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Quantum Algebra*

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Abstract

By considering a set of N anyonic oscillators (non-local, intrinsically two-dimensional objects interpolating between fermionic and bosonic oscillators) on a two-dimensional lattice, we realize the $SU_q(N)$ quantum algebra by means of a generalized Schwinger construction. We find that the deformation parameter q of the algebra is related to the anyonic statistical parameter ν by $q = \exp(i\pi\nu)$.

Key-words: Quantum algebras; Anyons.

1. Introduction and Conclusions

Quasitriangular Hopf algebra [1-3] has attracted a lot of attention of physicists and mathematicians in the last few years [4]. Interesting examples of this structure are deformations of Lie algebras and Lie groups [1-4].

This kind of algebra has left its track in several areas of physics [5]. On the other hand, anyons [6-9], *i.e.* two-dimensional objects with arbitrary statistics interpolating between bosons and fermions, have also deserved considerable interest mainly because their possible connection to planar condensed matter phenomena, particularly the fractional quantum Hall effect [10].

Deformed Lie algebras and anyons may have a deep connection as indicated by the role played by the braid group in both these structures, therefore it is possible that the characteristic symmetries of anyonic systems could be described by deformed Lie algebras.

It was shown in ref. [11] an interesting relation between these two structures. Intrinsically two-dimensional operators, interpolating between fermionic and bosonic oscillators, were constructed on a two-dimensional lattice. These anyonic oscillators, which carry braiding properties, were used to build explicitly the deformed algebra of $SU_q(2)$ in a sort of generalized Schwinger construction [12].

Actually, a generalized Schwinger approach was built a few years ago, using operators obeying q -commutation relations, in order to realize the quantum enveloping algebra of the type A_n , B_n , C_n , D_n [13-17] and the quantum exceptional algebras [18]. These q -oscillators are different from the anyonic oscillators because they can live in any dimension and are local operators. Therefore, it seems sensible to ask to what extent the role of the building blocks of quantum algebras can be played by anyonic oscillators.

In this letter by considering a set of N anyonic oscillators we realize à la Schwinger the quantum enveloping algebra of $SU_q(N)$. The main difference from the $SU_q(2)$ case [11] comes from a trilinear relation among the quantum generators called *deformed Serre's relation*, which is proved in section 3 by non trivial use of the braiding properties of the generator's density of $SU_q(N)$.

It comes out from [11] and from the case we have analysed to be essential to deal with hard core anyonic oscillators to construct the deformed algebras. Therefore we believe that for all deformed algebras which can be realized with q -deformed fermionic oscillators the same role could be played by anyonic oscillators.

In section 2 we review briefly the main results concerning anyonic oscillators, in section 3 we show how to realize the quantum enveloping algebra of $SU_q(N)$ with these oscillators and we find that the deformation parameter is connected to the anyonic statistical parameter ν as $q = \exp(i\pi\nu)$.

2. Lattice Angle Function and Anyonic Oscillators

In this section we are going to review the construction of anyonic oscillators of ref. [11], on a two-dimensional square lattice of lattice spacing one.

Anyonic oscillators are two-dimensional non-local objects [19-24], interpolating between bosonic and fermionic oscillators which will be constructed on a square lattice Ω by means of the Jordan-Wigner transformation [25] that transmutes, in our case, fermionic oscillators into anyonic oscillators. Its basic ingredients are a lattice angle function and fermionic oscillators.

The lattice angle function was built in a very general way in ref. [21], but we shall describe concisely the special but very useful way of defining it of ref. [11].

We start by defining for each point x of the two dimensional lattice Ω a cut γ_x , made with bonds of the dual lattice $\tilde{\Omega}$ from minus infinity to $x^* = x + o^*$ along x -axis, with $o^* = (\frac{1}{2}, \frac{1}{2})$ the origin of the dual lattice, and we denote by x_γ the point $x \in \Omega$ with its associated cut γ_x .

Given any two distinct points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ on Ω , and their associated cuts γ_x and γ_y , it is possible to show that [11]

$$\Theta_{\gamma_x}(x, y) - \Theta_{\gamma_y}(y, x) = \begin{cases} \pi \operatorname{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2, \\ \pi \operatorname{sgn}(x_1 - y_1) & \text{for } x_2 = y_2, \end{cases} \quad (2.1)$$

with $\Theta_{\gamma_x}(x, y)$ being the angle of the point x measured from the point $y^* \in \tilde{\Omega}$ with respect to a line parallel to the positive x -axis.

In the formula (2.1) we are neglecting a term which depends on x and y [21] and, as shown in ref. [11], as this term is a lattice feature this is done by a sort of continuum limit. This term can be neglected for all $x, y \in \Omega$.

We notice that the elimination of this term is essential to get q -commutation relations for anyonic oscillators with a constant q -factor as will be clear in a moment.

The term depending on x and y which is missing in (2.1) can be seen [11] as being the angle between x and $x + 2o^*$ as measured from y^* , which is clearly negligible when x and y are very far apart from each other. However, this is not the case when x and y are close to each other.

When x and y are close to each other we embed the lattice Ω into another lattice Λ whose lattice spacing ϵ is taken to be much smaller than one. As Ω is a sublattice of Λ , the quantities defined on Ω can be viewed as the restriction to Ω of quantities defined on Λ . In this case the missing term can be represented as the angle between x and $x + 2\epsilon o^*$ as measured from $\tilde{y}^* = y + \epsilon o^*$ which is negligible. Thus the missing term can be neglected $\forall x, y \in \Omega$ [11].

Equation (2.1), which relates the angle of two distinct points on Ω , can be used to endow the lattice with an ordering which will be very useful in handling anyonic oscillators. We define $x_\gamma > y_\gamma$ by making the choice of the positive sign of equation (2.1), i.e.

$$x_\gamma > y_\gamma \iff \begin{cases} x_2 > y_2, \\ x_2 = y_2, x_1 > y_1. \end{cases} \quad (2.2)$$

Even if it is unambiguous, this definition is not unique, it depends on the choice made for the cuts. Suppose, now, instead of choosing γ_x we choose for each point of the lattice a cut δ_x , made with bonds of the dual lattice from plus infinity to *x along x -axis, with $^*x = x - o^*$. In this case it can be shown that the relation between the angle of two distinct points $x, y \in \Omega$ becomes [11]

$$\bar{\Theta}_{\delta_x}(x, y) - \bar{\Theta}_{\delta_y}(y, x) = \begin{cases} -\pi \operatorname{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2, \\ -\pi \operatorname{sgn}(x_1 - y_1) & \text{for } x_2 = y_2. \end{cases} \quad (2.3)$$

Notice that $\bar{\Theta}_{\delta_x}(x, y)$ is now the angle of x as seen from $^*y \in \bar{\Omega}$ with respect to a line parallel to the negative x -axis.

Denoting by x_δ the point x with its associated cut δ_x we can define $x_\delta > y_\delta$ for the positive sign of eq. (2.3) which in this case gives

$$x_\delta > y_\delta \iff \begin{cases} x_2 < y_2, \\ x_2 = y_2, x_1 < y_1, \end{cases} \quad (2.4)$$

that is the opposite of the ordering induced by the cut γ .

We can also have the relation between Θ_γ and $\bar{\Theta}_\delta$. Using the definition we have given for these angles we get [11]

$$\bar{\Theta}_{\delta_x}(x, y) - \Theta_{\gamma_x}(x, y) = \begin{cases} -\pi \operatorname{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2, \\ -\pi \operatorname{sgn}(x_1 - y_1) & \text{for } x_2 = y_2, \end{cases} \quad (2.5)$$

and using (2.3) and (2.5) it follows that

$$\bar{\Theta}_{\delta_x}(x, y) - \Theta_{\gamma_y}(y, x) = 0, \quad (2.6)$$

which is valid also when $x = y$, *i.e.*

$$\bar{\Theta}_{\delta_x}(x, x) - \Theta_{\gamma_x}(x, x) = 0. \quad (2.7)$$

We are going to use now, the two above angle functions $\Theta_{\gamma_x}(x, y)$ and $\bar{\Theta}_{\delta_x}(x, y)$ to define two kinds of parity related anyonic oscillators.

We define anyonic oscillators of type γ and δ as

$$a_i(x_\alpha) = K_i(x_\alpha) c_i(x) \quad (\text{no sum over } i) \quad (2.8)$$

where the disorder operators $K_i(x_\alpha)$ [26] are given by

$$K_i(x_\alpha) = e^{i\nu \sum_{y \in \Omega} \Theta_{\alpha_x}(x, y) c_i^\dagger(y) c_i(y)} \quad (2.9)$$

with $\alpha_x = \gamma_x, \delta_x$, $i = 1, \dots, N$, ν is the statistical parameter and c_i, c_i^\dagger are fermionic oscillators defined on Ω which have the well-known anticommutation relations

$$\begin{aligned} \{c_i(x), c_j^\dagger(y)\} &= \delta_{i,j} \delta(x, y) \\ \{c_i(x), c_j(y)\} &= 0 \\ \{c_i^\dagger(x), c_j^\dagger(y)\} &= 0 \end{aligned} \quad (2.10)$$

with $\delta(\mathbf{x}, \mathbf{y})$ the delta function on Ω , i.e. zero for different points of the lattice and one if $\mathbf{x} = \mathbf{y}$.

Using (2.1) and (2.10) we get for type γ anyonic oscillators the following commutation relations

$$a_i(\mathbf{x}_\gamma) a_i(\mathbf{y}_\gamma) + q^{-1} a_i(\mathbf{y}_\gamma) a_i(\mathbf{x}_\gamma) = 0, \quad (2.11a)$$

$$a_i(\mathbf{x}_\gamma) a_i^\dagger(\mathbf{y}_\gamma) + q a_i^\dagger(\mathbf{y}_\gamma) a_i(\mathbf{x}_\gamma) = 0, \quad (2.11b)$$

for $\mathbf{x} > \mathbf{y}$ (from now on $\mathbf{x} > \mathbf{y} \iff \mathbf{x}_\gamma > \mathbf{y}_\gamma$) and $q = \exp(i\pi\nu)$. If $\mathbf{x} = \mathbf{y}$ we have

$$(a_i(\mathbf{x}_\gamma))^2 = 0, \quad (2.12a)$$

$$a_i(\mathbf{x}_\gamma) a_i^\dagger(\mathbf{x}_\gamma) + a_i^\dagger(\mathbf{x}_\gamma) a_i(\mathbf{x}_\gamma) = 1. \quad (2.12b)$$

Eqs. (2.11-12) tell us that these hard core oscillators obey q -commutation relations for different points of the lattice but the standard anticommutation relations for oscillators at the same point of the lattice. We notice that we can get the complete set of relations for these oscillators by taking the hermitean conjugate of formulas (2.11-12).

It is very easy to see that different oscillators obey the well known anticommutation relations

$$\begin{aligned} \{a_i(\mathbf{x}_\gamma), a_j(\mathbf{y}_\gamma)\} &= \{a_i(\mathbf{x}_\gamma), a_j^\dagger(\mathbf{y}_\gamma)\} = 0 \\ \forall \mathbf{x}, \mathbf{y} \in \Omega \quad \text{and} \quad \forall i, j, \quad i \neq j = 1, \dots, N. \end{aligned} \quad (2.13)$$

The commutation relations among anyonic oscillators of type δ can be obtained from the previous ones, (2.11-13), by replacing q by q^{-1} and γ by δ . This is due to the fact that $\mathbf{x} > \mathbf{y}$ means $\mathbf{x}_\gamma > \mathbf{y}_\gamma$ and thus $\mathbf{x}_\delta < \mathbf{y}_\delta$; alternatively we can say that δ ordering can be obtained from γ ordering by a parity transformation which, as is well known, changes the braiding phase q into q^{-1} (see for instance [9]).

To complete our discussion on the commutation relations of anyonic oscillators we can compute these relations among different types of oscillators, i.e. between type γ and type δ oscillators. By using eqs. (2.5-8) we have

$$\{a_i(\mathbf{x}_\delta), a_i(\mathbf{y}_\gamma)\} = 0 \quad \forall \mathbf{x}, \mathbf{y}, \quad (2.14a)$$

$$\{a_i(\mathbf{x}_\delta), a_i^\dagger(\mathbf{y}_\gamma)\} = 0 \quad \forall \mathbf{x}, \mathbf{y} \quad \mathbf{x} \neq \mathbf{y}, \quad (2.14b)$$

and finally

$$\{a_i(\mathbf{x}_\delta), a_i^\dagger(\mathbf{x}_\gamma)\} = q^{-\left[\sum_{\mathbf{y} < \mathbf{x}} - \sum_{\mathbf{y} > \mathbf{x}}\right] c_i^\dagger(\mathbf{y}) c_i(\mathbf{y})} \quad (2.15)$$

and those coming from taking the hermitean conjugate of relations (2.14-15).

It should be clear from the above discussion that anyonic oscillators does not have anything to do with q -oscillators introduced a few years ago (ref. [13-18]). The main reason is, as we have seen, that anyonic oscillators depend on the introduction of an angle function, with its associated cut, which is an intrinsically two-dimensional object thus, as it should be, these oscillators are non-local objects which can be introduced only in two dimensions, contrarily to the deformed q -oscillators which are local objects that can be defined in any dimension.

3. Realization of $SU_q(N)$ Algebra

We are going to realize, in this section, the quantum enveloping algebra $SU_q(N)$ [1,2] with the anyonic oscillators defined in the previous section.

Let $\alpha_1, \dots, \alpha_{N-1}$ be the simple roots of the simple Lie algebra $SU(N)$ and a_{ij} , for $i, j = 1, \dots, N-1$, its Cartan matrix. The quantum enveloping algebra $SU_q(N)$ is a Hopf algebra with unit $\mathbf{1}$ and generators $E_{\pm\alpha_i}$, H_{α_i} defined through the commutation relations in the Chevalley basis

$$[H_{\alpha_i}, H_{\alpha_j}] = 0, \quad (3.1a)$$

$$[H_{\alpha_i}, E_{\pm\alpha_j}] = \pm a_{ij} E_{\pm\alpha_j}, \quad (3.1b)$$

$$[E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \quad (3.1c)$$

$$[E_{\alpha_i}, E_{\alpha_j}] = 0 \quad \text{if } a_{ij} = 0, \quad (3.1d)$$

and the deformed Serre's relation

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_q (E_{\pm\alpha_i})^{1-a_{ij}-n} E_{\pm\alpha_j} (E_{\pm\alpha_i})^n = 0 \quad (3.2)$$

for $j = i \pm 1$, with the q -binomial coefficients $\begin{bmatrix} m \\ n \end{bmatrix}_q$, given by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q! [n]_q!} \quad (3.3)$$

where $[m]_q! = [m]_q [m-1]_q \cdots [1]_q$, and

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}. \quad (3.4)$$

To complete the definition the comultiplication Δ and antipode S are given by

$$\begin{aligned} \Delta(H_{\alpha_i}) &= H_{\alpha_i} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\alpha_i}, \\ \Delta(E_{\pm\alpha_i}) &= E_{\pm\alpha_i} \otimes q^{-H_{\alpha_i}/2} + q^{H_{\alpha_i}/2} \otimes E_{\pm\alpha_i}, \\ S(H_{\alpha_i}) &= -H_{\alpha_i}, \\ S(E_{\pm\alpha_i}) &= q^{-\rho} E_{\pm\alpha_i} q^{\rho}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} H_{\alpha} \end{aligned} \quad (3.5)$$

with Δ_+ the set of positive roots of $SU(N)$, and the co-unit

$$\epsilon(\mathbf{1}) = 1, \quad \epsilon(E_{\pm\alpha_i}) = \epsilon(H_{\alpha_i}) = 0. \quad (3.6)$$

Now, we realize the generators of $SU_q(N)$ quantum enveloping algebra through the N anyonic oscillators, defined on the lattice, of the previous section. Generalizing the $SU_q(2)$ case [11] the generators should be given by the sum over the lattice of density operators. Since type γ and type δ oscillators are related by a parity transformation which changes q into q^{-1} , and as we expect that

$$E_{+\alpha_i}(\mathbf{x})_q = \left[E_{-\alpha_i}(\mathbf{x})_{q^{-1}} \right]^\dagger, \quad (3.7)$$

for q complex, we are led to construct $E_{\alpha_i}(\mathbf{x})$ and $E_{-\alpha_i}(\mathbf{x})$ with different type of anyonic oscillators.

The densities of quantum generators are given by

$$E_{\alpha_i}(\mathbf{x}) = a_i^\dagger(\mathbf{x}_\gamma) a_{i+1}(\mathbf{x}_\gamma), \quad (3.8a)$$

$$E_{-\alpha_i}(\mathbf{x}) = a_{i+1}^\dagger(\mathbf{x}_\delta) a_i(\mathbf{x}_\delta), \quad (3.8b)$$

$$\begin{aligned} H_{\alpha_i}(\mathbf{x}) &= a_i^\dagger(\mathbf{x}_\gamma) a_i(\mathbf{x}_\gamma) - a_{i+1}^\dagger(\mathbf{x}_\gamma) a_{i+1}(\mathbf{x}_\gamma) = \\ &= a_i^\dagger(\mathbf{x}_\delta) a_i(\mathbf{x}_\delta) - a_{i+1}^\dagger(\mathbf{x}_\delta) a_{i+1}(\mathbf{x}_\delta), \end{aligned} \quad (3.8c)$$

the last identity in eq. (3.8 c) comes from the cancellation of the disorder operators, as it may be easily checked using eq. (2.8), and the generators are given by a sum on the lattice

$$\begin{aligned} E_{\pm\alpha_i} &= \sum_{\mathbf{x} \in \Omega} E_{\pm\alpha_i}(\mathbf{x}), \\ H_{\alpha_i} &= \sum_{\mathbf{x} \in \Omega} H_{\alpha_i}(\mathbf{x}). \end{aligned} \quad (3.9)$$

We can easily see that the equations (3.1 a) and (3.1 b) are satisfied if we realize that H_{α_i} counts the number of operators on the lattice of the type i minus those of type $i + 1$, independently if they are type γ or type δ operators. The relation (3.1 a) is really trivial, and separating eq. (3.1 b) in four cases where $[H_{\alpha_i}, E_{\pm\alpha_i}]$, $[H_{\alpha_i}, E_{\pm\alpha_{i+1}}]$, $[H_{\alpha_i}, E_{\pm\alpha_{i-1}}]$, $[H_{\alpha_i}, E_{\pm\alpha_j}]$ for $|i - j| \geq 2$, we immediately realize that we get respectively ± 2 , ∓ 1 , ∓ 1 and zero, which is the desired result.

It is again easy to get convinced that eq. (3.1 d) is satisfied. The Cartan matrix a_{ij} is equal to zero for $|i - j| \geq 2$ and in this case the generators E_{α_i} and E_{α_j} are made up with different anyonic oscillators, which anticommute (eq. (2.13)), giving thus the correct result.

The proof of (3.1 c) is analogous to the $SU_q(2)$ case [11]. We consider firstly the commutator of the generator's densities $E_{\alpha_i}(\mathbf{x})$ and $E_{-\alpha_i}(\mathbf{y})$, which gives zero if $\mathbf{x} \neq \mathbf{y}$, and using (2.14-15) we have

$$\begin{aligned} [E_{\alpha_i}(\mathbf{x}), E_{-\alpha_i}(\mathbf{y})] &= q \left[\sum_{\mathbf{y} < \mathbf{x}} - \sum_{\mathbf{y} > \mathbf{x}} \right] c_{i+1}^\dagger(\mathbf{y}) c_{i+1}(\mathbf{y}) a_i^\dagger(\mathbf{x}_\gamma) a_i(\mathbf{x}_\delta) \\ &\quad - q \left[\sum_{\mathbf{y} < \mathbf{x}} - \sum_{\mathbf{y} > \mathbf{x}} \right] c_i^\dagger(\mathbf{y}) c_i(\mathbf{y}) a_{i+1}^\dagger(\mathbf{x}_\delta) a_{i+1}(\mathbf{x}_\gamma). \end{aligned} \quad (3.10)$$

Substituting the explicit definition of the anyonic oscillators in the right hand side, using (2.5) and (2.7) and taking into account that the above commutator is zero for $\mathbf{x} \neq \mathbf{y}$, we obtain

$$[E_{\alpha_i}, E_{-\alpha_i}] = \sum_{\mathbf{x} \in \Omega} \left(\prod_{\mathbf{y} < \mathbf{x}} q^{-H_{\alpha_i}(\mathbf{y})} H_{\alpha_i}(\mathbf{x}) \prod_{\mathbf{z} > \mathbf{x}} q^{H_{\alpha_i}(\mathbf{z})} \right). \quad (3.11)$$

Analogously to the $SU_q(2)$ case, by considering that for each point of the lattice Ω $H_{\alpha_i}(\mathbf{x})$ admits only the eigenvalues 0 and $\pm \frac{1}{2}$, we can prove by complete induction that the right hand side of (3.11) is indeed the right hand side of (3.1 c) for $i = j$. We would like to stress here that it is essential to choose hard core oscillators to get this result. When i is different from j , let us consider j as given by $|i - j| \geq 1$. For the equal sign case, E_{α_i} and $E_{-\alpha_{i \pm 1}}$ share only $a_{i+1}(\mathbf{x}_\gamma)$ and $a_{i+1}(\mathbf{y}_\delta)$ or $a_i^\dagger(\mathbf{x}_\gamma)$ and $a_i^\dagger(\mathbf{y}_\delta)$ respectively as oscillators of the same type, all the others are of different types. But from (2.14 a) these oscillators anticommute $\forall \mathbf{x}, \mathbf{y}$ thus E_{α_i} commutes with $E_{-\alpha_{i \pm 1}}$. When $|i - j| > 1$ all the oscillators which are present are of different kind (anticommute among themselves) and as a result E_{α_i} commutes trivially with E_{α_j} . Equation (3.1 c) is proved.

The deformed Serre's relation can be written explicitly for the case where $j = i + 1$ as

$$(E_{\pm \alpha_i})^2 E_{\pm \alpha_{i+1}} - (q + q^{-1}) E_{\pm \alpha_i} E_{\pm \alpha_{i+1}} E_{\pm \alpha_i} + E_{\pm \alpha_{i+1}} (E_{\pm \alpha_i})^2 = 0. \quad (3.12)$$

To prove (3.12) we have to write the generators in the above formula in terms of the generator's densities by the use of (3.9). Our strategy will be that of dividing the sum over the lattice into two parts, the sum where the generator's densities are at different lattice points and that one where at least two of the generator's densities are at the same lattice point. Each part can be compared separately in formula (3.12).

Let us consider the first part. The first term of (3.12) can be written as

$$\begin{aligned} I_1 &= \sum_{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}} E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) E_{\alpha_{i+1}}(\mathbf{z}) = \\ &= \left(\sum_{\substack{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \\ \mathbf{y} > \mathbf{z}}} + \sum_{\substack{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \\ \mathbf{y} < \mathbf{z}}} \right) E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) E_{\alpha_{i+1}}(\mathbf{z}); \end{aligned} \quad (3.13)$$

now using

$$E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{y}) = \begin{cases} q E_{\alpha_{i+1}}(\mathbf{y}) E_{\alpha_i}(\mathbf{x}) & \text{for } \mathbf{x} > \mathbf{y} , \\ q^{-1} E_{\alpha_{i+1}}(\mathbf{y}) E_{\alpha_i}(\mathbf{x}) & \text{for } \mathbf{x} < \mathbf{y} , \end{cases} \quad (3.14)$$

we have

$$I_1 = \left(q \sum_{\substack{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \\ \mathbf{y} > \mathbf{z}}} + q^{-1} \sum_{\substack{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \\ \mathbf{y} < \mathbf{z}}} \right) E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y}). \quad (3.15)$$

Using the same procedure for the third term of (3.12) we get

$$\begin{aligned}
 I_2 &= \sum_{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}} E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) = \\
 &= \left(q \sum_{\substack{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \\ \mathbf{z} > \mathbf{x}}} + q^{-1} \sum_{\substack{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z} \\ \mathbf{z} < \mathbf{x}}} \right) E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y})
 \end{aligned} \tag{3.16}$$

and it is easy to show using (3.14) and

$$E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) = \begin{cases} q^{-2} E_{\alpha_i}(\mathbf{y}) E_{\alpha_i}(\mathbf{x}) & \text{for } \mathbf{x} > \mathbf{y} , \\ q^2 E_{\alpha_i}(\mathbf{y}) E_{\alpha_i}(\mathbf{x}) & \text{for } \mathbf{x} < \mathbf{y} , \end{cases} \tag{3.17}$$

that

$$\sum_{\mathbf{x} < \mathbf{z} < \mathbf{y}} E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y}) = \sum_{\mathbf{y} < \mathbf{z} < \mathbf{x}} E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y}). \tag{3.18}$$

In the right hand side of (3.16) there is a term proportional to q which is the sum over $\mathbf{x} < \mathbf{z} < \mathbf{y}$ that is already present in (3.15), but from (3.18) this term is equal to the sum over $\mathbf{y} < \mathbf{z} < \mathbf{x}$. The same happens in (3.16) for the term proportional to q^{-1} where the sum over $\mathbf{y} < \mathbf{z} < \mathbf{x}$ is already in (3.15), but again from (3.18) this sum is equal to the one over $\mathbf{x} < \mathbf{z} < \mathbf{y}$. Taking into account these two observations we realize that (3.16) is the missing part in (3.15) to make the opposite of the second term in (3.12), i.e.

$$\begin{aligned}
 &\sum_{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}} \left(E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) E_{\alpha_{i+1}}(\mathbf{z}) + E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) \right) = \\
 &= \left(q + q^{-1} \right) \sum_{\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}} E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y}) .
 \end{aligned} \tag{3.19}$$

To complete the proof we have to consider the part of the sum where at least two of the generator's densities are at the same lattice point. The easiest way to handle this part is never cross generator's densities at the same lattice point. The computation goes as follows. We first define the simbol $\sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^c$ as denoting the sum over $\mathbf{x}, \mathbf{y}, \mathbf{z}$ when at least two of the lattice points are equal; notice that in our case we cannot have all of them being equal because $(E_{\alpha_i}(\mathbf{x}))^2 = 0$. Writing explicitly this definition for the first term of (3.12) and using the braiding relations (3.14) and

(3.17) we have

$$\begin{aligned}
& \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^e E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) E_{\alpha_{i+1}}(\mathbf{z}) \equiv \\
& \equiv \left(\sum_{\mathbf{y} < \mathbf{x} = \mathbf{s}} + \sum_{\mathbf{y} > \mathbf{x} = \mathbf{s}} \right) (E_{\alpha_i}(\mathbf{z}) E_{\alpha_i}(\mathbf{y}) E_{\alpha_{i+1}}(\mathbf{z})) + \\
& + \left(\sum_{\mathbf{x} < \mathbf{y} = \mathbf{s}} + \sum_{\mathbf{x} > \mathbf{y} = \mathbf{s}} \right) (E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{z}) E_{\alpha_{i+1}}(\mathbf{z})) = \\
& = \left(q^{-1} \sum_{\mathbf{y} < \mathbf{x} = \mathbf{s}} + q \sum_{\mathbf{y} > \mathbf{x} = \mathbf{s}} + q \sum_{\mathbf{y} < \mathbf{x} = \mathbf{s}} + q^{-1} \sum_{\mathbf{y} > \mathbf{x} = \mathbf{s}} \right) (E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y})) .
\end{aligned} \tag{3.20}$$

Repeating the above steps for the third term of (3.12) we realize, again, that this term is what is missing in (3.20) to make the opposite of the second term, for the same kind of sum, in (3.12), *i.e.*

$$\begin{aligned}
& \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^e \left(E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) E_{\alpha_{i+1}}(\mathbf{z}) + E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{x}) E_{\alpha_i}(\mathbf{y}) \right) = \\
& = (q + q^{-1}) \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^e E_{\alpha_i}(\mathbf{x}) E_{\alpha_{i+1}}(\mathbf{z}) E_{\alpha_i}(\mathbf{y}) .
\end{aligned} \tag{3.21}$$

Putting together (3.19) and (3.21) we get the desired result. The part with minus sign in (3.12) is trivially obtained if we recall that under a parity transformation $E_{\alpha_i} \rightarrow E_{-\alpha_i}$ and $q \leftrightarrow q^{-1}$, and that (3.12) is invariant under $q \leftrightarrow q^{-1}$.

The case for $j = i - 1$ is similar.

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