0/2} [n+1]^{1/2} |n+1\rangle, \\ a |n\rangle &= q^{-\nu_0/2} [n] |n-1\rangle, \\ N |n\rangle &= (\nu_0 + n) |n\rangle, \end{aligned} \quad (3.1)$$

where ν_0 is a real free parameter which goes to zero when $q \rightarrow 1$. When $\nu_0 = 0$, N is interpreted as the usual particle number operator for the state $|n\rangle$. This is not anymore the case for $\nu_0 \neq 0$ [22]; its eigenvalue is interpreted as the sum of the number of particles n , in the state $|n\rangle$, plus a background effect ν_0 . The operator $\hat{N} = N - \nu_0$ is the number operator, $\hat{N} |n\rangle = n |n\rangle$, for the representations in (3.1) characterized by ν_0 .

For $q > 1$, as ν_0 is the lowest bound of the spectrum of N , it classifies inequivalent representations of the algebra (2.1) [22]. In fact, it can be easily verified that

$$\mathcal{C} = q^{-N} ([N] - a^\dagger a) \quad (3.2)$$

is a Casimir operator for the algebra (2.1) and in the representation (3.1) one has

$$\mathcal{C} |n\rangle = q^{-\nu_0} [\nu_0] |n\rangle. \quad (3.3)$$

As (3.2) is different from zero only for $q \neq 1$, one sees from (2.4) that when $q = 1$, ν_0 is necessarily zero.

Let us now consider an ideal deformed system described by the Hamiltonian

$$H = \sum_i \omega_i a_i^\dagger a_i = \sum_i \omega_i ([N_i] - q^{N_i} \mathcal{C}_i) \quad (3.4)$$

where N_i is an operator that can be interpreted as the number operator of particles in levels i when $\nu_0 = 0$. a_i, a_i^\dagger and N_i satisfy algebra (2.1) and commute for different levels.

In the grand canonical partition function (2.14), we have now

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i q^{-\nu_0} [n] - \mu n)}. \quad (3.5)$$

When ν_0 is assumed to be the same for all levels i , we obtain the low density virial expansion:

$$\begin{aligned} P &= \frac{n}{\beta} \left[1 - q^{-\nu_0 d/\alpha} \left(-\frac{1}{2^{d/\alpha+1}} + \frac{1}{(q+q^{-1})^{d/\alpha}} \right) (n \Lambda^d) \right. \\ &\left. + q^{-2\nu_0 d/\alpha} 2 \left(\frac{1}{2^{2d/\alpha+1}} - \frac{1}{3^{d/\alpha+1}} - \frac{2}{2^{d/\alpha}(q+q^{-1})^{d/\alpha}} + \frac{1}{(1+q+q^{-1})^{d/\alpha}} \right) (n \Lambda^d)^2 + \dots \right]. \end{aligned} \quad (3.6)$$

When the energy spectrum of the q -particles obeys the dispersion law (2.16), the pressure P and the density n are given by (2.19) and (2.20), where now $\Lambda_q^{-\nu} = \frac{\pi^{\nu/2} \Gamma(\frac{\nu}{\alpha} + 1)}{\Gamma(\frac{\nu}{2} + 1) h^\nu (\beta \gamma_q)^\nu}$ is the modified thermal wavelength with $\gamma_q = q^{-\nu_0} \gamma$. Also, in the low temperature regime, when the fugacity $z \rightarrow 1$ the temperature attains its critical value T_c^q ,

$$T_c^q = \frac{\gamma_q \Gamma^{\alpha/\nu} \left(\frac{\nu}{2} + 1 \right) h^\alpha n^{\alpha/\nu}}{k \pi^{\alpha/2} \Gamma^{\alpha/\nu} \left(\frac{\nu}{\alpha} + 1 \right) y_q^{\alpha/\nu}(1)}, \quad (3.7)$$

and (2.28) becomes here

$$\frac{T_c^q}{T_c} = \left(\frac{2.61}{y_q(1)} \right)^{\alpha/\nu} q^{-\nu_0} . \quad (3.8)$$

Similarly to the $\nu_0 = 0$ case, the basic equations are

$$P(T, z) = \beta^{-1} \Lambda_q^{-\nu} Y_q(z) \quad (3.9)$$

and

$$n(T, z) = \frac{1}{V} \frac{z}{1-z} + \Lambda_q^{-\nu} y_q(z) ; \quad (3.10)$$

the first term on the right-hand side of (3.10), which is due to the contribution of zero energy, is relevant only for $T \leq T_c^q$. In this region z remains equal to one, as in the standard case. For $\nu_0 = 0$, (2.27), (2.28) and (2.29a and b) are recovered.

The specific heat per mole C_V , (2.36) and (2.37), is then

$$C_V/k = \frac{\nu}{\alpha} \left(\frac{\nu}{\alpha} + 1 \right) (\Lambda_q^\nu n)^{-1} Y_q(z) - \left(\frac{\nu}{\alpha} \right)^2 \frac{y_q(z)}{z y_q'(z)}, \quad T > T_c^q , \quad (3.11a)$$

$$C_V/k = \frac{\nu}{\alpha} \left(\frac{\nu}{\alpha} + 1 \right) (\Lambda_q^\nu n)^{-1} Y_q(1) , \quad T < T_c^q . \quad (3.11b)$$

It is interesting to observe that the presence of the parameter ν_0 , which characterizes the different representations of the q -oscillators algebra (2.1), changes the shape of the specific heat C_V in (3.11) but leaves the “ λ -point discontinuity” invariant [22]. We remark also that as in the “fundamental” representation ($\nu_0 = 0$), the q -gas presents Bose-Einstein condensation but now the critical temperature depends on the representation under consideration.

4 A deformed model for phonons in 4He

We propose now that the phonons in 4He are described by a q -gas and take for the phonon gas the Hamiltonian (3.4), that is,

$$H = \sum_i \omega_i a_i^\dagger a_i = \sum_i \omega_i \left([N_i] - q^{N_i} \mathcal{C} \right) , \quad (4.1)$$

As $\mu = 0$ for phonons [31], our canonical potential is

$$\Omega = -\frac{1}{\beta} \sum_i \ln \sum_{n=0}^{\infty} e^{-\beta \omega_i q^{\nu_0^i [n]}} . \quad (4.2)$$

We will now abandon the assumption that ν_0^i is the same for all levels i . We propose then a different ν_0^i for each level such that in the continuum limit [10]

$$\nu_0^i \rightarrow \nu_0(p) = \frac{\delta^2}{\theta} p^2 = \frac{p^2/2m}{E_\lambda} , \quad (4.3)$$

where $q = e^\theta$. As ν_o is dimensionless, the algebraic constant δ has dimensions $[\delta] = gr^{-1}cm^{-1} \text{ sec}$. Besides, it appears in (4.3) an energy scale E_λ . We take it as $E_\lambda = k_B T_\lambda$, where $T_\lambda = 2.18K$ is the λ -point temperature for 4He , and $m = m_{{}^4He}$. The choice of T_λ seems the most natural, as superfluidity in 4He only happens below this temperature. It is as natural to choose $m = m_{{}^4He}$ since we have the non-relativistic classical dispersion law for $\nu_o(p)$.

We take for the phonon the anomalous dispersion relation, that is [6],

$$\omega_i \rightarrow \omega(p) = c_0(1 - \alpha p^2) , \quad (4.4)$$

where c_0 is the sound velocity and α is some dimensional constant to be determined latter. Therefore, our energy-momentum relation for the phonon in 4He is

$$\omega^{ph}(p) = q^{\nu_o(p)} \omega(p) = e^{\delta^2 p^2} c_0 p (1 - \alpha p^2) ; \quad (4.5)$$

for small phonon momenta it can be expanded as

$$\omega^{ph}(p) = c_0 p (1 - (\alpha - \delta^2) p^2 - (\alpha \delta^2 - \frac{1}{2} \delta^4) p^4 + (\delta^6 - \alpha \delta^4) p^6 + \dots) . \quad (4.6)$$

This relation is to be compared with the small momenta ad-hoc phonon dispersion relation usually found in the literature [6–8], that is,

$$\omega^{ph}(p) = c_0 p (1 - \gamma_2 p^2 - \gamma_4 p^4 + \dots) , \quad (4.7)$$

where the γ 's are free parameters that are determined in measurements of 4He specific heat. We are thus presenting an algebraic interpretation for dispersion relation (4.7). From (4.5) and (4.6) we see that the presence of deformation alters the usual anomalous phonon dispersion relation.

In order to obtain the canonical potential in the continuum limit, we follow the procedure of section 2 and substitute (4.6) in (4.2); we get

$$\Omega = -\frac{V}{h^3 \beta} \int d^3 p \ln \sum e^{\beta c_0 p (1 - (\alpha - \delta^2) p^2 - (\alpha \delta^2 - \frac{1}{2} \delta^4) p^4 + \dots)} . \quad (4.8)$$

For low temperatures, we can expand the exponential and the logarithm in the integral of (4.8). Then, defining a new variable $y = \beta c_0 p$, we have for the pressure,

$$P = \frac{4\pi}{\beta^4 h^3 c_0^3} \left[\frac{\omega^{(3)}}{3} - \frac{\omega^{(5)}}{\beta^2} + \frac{(4b_2 - b_4)}{\beta^4} \omega^{(7)} + \frac{(-55b_2^3 + 30b_2 b_4 - 3b_6)}{\beta^6} \omega^{(9)} + \dots \right] , \quad (4.9)$$

where

$$\omega^{(m)} = \int dy y^{m-1} \frac{\sum_n [n] e^{-y[n]}}{\sum_n [n] e^{-y[n]}} \quad (4.10)$$

and

$$b_2 = \frac{\delta^2 - \alpha}{c_0^2} , \quad b_4 = \frac{\delta^4/2 - \alpha \delta^2}{c_0^4} , \quad b_6 = \frac{\delta^6/3 - \alpha \delta^4}{2c_0^6} . \quad (4.11)$$

Finally, from (2.30) and substituting (4.9) in (2.32), the low temperature q -phonon specific heat per mole is

$$C_{V,q}^{phonon} = \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + \tilde{G}T^9 + \dots, \quad (4.12)$$

where

$$\begin{aligned} \tilde{A} &= \frac{2k_B^4 V}{\pi^2 \hbar^3 c_0^3} \omega^{(3)}; \quad \tilde{B} = \frac{15k_B^6 (\alpha - \delta^2) V}{\pi^2 \hbar^3 c_0^5} \omega^{(5)}, \quad \tilde{D} = \frac{28k_B^8 \left(\frac{7}{2} \alpha^4 + 4\delta^2 - 7\alpha\delta^2\right) V}{\pi^2 \hbar^3 c_0^7} \omega^{(7)} \\ \tilde{G} &= \frac{15k_B^{10} (-81\delta^6 + 110\alpha^3 + 243\alpha\delta^4 - 270\delta^2\alpha^2) V}{2\pi^2 \hbar^3 c_0^9} \omega^{(9)}. \end{aligned} \quad (4.13)$$

5 Comparison with experimental data and final comments

In this section we show that using for A, B and D values experimentally determined by fitting the low temperature phonon specific heat

$$C_V^{ph} = AT^3 + BT^5 + DT^7 + \dots \quad (5.1)$$

with measured specific heat data of 4He at the temperature range $0.14 K \leq T \leq 0.86 K$, [8], our model leads to unstable phonons for all the analysed values of the pressure [10]. In that temperature range, we take the 4He total molar specific heat

$$C_V = C_{V,q}^{ph} + C_V^{roton}, \quad (5.2)$$

with $C_{V,q}^{ph}$ given by and (4.12) and (4.13). The rotons contribution to the total specific heat is negligible for almost the whole analysed temperature range ($0.14 K \leq T \leq 0.86 K$) [8]. Therefore we will take the usual (non-deformed) roton molar specific heat,

$$C_V^{roton} = \frac{2Vm^{1/2}p_0^2\Delta^2}{(2\pi)^{3/2}\hbar^3k_B^{1/2}T^{3/2}} \left(1 + k_B T/\Delta + \frac{3}{4}(k_B T/\Delta)^2\right) e^{-\Delta/k_B T}, \quad (5.3)$$

where Δ is the energy gap, p is the position of the energy minimum and m is the effective mass of the roton. Equation (5.2) is then

$$\begin{aligned} C_V &= \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + \tilde{G}T^9 + \dots + \\ &\quad \frac{2Vm^{1/2}p_0^2\Delta^2}{(2\pi)^{3/2}\hbar^3k_B^{1/2}T^{3/2}} \left(1 + k_B T/\Delta + \frac{3}{4}(k_B T/\Delta)^2\right) e^{-\Delta/k_B T}. \end{aligned} \quad (5.4)$$

We compare (5.4) with the data of analysis 2 in reference [8]. As the T^7 coefficients present very large errors for the samples 10-16, we restrict our analysis to the samples 6-10. Taking for the coefficients \tilde{A} , \tilde{B} and \tilde{D} in (5.4) the least-squares fits of the measured specific heat data for samples 6-10 in analysis 2 (and $\tilde{G} = 0$), we obtain q , α , δ and c_0 ; α , c_0 and δ are determined from the three first relations (4.13), but as a consequence of

assumption (4.3), the deformation parameters q and δ are not independent. Therefore q is determined from

$$\ln q = 2m_{4He}\delta^2 k_B T_\lambda . \quad (5.5)$$

These results are listed in Table I. We see that the values of q increase with the pressure and that the values of c_0 are almost 5% lower than the directly measured sound velocities [9]. We remark that our q -phonon model has interesting properties. Besides reproducing the experimental results of 4He C_V measurements, the parameters have an algebraic nature and are not introduced ad-hoc as in the usual model. Its greatest advantage is that it reconciles the theoretical prediction [6] for the anomalous dispersion of phonons with the unstable character of phonon spectrum in 4He [7–9]. *Indeed, inspection of Table I shows that α is always positive, but the coefficient of p^3 in dispersion relation (4.6), that is, $(\alpha - \delta^2)$, is always negative.*

Let us now take the total molar specific heat (5.4) and consider non-zero values for \tilde{G} . From the definition (4.13), we see that it is given in terms of the parameters α , δ and c_0 which are completely determined once \tilde{A} , \tilde{B} and \tilde{D} are known. We have then a “theoretical” \tilde{G}_T , which is calculated from the c_0 , α and δ determined from \tilde{A} , \tilde{B} and \tilde{D} . Therefore, considering terms up to T^9 in our C_V^{ph} expansion is crucial to prove the consistency of our model. In fact, we find different sets of values of α , δ and c_0 such that the “theoretical” values \tilde{G}_T are equal to the “experimental” ones, \tilde{G}_E , that is, those values of \tilde{G} obtained by fitting the theoretical C_V (5.4) with the curves resulting from least-squares fits of C_V measured data for samples 6-10 in analysis 2 [8] within 5% of accuracy.

In Tables II and III we show sets of parameters for which $\tilde{G}_T = \tilde{G}_E$. Table II lists three difference sets for sample 6 in analysis 2 [8] and Table III, a set of parameters for each of samples 7-10 in analysis 2 [8]. We see that the c_0 values are then more in accordance with the experimental ones. In Figs. 1 and 2, we compare the curves obtained in our model with the ones resulting from least-squares fits of C_V data for samples 6-10 in analysis 2 [8]. The curves in Fig. 1 correspond to the four different sets of parameters for sample 6 shown in Table II and those in Fig. 2, to the sets of parameters for samples 6-10 listed in Table III. In Fig. 3, we show the deviation from Greywall’s experimental values C_{exp} in the C_V values predicted in our model, for samples 6-10 in analysis 2 [8] (see Table III). We note that it was recently shown that the specific heat calculated from Tsallis statistics [32] also presents good results when compared with experimental data [33].

An intriguing question remains: why 4He superfluid excitations should be considered as deformed objects? There have been interesting indications that the continuum descriptions of physical quantities breakdown both in a convergent fluid [34] and, more recently, in superfluid 4He [35]. As a similar mechanism has been observed in connection with deformed algebras [36] we believe that this is the underlying reason why our approach applies so well to 4He superfluid. Finally, we recall that the roton contribution, which starts to be relevant for $T > 0.6 K$, becomes dominant for $T \leq 1 K$. Therefore, we believe that in order to have a good model for temperatures closer to the λ -point it will be necessary to deform the roton as well.

Acknowledgements

The authors thank C. Tsallis for helpful suggestions.

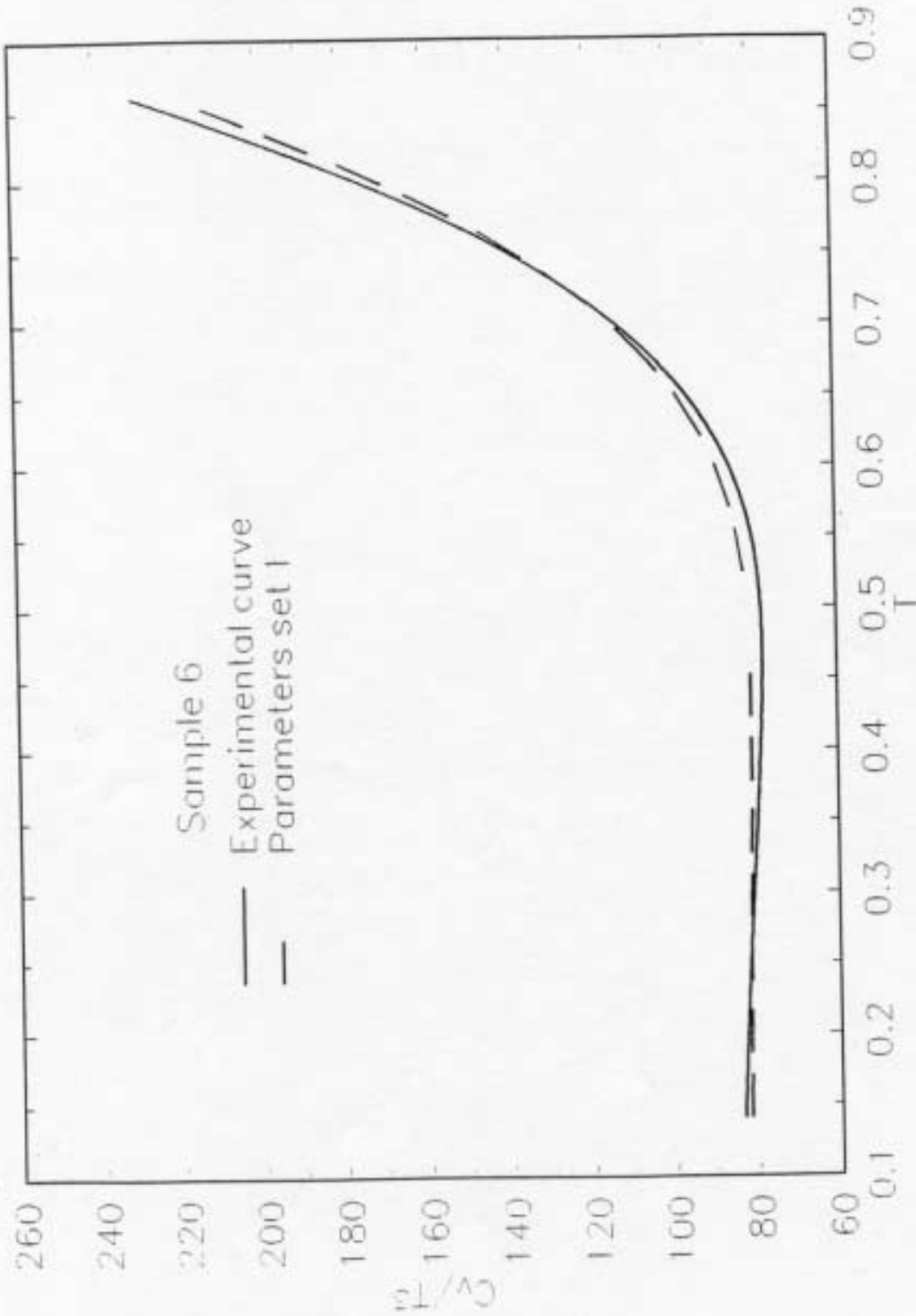


Fig.1 a-d Comparison of the curves obtained in our model for the sets of parameters in Table II and Greywall's experimental curve, for the sample 6 in analysis 2 (reference 8).

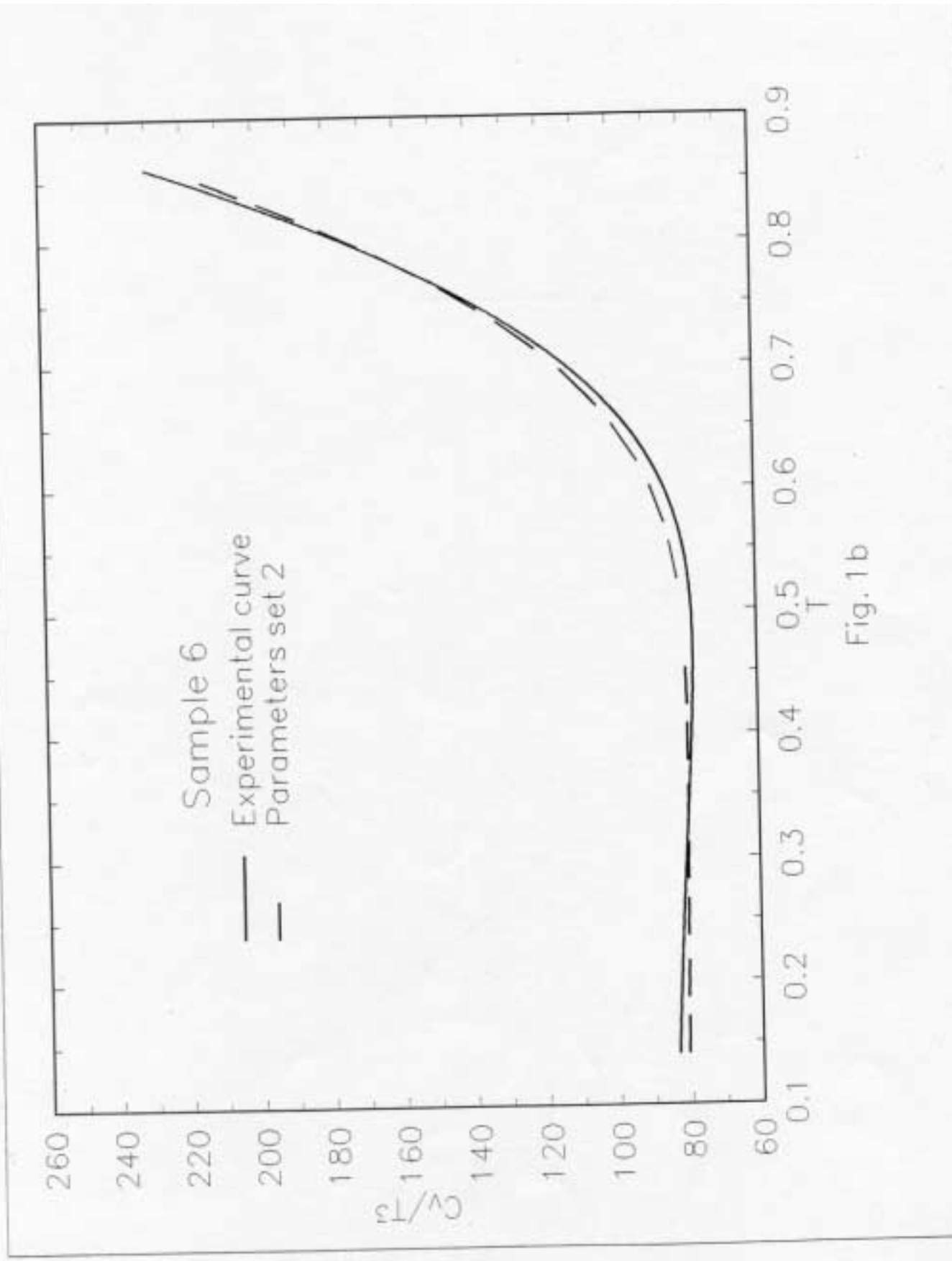
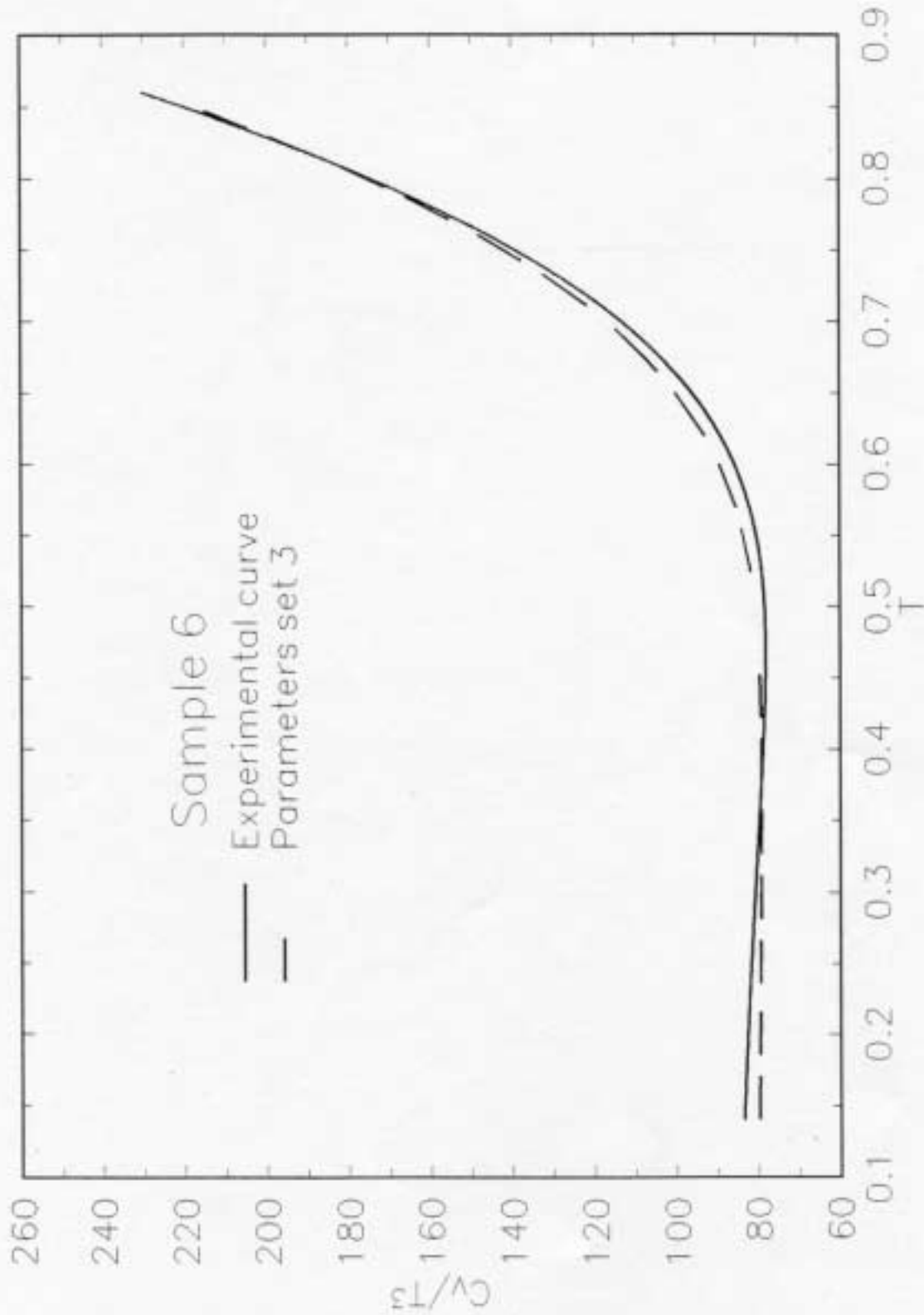


Fig.1 b



Sample 6
— Experimental curve
- - Parameters set 3

Fig. 1c

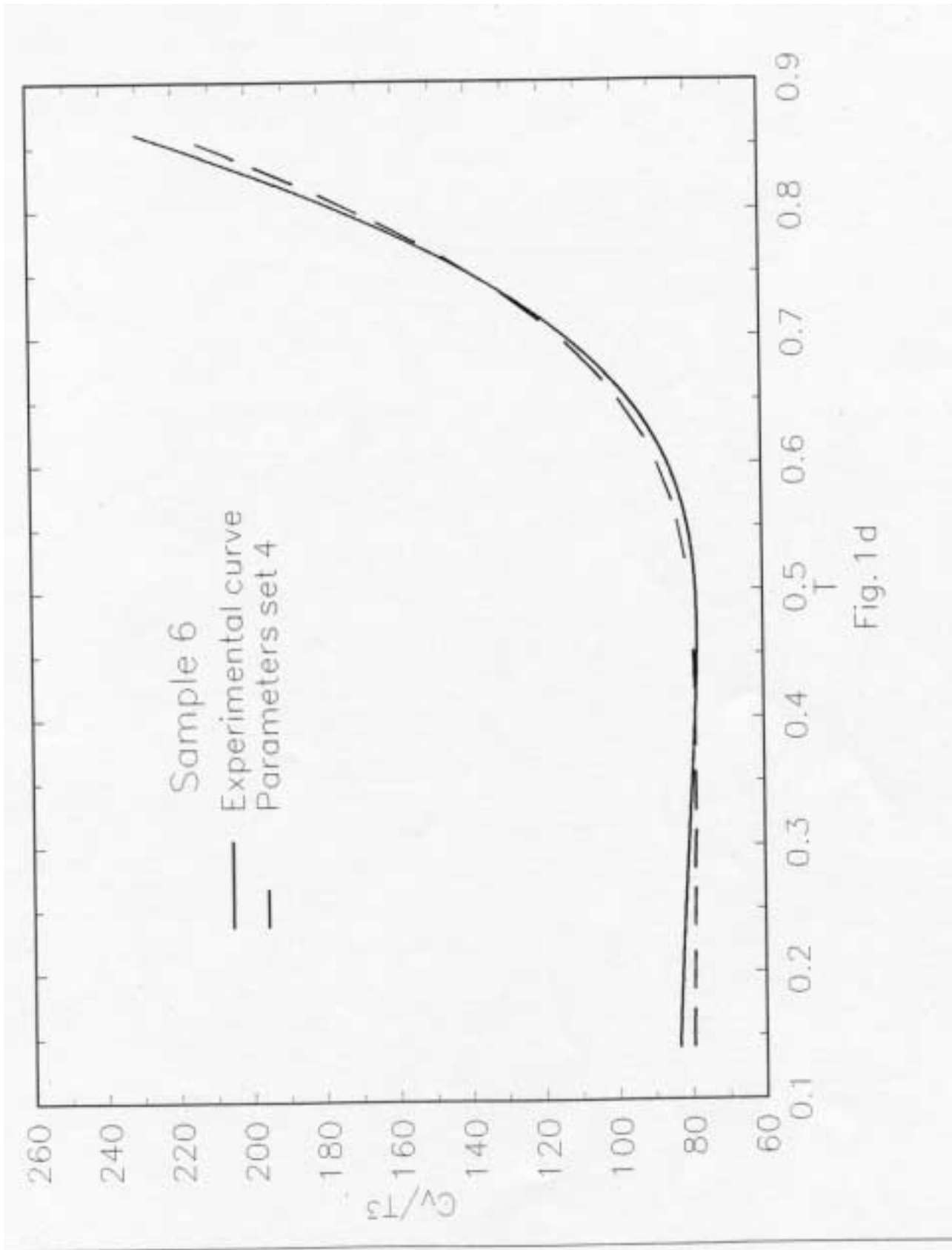


Fig. 1d

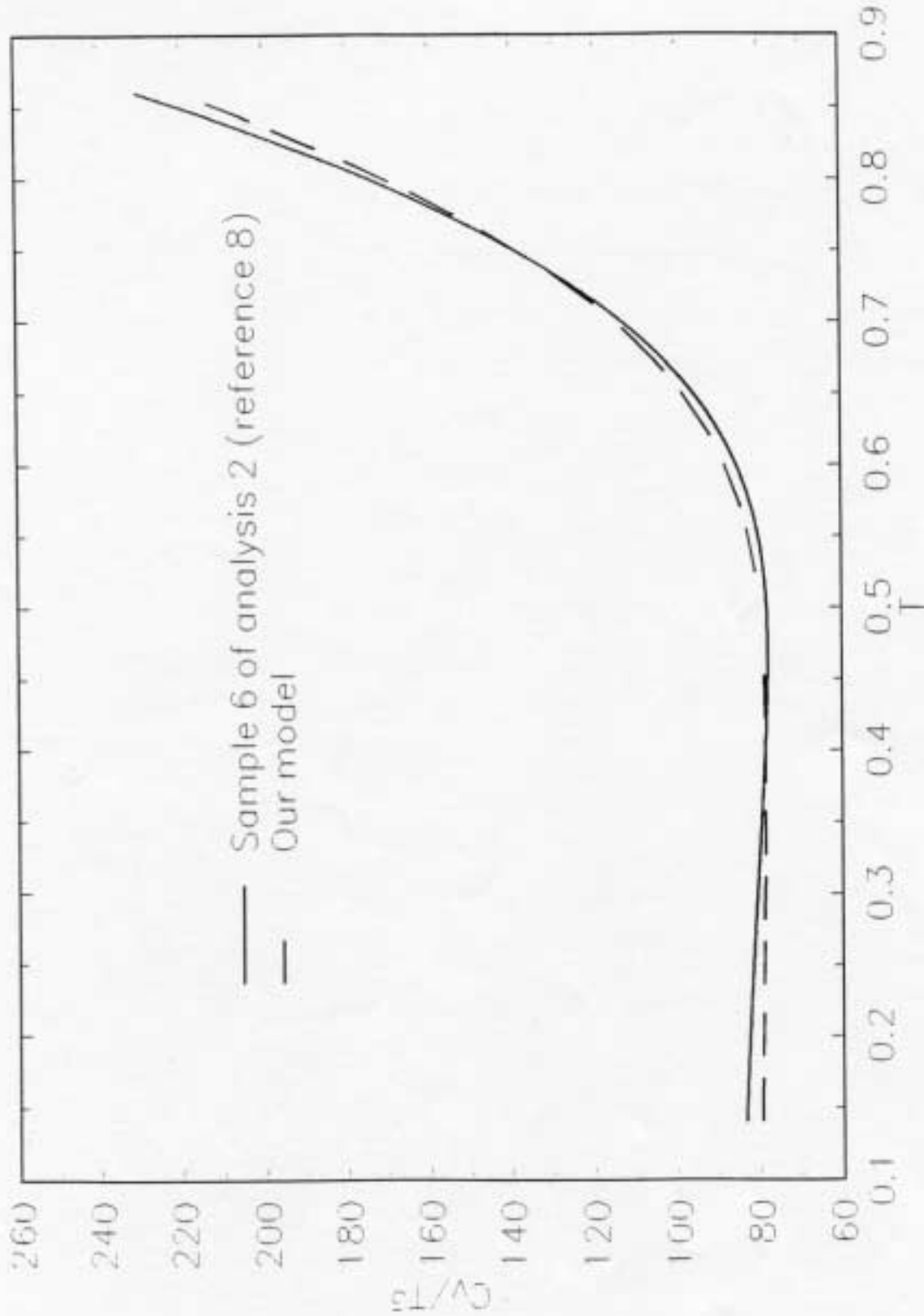


Fig. 2a - eComparison of the curves obtained in our model and Greywall's experimental curves, for samples 6-10 in analysis 2 (reference 8). See Table III for the values of the parameters.

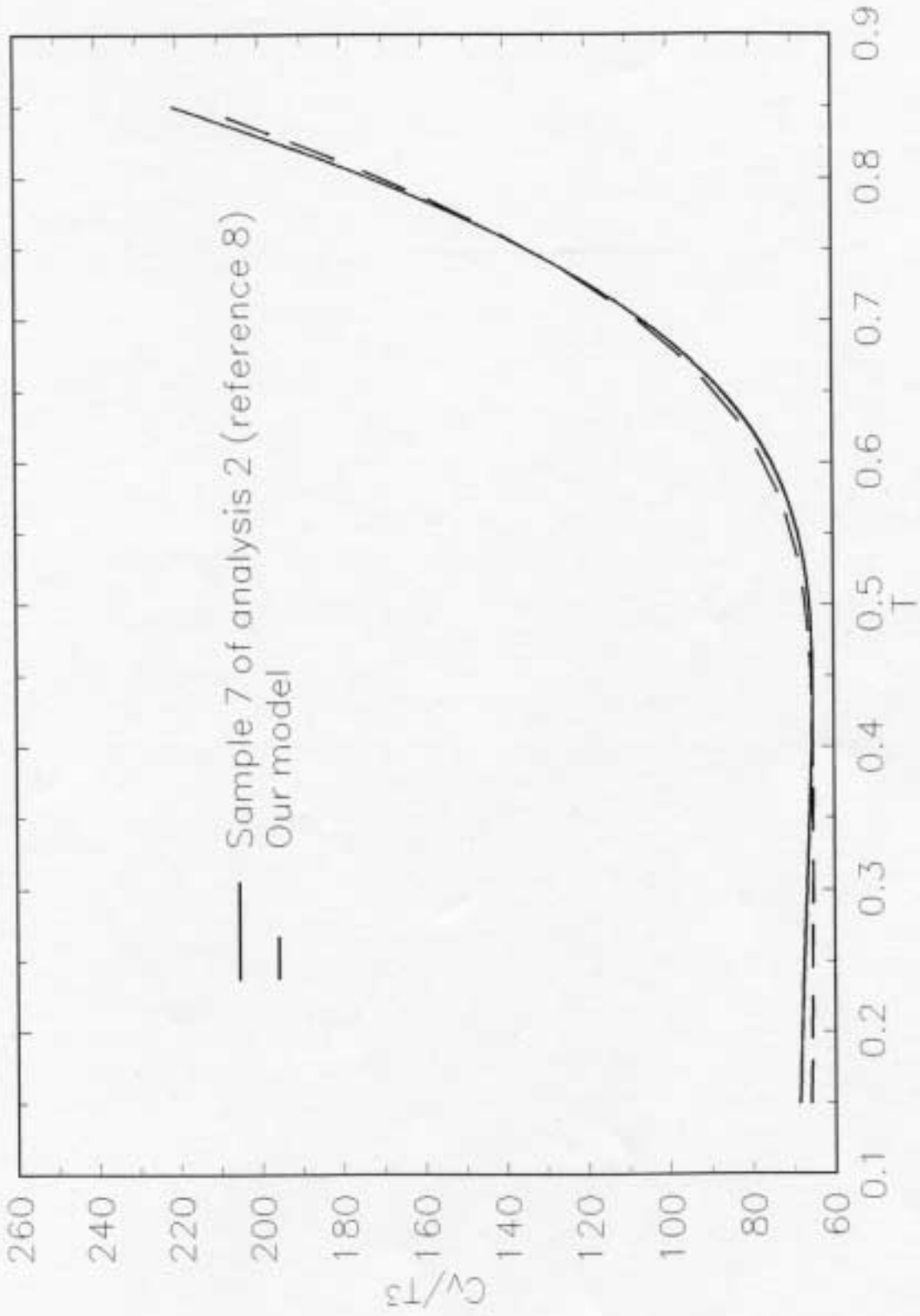


Fig. 2b

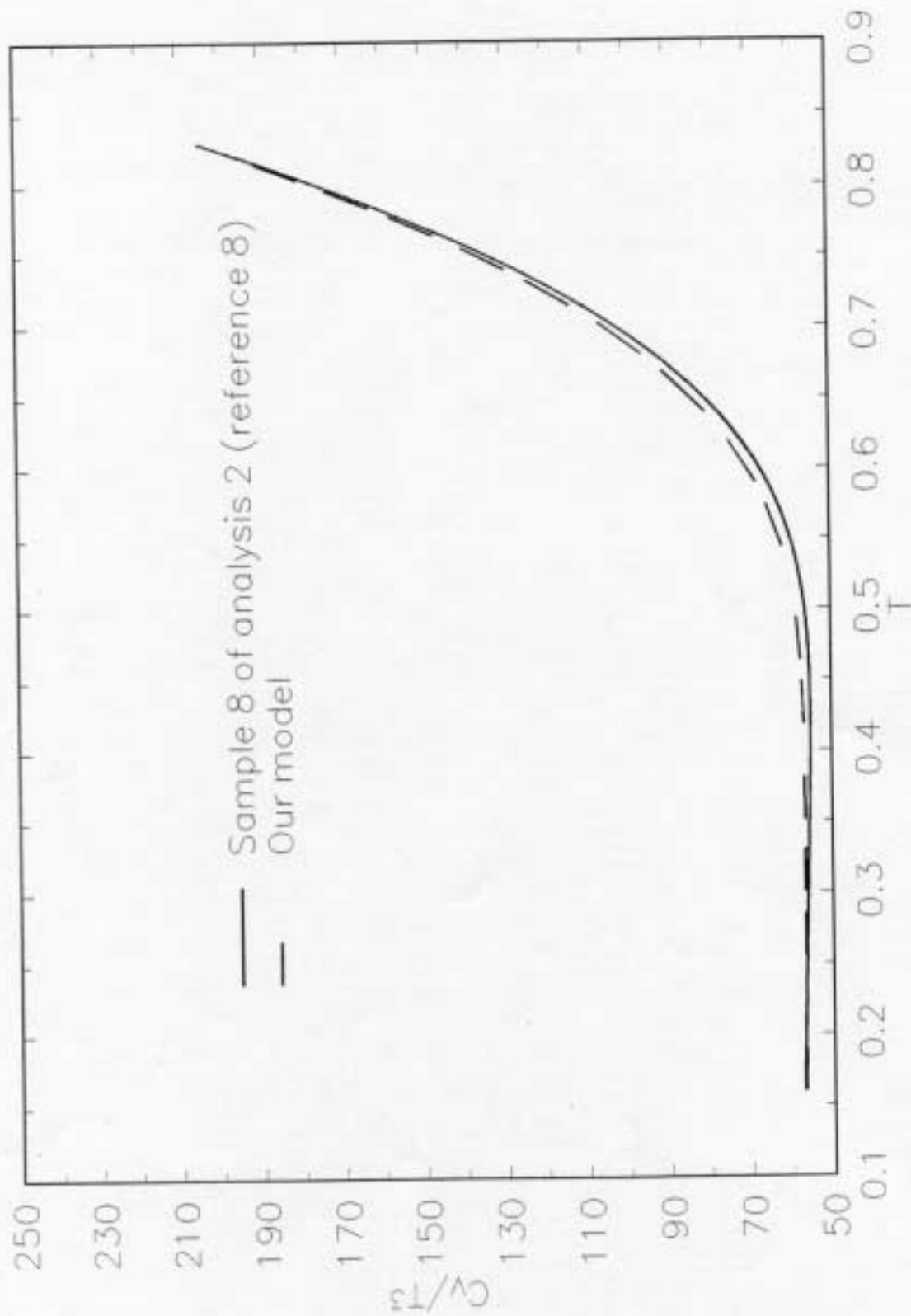


Fig. 2c

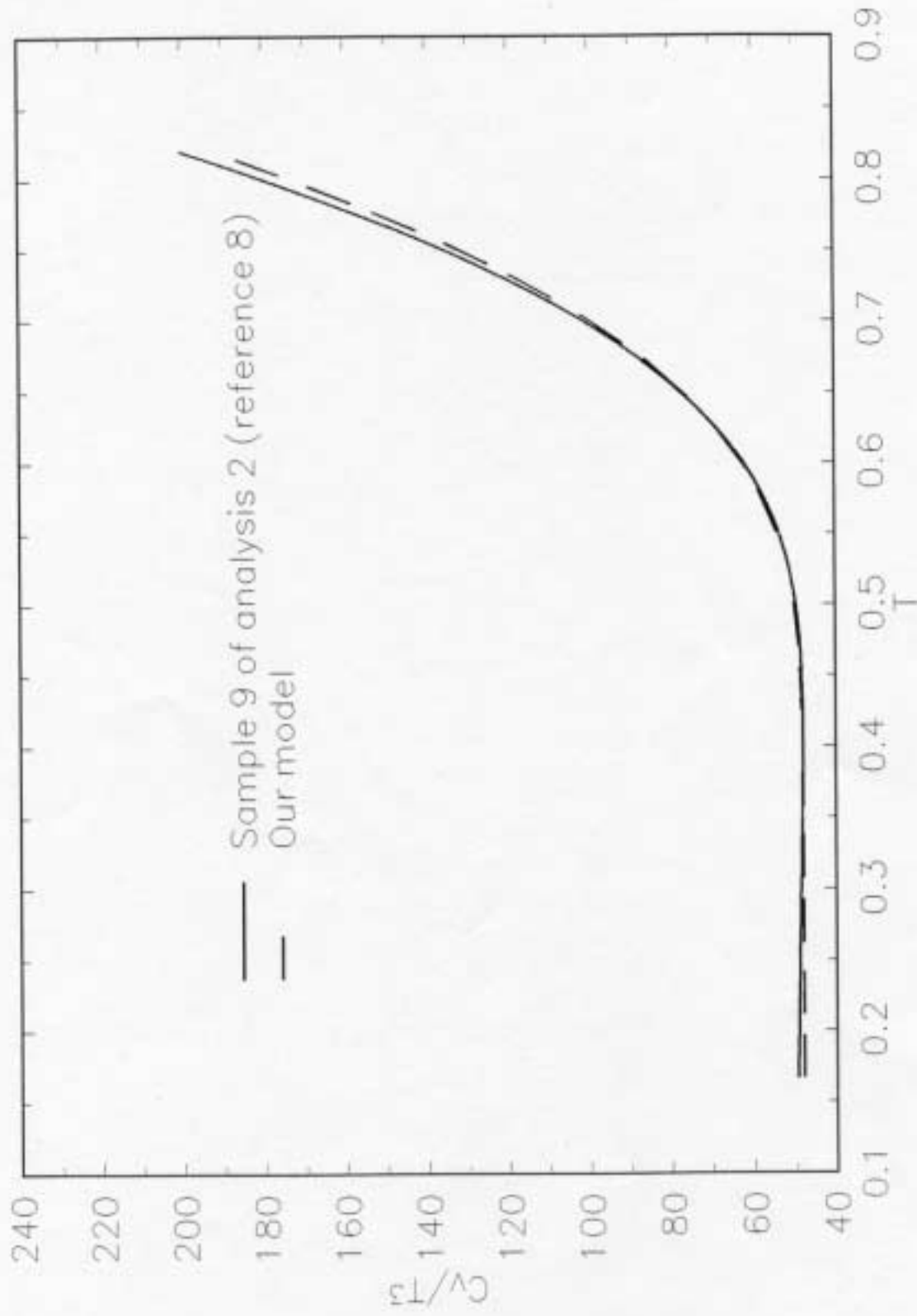


Fig. 2d

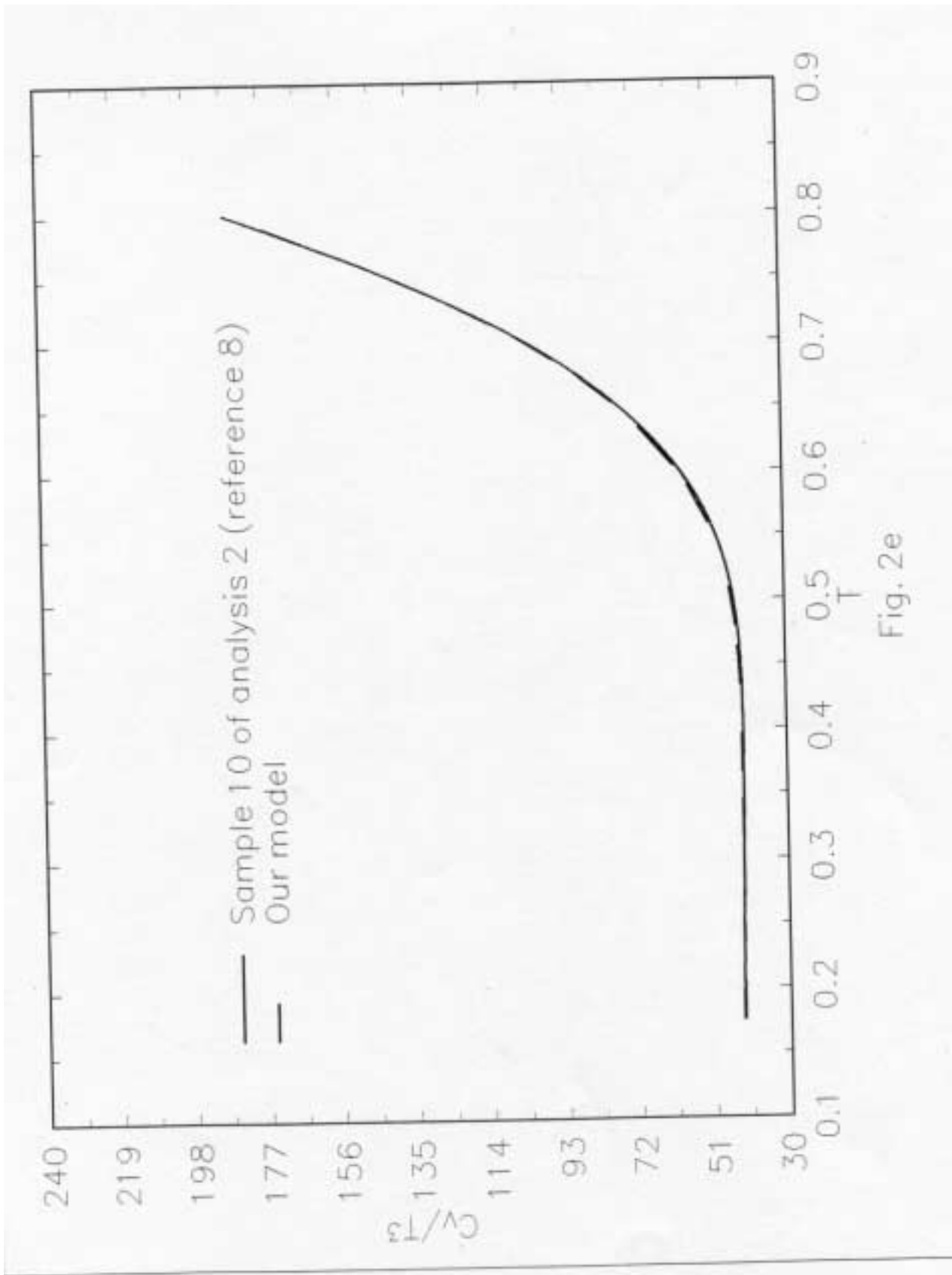


Fig. 2e

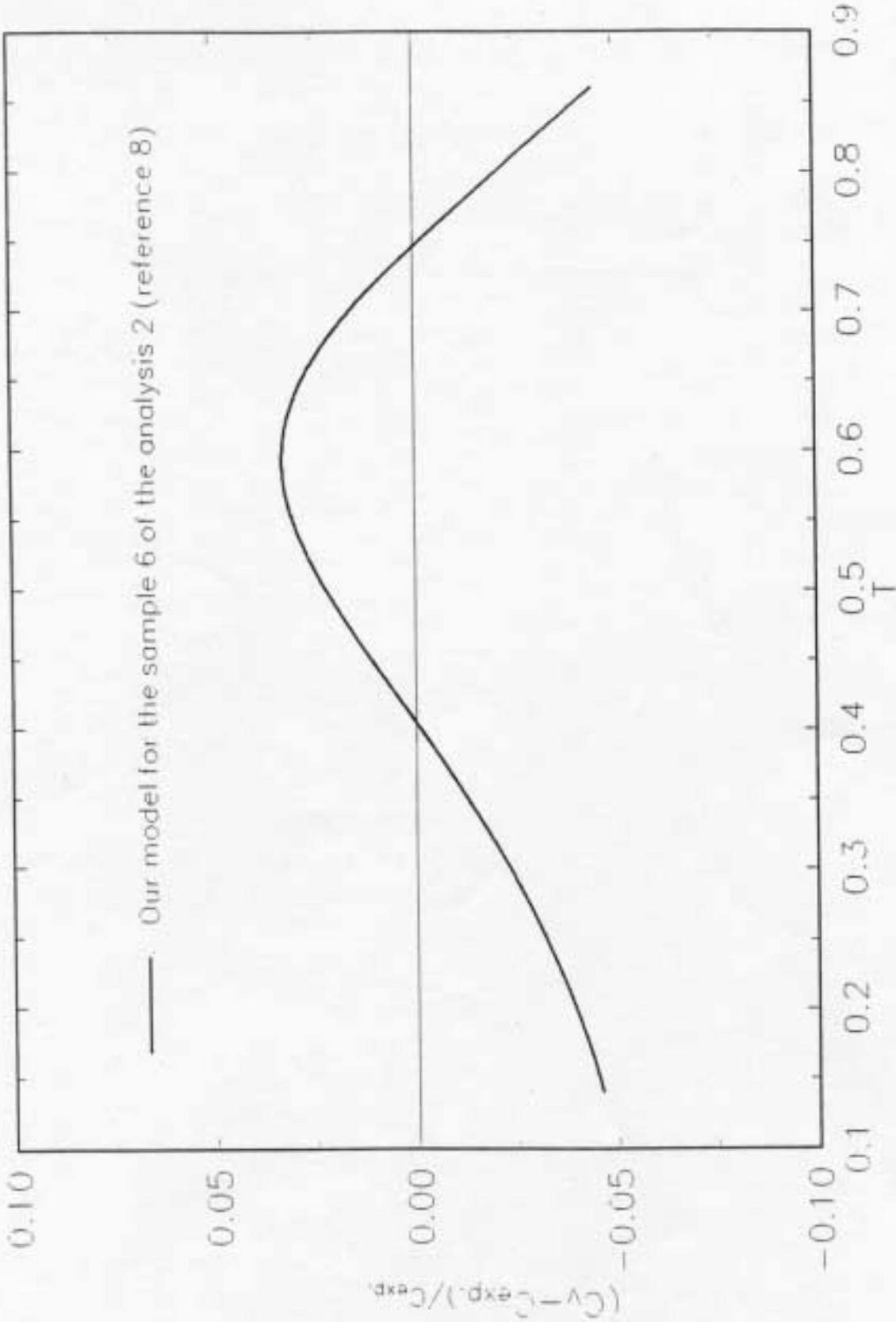
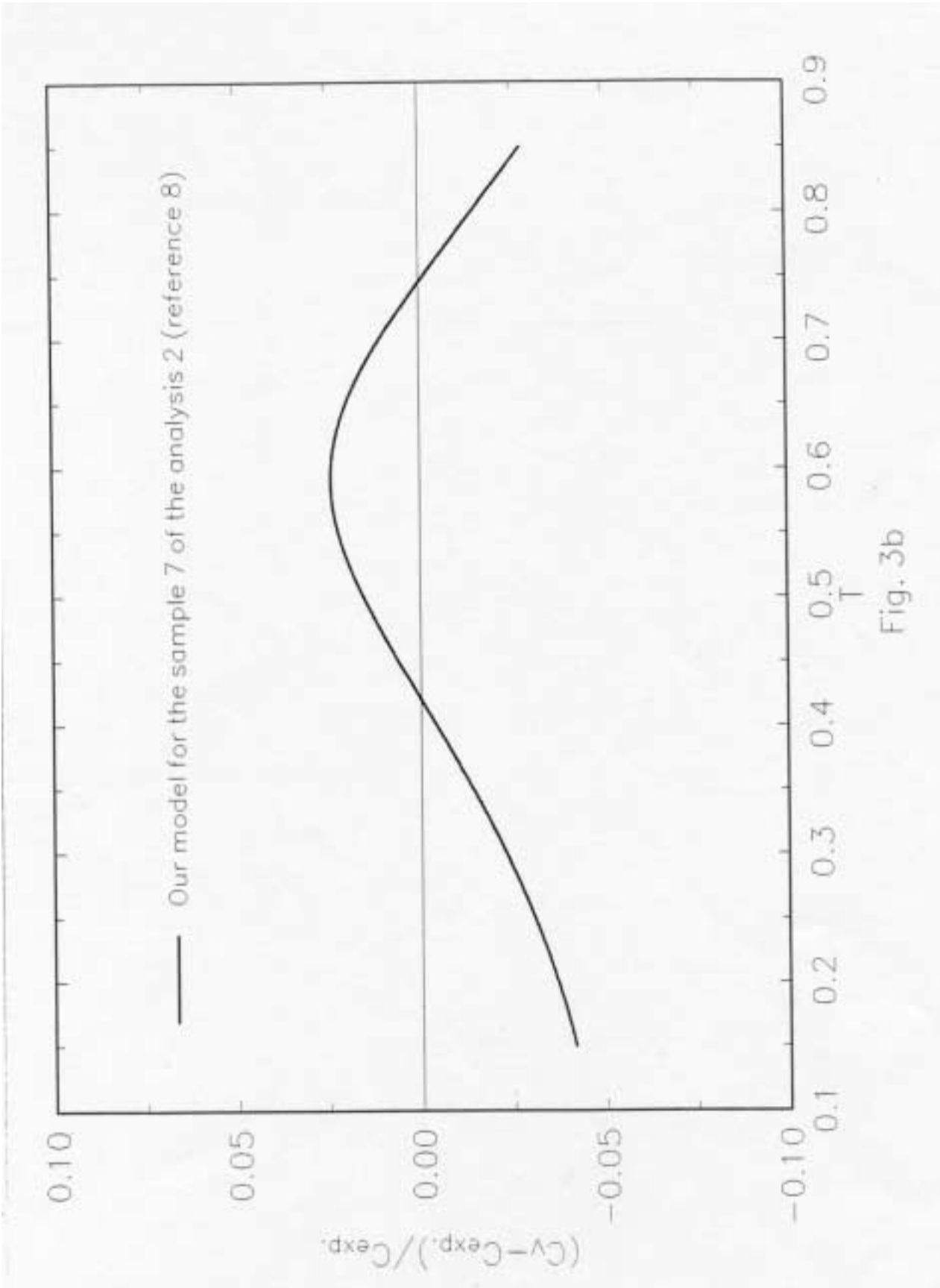


Fig. 3a-e Deviations from Greywall's experimental values $C_{exp.}$ in the molar specific heat (C_v) values predicted in our model with the sets of parameters shown in Table III for samples 6-10 in analysis 2 of the reference 8.



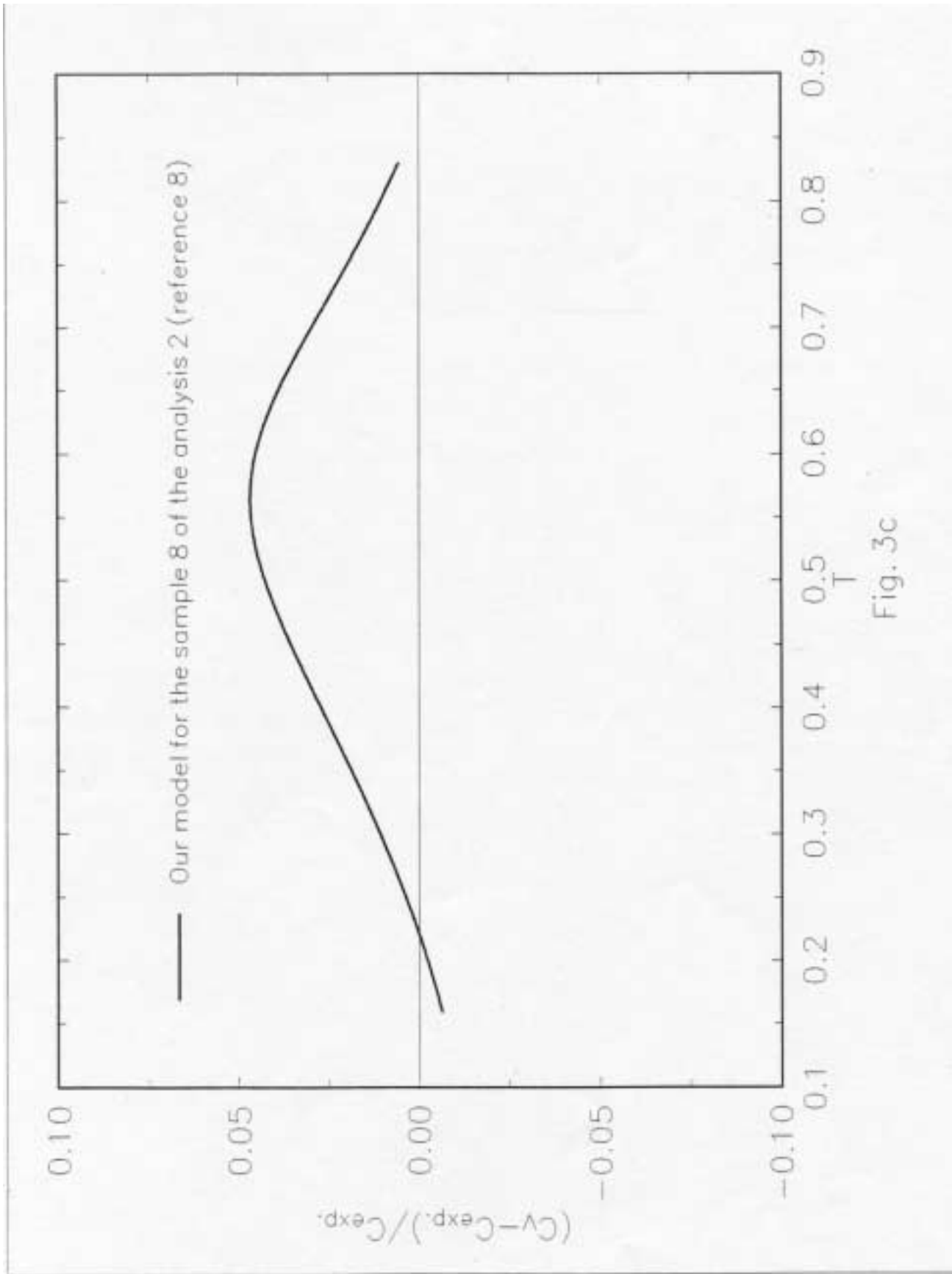


Fig. 3c

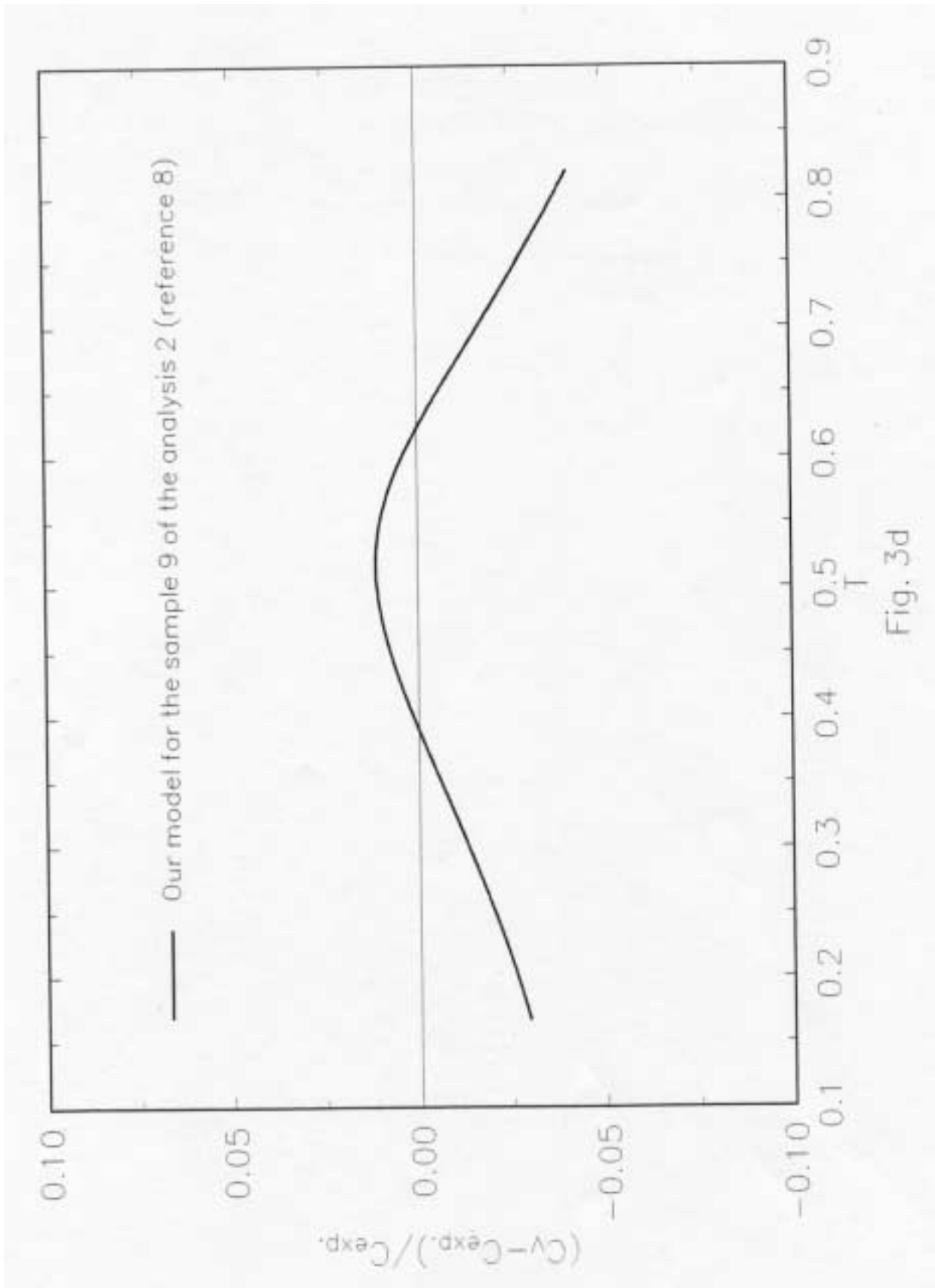


Fig. 3d

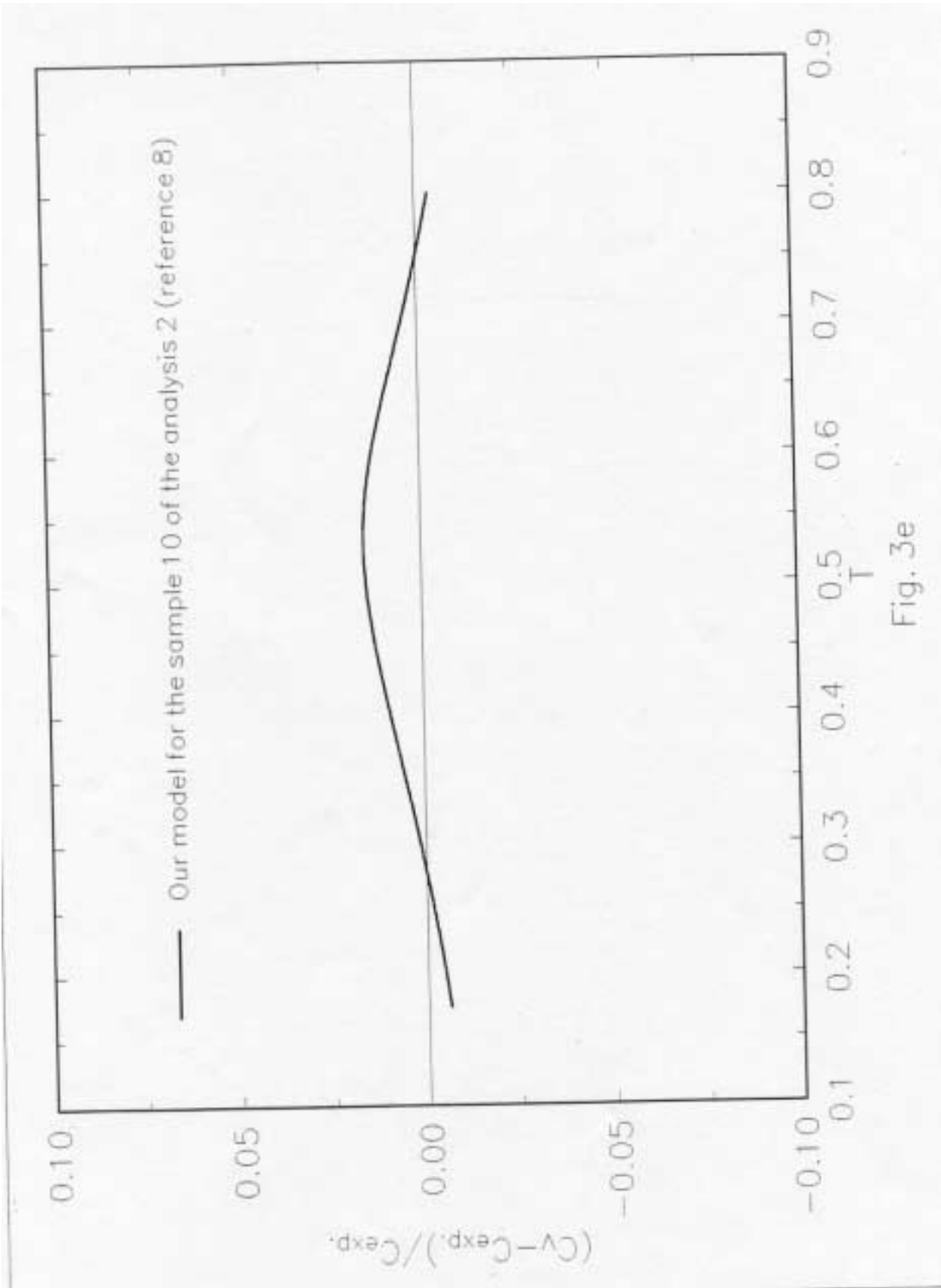


Fig. 3e

Sample	$\tilde{A}/10^4$ (<i>erg/mol K</i> ⁴)	$\tilde{B}/10^4$ (<i>erg/mol K</i> ⁶)	$\tilde{D}/10^4$ (<i>erg/mol K</i> ⁸)	V (<i>cm</i> ³)	$c_0/10^4$ (<i>cm/sec</i>)	$\alpha/10^{38}$ (<i>gr</i> ⁻² <i>cm</i> ⁻² <i>sec</i> ²)	δ (<i>gr</i> ⁻¹)
6	84.42	-49.8	83	27.5790	2.2854	2.1	1
7	69.3	-36.10	67	26.9650	2.4177	2.7178	1
8	57.77	-25.4	49	26.4240	2.5501	3.0873	2
9	49.85	-18.8	38	25.9760	2.6625	3.3821	2
10	43.44	-13.0	25	25.2000	2.7739	3.3206	2

Table I - Values of c_0 , α , δ and q resulting from the least-square fits of the specific heat data with the expression $C_V = \tilde{A}T^3 + \tilde{B}T^5 + \tilde{D}T^7 + C_V^{roton}$, for samples 6-10 in analysis 2 of ref. [8]; the roton data are those of [8].

Sample	$\tilde{A}/10^4$ (<i>erg/mol K</i> ⁴)	$\tilde{B}/10^4$ (<i>erg/mol K</i> ⁶)	$\tilde{D}/10^4$ (<i>erg/mol K</i> ⁸)	$\tilde{G}/10^4$ (<i>erg/mol K</i> ¹⁰)	V (<i>cm</i> ³)	$c_0/10^4$ (<i>cm/sec</i>)	$\alpha/1$ (<i>gr</i> ⁻² <i>cm</i>)
6	84.42	-49.8	83	0	27.579	2.2854	2.
6	80	-15	59	-27.2	27.579	2.3250	2.97
6	82	-11	23	-9.76	27.579	2.3272	1.83
6	81	-15	51	-26	27.579	2.3182	2.72

Table II - In the upper row, we repeat the values of table I for sample 6. In the lower ones, we have three sets of values of c_0 , α , δ and q obtained taking for \tilde{A} , \tilde{B} , \tilde{C} and \tilde{G} (see relations (4-12) and (4-13)) values that reproduce, within 5% of accuracy, the curves resulting from least-squares fits of the specific heat data [8] with the expression $C_V = AT^3 + BT^5 + DT^7 + C_V^{roton}$. These values are such that $\tilde{G}_T = \tilde{G}_E$, where \tilde{G}_T is given by the last relation (4.13).

Sample	$\tilde{A}/10^4$ (<i>erg/mol K</i> ⁴)	$\tilde{B}/10^4$ (<i>erg/mol K</i> ⁶)	$\tilde{D}/10^4$ (<i>erg/mol K</i> ⁸)	$\tilde{G}/10^4$ (<i>erg/mol K</i> ¹⁰)	<i>V</i> (<i>cm</i> ³)	<i>c</i> ₀ /10 ⁴ (<i>cm/sec</i>)	$\alpha/1$ (<i>gr</i> ⁻² <i>cm</i>)
6	80	-15	59	-27.2	27.579	2.3250	2.97
7	66	-16	67	-46	26.965	2.4546	3.84
8	57	-9	37	-11.3	26.424	2.5631	3.42
9	48	-4	26	-35.82	25.976	2.6992	3.55
10	43	-7	21	-9.31	25.568	2.7847	3.44

Table III - Values of c_0 , α , δ and q obtained taking for \tilde{A} , \tilde{B} , \tilde{D} and \tilde{G} values that reproduce within 5% of accuracy the curves resulting from least-squares fits of C_V data [8] with the expression $C_V = AT^3 + BT^5 + DT^7 + C_V^{roton}$. These values are such that $\tilde{G}_T = \tilde{G}_E$, where \tilde{G}_T is given by the last relation (4.13).

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