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Physical Variables for the Chern-Simons-Maxwell Theory without Matter

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Abstract

We work out the physical field variables and write down the physical Hamiltonian for the Chern-Simons-Maxwell theory by working with the symplectic projector method.

1 Introduction

Some years ago, we developed a method based on the so-called symplectic projectors to work in the framework of gauge field theories[1,2]; the idea of the procedure is to pick out from the original set of field variables those which are the "true" or "physical" variables. This would be the first step to treat a gauge theory in a strictly canonical way [3,4,5].

We show in this letter how to derive the physical Hamiltonian for the $D = 3$-Chern-Simons-Maxwell (CSM) model with the Coulomb gauge conditions without coupling to matter fields. Its expression is closely related to the one previously found in a work by Devecchi et al[6], where the Dirac bracket quantization procedure (DBQP) has been adopted. Here, we proceed along a different way and try to check the efficacy of the symplectic projector method by applying it to a 3 - dimensional gauge theory.

2 The Physical Hamiltonian for the CSM theory

We start off from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma, \tag{1}$$

where the metric $(-1, 1, 1)$ is adopted.

The generalised Hamiltonian has the following canonical form:

$$\mathcal{H} = \int d^2x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\left(\varepsilon^{ij}\partial^i A^j\right)^2 + \frac{1}{2}m^2 A^k A^k + m\varepsilon^{ij}A^i\pi^j\right], \tag{2}$$

with the (second class) constraint relations:

$$\Omega^1 = \pi^0 = 0, \tag{3}$$

$$\Omega^2 = \partial^i \pi^i + m\, \varepsilon^{ij} \partial^j A^i = 0, \tag{4}$$

$$\Omega^3 = A^0 = 0, \tag{5}$$

$$\Omega^4 = \partial^i A^i = 0. \tag{6}$$

To set up a symplectic structure, we rename field variables according to the following correspondence:

$$\left(A^0,\, A^1,\, A^2,\, \pi^0,\, \pi^1,\, \pi^2\right) \Leftrightarrow \left(\xi^1,\, \xi^2,\, \xi^3,\, \xi^4,\, \xi^5,\, \xi^6\right). \tag{7}$$

The constraints Ω^i define a local metric, g_{ij}, which is the inverse of $g^{ij}\,(x,y) = \{\Omega^i\,(x)\,,\, \Omega^j\,(y)\,\}$, and reads formally as below:

$$g^{-1} = \begin{pmatrix} 0 & 0 & \delta^2\,(x-y) & 0 \\ 0 & 0 & 0 & \nabla^{-2} \\ -\,\delta^2\,(x-y) & 0 & 0 & 0 \\ 0 & -\,\nabla^{-2} & 0 & 0 \end{pmatrix}. \tag{8}$$

The general form for the symplectic projectors is given by the expression that follows[1] :

(9) $\Lambda^{\mu}_{\nu}\,(x,\,y) = \delta^{\mu}_{\nu}\,\delta^{2}\,(x-y) - \varepsilon^{\mu\alpha}\int d^{2}\,r\,d^{2}\,\varpi\,g_{ij}\,(r,\,\varpi)\;\delta_{\alpha(x)}\,\Omega^{i}\,(r)\;\delta_{\nu(y)}\,\Omega^{j}\,(\varpi)\,,$

with $\delta_{\alpha(x)}\,\Omega^{i}\,(r) \equiv \frac{\delta\,\Omega^{i}\,(r)}{\delta\,\xi^{\alpha}\,(x)}$.

After a lengthy but straightforward calculation, we find:

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^{2}\,(x-y) - \frac{\partial^{x}_{1}\,\partial^{y}_{1}}{\nabla^{2}} & -\frac{\partial^{x}_{1}\partial^{y}_{2}}{\nabla^{2}} & 0 & 0 & 0 \\ 0 & -\frac{\partial^{x}_{2}\,\partial^{y}_{1}}{\nabla^{2}} & \delta^{2}\,(x-y) - \frac{\partial^{x}_{2}\,\partial^{y}_{2}}{\nabla^{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\,m\,\delta^{2}\,(x-y) & 0 & \delta^{2}\,(x-y) - \frac{\partial^{x}_{1}\,\partial^{y}_{1}}{\nabla^{2}} & -\frac{\partial^{x}_{1}\,\partial^{y}_{2}}{\nabla^{2}} \\ 0 & m\,\delta^{2}\,(x-y) & 0 & 0 & -\frac{\partial^{x}_{2}\,\partial^{y}_{1}}{\nabla^{2}} & \delta^{2}\,(x-y) - \frac{\partial^{x}_{2}\,\partial^{y}_{2}}{\nabla^{2}} \end{pmatrix} .(10)$$

Getting the physical variables, $\xi^{*}_{\mu}\,(x)$, is a simple matter of applying the prescription

(11) $$\xi^{\mu\,*}\,(x) = \int d^{2}\,y\,\Lambda^{\mu}_{\nu}\,(x,y)\;\xi^{\nu}\,(y)\,;$$

we get thereby:

(12) $$\xi^{1*}\,(x) = \xi^{4*}\,(x) = 0,$$

(13) $$\xi^{2*}\,(x) = A^{\perp}_{1}\,(x)\,,$$

(14) $$\xi^{3*}\,(x) = A^{\perp}_{2}\,(x)\,,$$

(15) $$\xi^{5*}\,(x) = \pi^{\perp}_{1}\,(x) - m\,A^{\perp}_{2}\,(x)\,,$$

$$\xi^{6*}(x) = \pi_2^{\perp}(x) + m\,A_1^{\perp}(x)\,. \tag{16}$$

Now, our original constrained Hamiltonian written in symplectic coordinates takes over the form:

$$\mathcal{H} = \int d^2x\,\left[\frac{1}{2}\left(\xi_5^2 + \xi_6^2\right) + \frac{1}{2}\left(\partial_1\xi_3 - \partial_2\xi_2\right)^2 + \frac{1}{2}m^2\left(\xi_2^2 + \xi_3^2\right) + m\left(\xi_2\xi_6 - \xi_3\xi_5\right)\right]; \tag{17}$$

on the other hand, the projected Hamiltonian becomes as below:

$$\mathcal{H}^* = \int d^2x\left[\frac{1}{2}\left(\xi_5^{*2} + \xi_6^{*2}\right) + \frac{1}{2}\left(\partial_1\xi_3^* - \partial_2\xi_2^*\right)^2 + \frac{1}{2}m^2\left(\xi_2^{*2} + \xi_3^{*2}\right) + m\left(\xi_2^*\xi_6^* - \xi_3^*\xi_5^*\right)\right]. \tag{18}$$

Coming back to the original phase - space notation, with the help of $eqs.\,(12)-(15)$, we finally conclude that the projected Hamiltonian reads as below:

$$\mathcal{H}^* = \int d^2x\,\left[\frac{1}{2}\left(\pi_i^{\perp}\,\pi_i^{\perp} + 4\,m^2A_i^{\perp}\,A_i^{\perp}\right) + \frac{1}{2}\left(\xi^{ij}\,\partial_i\,A_j^{\perp}\right)^2 + 2\,m\,\left(A_1^{\perp}\,\pi_2^{\perp} - A_2^{\perp}\,\pi_1^{\perp}\right)\right]. \tag{19}$$

This is the Chern-Simons-Maxwell Hamiltonian written in terms of the so-called transverse expression, wich agrees with the results found in ref.[6] along a different line of arguments.

We wish to stress a very important point of these results: the physical Hamiltonian is the one given by eq.(18) in that the physical variables, those obeying canonical Poisson brackets, are the ξ^*'s and not the familiar transverse field variables. The only reason to write down $\mathcal{H}^*$ according to eq.(19) is to establish a bridge between our approach and the usual terminology.

Going over to the equations of motion and using the physical Hamiltonian within the framework of the Hamilton-Jacob equations, we find that:

$$\ddot{\xi}^*_2 = -2m^2 \xi^*_2 + \partial_2 \partial_2 \xi^*_2 - \partial_1 \partial_2 \xi^*_3 - 2m \xi^*_6, \tag{20}$$

$$\ddot{\xi}^*_3 = -2m^2 \xi^*_3 + \partial_1 \partial_1 \xi^*_3 - \partial_1 \partial_2 \xi^*_2 - 2m \xi^*_5, \tag{21}$$

$$\ddot{\xi}^*_5 = -2m^2 \xi^*_5 + \partial_2 \partial_2 \xi^*_5 - \partial_1 \partial_2 \xi^*_6 + m \left[2m^2 - \nabla^2\right] \xi^*_3, \tag{22}$$

$$\ddot{\xi}^*_6 = -2m^2 \xi^*_6 + \partial_1 \partial_1 \xi^*_6 - \partial_1 \partial_2 \xi^*_5 - m \left[2m^2 - \nabla^2\right] \xi^*_2. \tag{23}$$

Apparently, these equations might look rather strange; but, if we go back to the most familiar notation, by means of the correspondence between the A's, π's and ξ's ($eq.$(7)), we can cast them under the form:

$$\left(\Box + 4m^2\right) A^\perp_1 = -2m \pi^\perp_2, \tag{24}$$

$$\left(\Box + 4m^2\right) A^\perp_2 = 2m \pi^\perp_1, \tag{25}$$

$$\Box \pi^\perp_1 = 0, \tag{26}$$

$$\Box\, \pi_2^{\perp} = 0, \tag{27}$$

which amount to ensuring that

$$\Box\,\left(\Box + 4\,m^2\right)\, A_i^{\perp} = 0, \qquad (i = 1, 2)\,. \tag{28}$$

This guarantees that the physical excitation is a massive ($p^2 = 4\,m^{\,2}$) transverse vector; the massless pole is a spurious one: it has no dynamical rôle and does not correspond to any physical mode . Indeed, in coupling the A_μ - field propagator to a conserved external current, the current -current amplitude is such that the imaginary part of its residue taken at the pole $p^2 = 0$ vanishes, wich confirms that the latter does not correspond to any physical excitation.

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