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S-DUALITY IN $N = 4$ SUPERSYMMETRIC GAUGE THEORIES WITH ARBITRARY GAUGE GROUP

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ABSTRACT

The Goddard, Nuyts and Olive conjecture for electric-magnetic duality in Yang-Mills theory with an arbitrary gauge group G is extended by including a non-vanishing vacuum angle θ. This extended S-duality conjecture includes the case when the unbroken gauge group is non-abelian and a definite prediction for the spectrum of dyons results.

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In recent work [1,2,3,4,5,6,7,8], evidence has emerged that the electric-magnetic duality conjectured by Goddard, Nuyts and Olive (GNO) [9] is an exact relation between $N = 4$ supersymmetric gauge theories. In its original formulation, GNO duality has just one generator, which interchanges strong and weak coupling. This ordinary GNO duality requires that the spectrum of massive gauge bosons of a gauge theory for a gauge group G broken to H by an adjoint Higgs mechanism, is equal to the spectrum of magnetic monopoles for a dual gauge theory based on a dual gauge group G^* broken to H^* [9]. In actual fact, the duality can only be exact in the context of an $N = 4$ supersymmetric gauge theory (unless additional matter fields are added). The dual gauge group is not necessarily isomorphic to the original group. For groups with simply-laced Lie algebras $G^* \simeq G$; however for groups with non-simply-laced Lie algebras the groups $G \leftrightarrow G^*$ come in pairs: $\mathrm{SO}(2r+1) \leftrightarrow \mathrm{Sp}(r)$, $F_4 \leftrightarrow F_4'$ and $G_2 \leftrightarrow G_2'$. In the latter two cases the primes indicate that the corresponding dual groups are related by an exchange of long and short roots.

In the minimal case, where $G = \mathrm{SU}(2)$, the inclusion of non-zero vacuum (theta) angle leads to a larger group of duality transformations on the parameters of the theory [10]; namely the modular group $\mathrm{SL}(2,\mathbb{Z})$. In this letter, we will determine the corresponding duality group which extends GNO duality to the case of non-zero theta angle for an arbitrary gauge group. As in the case of gauge group $\mathrm{SU}(2)$, the additional symmetry comes from the θ periodicity of the partition function. The spectrum of states must also be invariant under this shift, which therefore provides a second symmetry generator. When combined with the original GNO duality, the new generator leads to an extended S-duality group which acts on an integer lattice of states. Our result is that this group in its most general form is a semi-direct product of a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ with the ordinary GNO duality group $\mathbb{Z}_2$. The main goal of the paper will be to elucidate the precise nature of this group. In particular, when the gauge group is simply-laced, the S-duality group reduces to the modular group itself; however, when the gauge group is non-simply-laced the action of the S-duality group is more complicated, but there is nevertheless a simple procedure for computing the conjectured spectrum of states. There results an enhanced duality conjecture, where dyonic states of the theory are conjectured to be gauge bosons of dual gauge theories with gauge group G or G^* in some characteristic pattern that we will elucidate. It remains a challenge to prove that these states exist within the semi-classical approximation. We should emphasize at this stage, however, that our picture of extended duality does not require any additional conjectures over and above the original GNO conjecture.

The exact duality that we are considering naturally occurs in the context of an $N = 4$ supersymmetric Yang-Mills theory with arbitrary gauge group G. We take all the fields to lie in a single sixteen dimensional supermultiplet. All the fields transform in the adjoint representation of the group and we take them to be Lie algebra valued. We can always work in a unitary gauge where the six real scalar fields are constant on a large sphere at infinity. The global SO(6) $\mathcal{R}$-symmetry is spontaneously broken to SO(5), and there are consequently five massless Goldstone bosons. The spectrum of massive states of the theory is completely determined by considering only the remaining real Higgs field ϕ, arbitrarily chosen up to an SO(6) $\mathcal{R}$-symmetry transformation. The bosonic part of the Lagrangian for this single scalar ϕ is then

$$\mathcal{L} = -\frac{e^2}{32\pi}\mathrm{Im}\left[\tau \mathrm{Tr}\left(F_{\mu\nu} + i^* F_{\mu\nu}\right)\left(F^{\mu\nu} + i^* F^{\mu\nu}\right)\right] + \frac{1}{2}\mathrm{Tr}\left(\mathcal{D}_\mu \phi \mathcal{D}^\mu \phi\right) - V(\phi), \tag{1}$$

where we use an othonormal basis for the algebra ($\mathrm{Tr}\left(\mathrm{T}^a \mathrm{T}^b\right) = \delta^{ab}$) and we have defined the complex coupling τ:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \tag{2}$$

The supersymmetric potential for ϕ vanishes which leads us naturally to the Prasad-Sommerfield limit $V(\phi) = 0$ [11].

Let ϕ_0 be the constant Higgs field on the large sphere at infinity in unitary gauge, chosen to lie within a Cartan subalgebra of the Lie algebra g:

$$\phi_0 = \boldsymbol{v} \cdot \boldsymbol{H}, \tag{3}$$

where $\boldsymbol{H}$ are the Cartan elements of g considered as an $r = \mathrm{rank}(g)$ vector. The simple roots $\boldsymbol{\alpha}_i$ of g can always be chosen such that $\boldsymbol{v} \cdot \boldsymbol{\alpha}_i \geq 0$. The Higgs field breaks the symmetry to a subgroup $H \subset G$ which consists of group elements which commute with ϕ_0:

$$H = \left\{ U \in G |\; U \phi_0 U^{-1} = \phi_0 \right\}. \tag{4}$$

Generically the unbroken gauge group will be the maximal torus of G; however, if $\boldsymbol{v}$ is orthogonal to any simple root of g then the unbroken gauge group has a non-abelian component. In general, therefore, H is locally of the form $\mathrm{U}(1)^{r'} \times K$, where K is a semi-simple Lie group of rank $r - r'$. The global definition of H, which requires the specification of a finite group, will not be required in what follows. The Lie algebra h of H consists of the generators of g commuting with ϕ_0.

The evidence for GNO duality begins with the mass formulae of the gauge bosons and monopoles in the theory. Associating the gauge bosons with the Cartan-Weyl basis of the Lie algebra g, the states corresponding to the Cartan elements are massless while the states associated to the step generators $E_{\boldsymbol{\alpha}}$ have a mass

$$M_{\boldsymbol{\alpha}} = e|\boldsymbol{v}\cdot\boldsymbol{\alpha}|, \tag{5}$$

Massive gauge bosons are associated to the roots of g with non-zero inner product with $\boldsymbol{v}$. We will denote this subset of the root system of g as $\Phi'(g)$. The states form multiplets of K and carry abelian charges with respect to the unbroken $\mathrm{U}(1)^{r'}$.

Monopole solutions in these theories were found originally in [12] (see also [13]) by embeddings of the SU(2) monopole. As with the gauge bosons, the monopole solutions are associated to roots of the Lie algebra and their mass spectrum is

$$\tilde{M}_{\boldsymbol{\alpha}} = \frac{4\pi}{e}\left|\boldsymbol{v}\cdot\boldsymbol{\alpha}^*\right|, \quad \boldsymbol{\alpha}^* = \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}. \tag{6}$$

where $\boldsymbol{\alpha}\in\Phi'(g)$ for a non-trivial solution. Notice that the spectrum of monopoles appears to be precisely equal to the spectrum of massive gauge bosons in a dual theory with gauge coupling $4\pi/\lambda e$ and gauge group G^*, whose Lie algebra g^* has roots $\lambda\boldsymbol{\alpha}/\boldsymbol{\alpha}^2$, where $\boldsymbol{\alpha}$ are the roots of g, and λ is a normalization constant. Actually this simplicity is somewhat illusory, as we discuss below. However, the above observation formed the original motivation for the GNO duality conjecture. The normalization constant λ is fixed to be [1]

$$\lambda = |\boldsymbol{\alpha}_{\mathrm{long}}||\boldsymbol{\alpha}_{\mathrm{short}}|. \tag{7}$$

It will be convenient to define

$$\eta = \frac{|\boldsymbol{\alpha}_{\mathrm{long}}|^2}{|\boldsymbol{\alpha}_{\mathrm{short}}|^2}, \tag{8}$$

where

$$\begin{aligned} \eta &= 1 \quad \text{for the simply laced algebras } su(r), so(2r), e_6, e_7, e_8,\\ \eta &= 2 \quad \text{for } so(2r+1), sp(r), f_4,\\ \eta &= 3 \quad \text{for } g_2. \end{aligned} \tag{9}$$

[1] In the following $|\boldsymbol{\alpha}_{\mathrm{long}}|$ and $|\boldsymbol{\alpha}_{\mathrm{short}}|$ are lengths of the long and short roots, respectively, of the Lie algebra g. We will fix the normalization of the roots of the dual algebra g^* by demanding that its long and short roots have the same length as those of g.

In the case of maximal symmetry breaking when $H = \mathrm{U}(1)^r$ dramatic new evidence [5,6,7] for the GNO conjecture has been found, by showing that a set of monopole states exists, within the semi-classical approximation, with the mass spectrum given in (6). The non-trivial observation is that monopoles associated to non-simple roots appear as bound-states at threshold of monopoles associated to simple roots. The situation with non-maximal symmetry breaking is not so clear. The point is that the symmetry between (5) and (6) hides an important difference. Remember that the gauge bosons form representations of K which lead to degeneracies in the mass spectrum (5). Superficially it appears as though these degeneracies are precisely mirrored in the monopole mass formula (6). However, it turns out that monopoles of equal mass are always part of a larger continuous degeneracy of solutions. For instance, whenever two monopoles correspond to different roots $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ which are related by a Weyl group transformation of k, then they are both contained in the same connected manifold of solutions (the moduli space). In fact, even when two monopoles are degenerate but their associated roots are not related by a Weyl group element, then there is a larger moduli space which connects the two solutions. This subtlety occurs in non-simply-laced cases for monopoles associated to short roots [13,14]. This phenomena is referred to as an 'accidental degeneracy' since there is no apparant symmetry which relates the two solutions. Hence the degeneracy of states implied by (6) is a pure illusion and in order to determine the true degeneracy of monopole states one should presumably perform a semi-classical quantization. In [3], some preliminary results indicate that indeed the monopoles carry a degeneracy which is consistent with GNO duality. For the present we shall simply assume that GNO duality is correct and examine the consequences for the spectrum of dyon states.

In the presence of non-zero θ angle, Witten [15] showed that the Noether charge for the electric U(1) transformations generated by the scalar field is

$$N = \frac{Q_e}{e} - \frac{\theta e}{8\pi^2} Q_m, \tag{10}$$

where Q_e and Q_m are the total electric and magnetic charge of the classical field defined as the following surface integrals of the electric and magnetic charges on the sphere at infinity

$$Q_e = \frac{1}{|\boldsymbol{v}|} \int_{S^2_\infty} dS_i \mathrm{Tr}\,(E_i \phi), \qquad Q_m = \frac{1}{|\boldsymbol{v}|} \int_{S^2_\infty} dS_i \mathrm{Tr}\,(B_i \phi). \tag{11}$$

The result (10) was derived in the context of SU(2) gauge theory, but is in fact independent of the gauge group. The GNO quantization condition states that the magnetic charge vector $\boldsymbol{\xi}_m$ has to be in the co-root lattice of g which is spanned by the duals of the simple roots $\boldsymbol{\alpha}_i^*$, so

$$Q_m = \frac{4\pi}{e}\hat{\boldsymbol{v}} \cdot \boldsymbol{\xi}_m, \tag{12}$$

where $\hat{\boldsymbol{v}} = \boldsymbol{v}/|\boldsymbol{v}|$. The electric charge vector $\boldsymbol{\xi}_e$ lies in the weight lattice of the representations under which the fields transform. Since in this case all the fields are in the adjoint representation of g, $\boldsymbol{\xi}_e$ has to lie in the root lattice of g which is spanned by the simple roots $\boldsymbol{\alpha}_i$, hence

$$N = \hat{\boldsymbol{v}} \cdot \boldsymbol{\xi}_e. \tag{13}$$

This result is modified in the presence of matter transforming under different representations of G, as is the case in the finite $N = 2$ Yang-Mills theories coupled to fundamental hypermultiplets. In the case at hand the Witten effect is recovered since, as in the SU(2) case, the monopoles acquire electric charge. The masses of Bogomol'nyi saturated states with a given electric and magnetic charge is then $|\boldsymbol{v}||Q_e + iQ_m|$, as found by Osborn [16]. Consequently, the mass formula for the monopoles is modified to

$$\tilde{M}_{\boldsymbol{\alpha}} = e\,|\tau\,(\boldsymbol{v}\cdot\boldsymbol{\alpha}^*)|\,. \tag{14}$$

The universal mass formula for all Bogomol'nyi saturated states in a theory with gauge group G can now be written as

$$M_G(X,\tilde{\tau}) = \sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\,|\boldsymbol{v}\cdot(\boldsymbol{\xi}_e + \tau\boldsymbol{\xi}_m)|\,, \tag{15}$$

where we have defined for later convenience

$$\tilde{\tau} = \frac{1}{|\boldsymbol{\alpha}_{\mathrm{long}}|^2}\left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\right), \qquad X = \begin{pmatrix} |\boldsymbol{\alpha}_{\mathrm{long}}|^{-1}\boldsymbol{\xi}_e \\ |\boldsymbol{\alpha}_{\mathrm{long}}|\boldsymbol{\xi}_m \end{pmatrix}. \tag{16}$$

The original GNO duality conjecture was made for the case of $\theta = 0$. It states that the monopoles of a theory with gauge group G can be thought of as the gauge bosons of an equivalent formulation of the theory with a gauge group G^* broken to $\mathrm{U}(1)^{r'} \times K^*$ with a dual coupling constant $4\pi/\lambda e$. The conjecture was made on the basis of the mass formulae (5) and (6). For the present purposes, we will assume GNO duality is an exact relation between theories and deduce the larger duality group which ensues when a non-zero vacuum angle is included.

We now consider the θ dependence of the action. The term F^*F can be written as a total derivative, so that $\int \mathrm{Tr}(F^*F)$ is a function of the gauge field on the large sphere at infinity S^3_∞. In fact, it is proportional to the winding number of the gauge field A_μ where A_μ maps S^3_∞ into the lie algebra g of G. It is known that for general gauge group[2]

$$\frac{e^2}{32\pi^2}\int \mathrm{Tr}(F^*F) = \frac{N}{|\boldsymbol{\alpha}_{\mathrm{long}}|^2} \tag{17}$$

where N is the integer winding number of the gauge field A_μ.

The partition function involves a sum over all integers N, implying that the θ periodicity of the partition function is simply

$$\theta \to \theta + 2\pi|\boldsymbol{\alpha}_{\mathrm{long}}|^2. \tag{18}$$

In terms of the complex coupling $\tilde\tau$ we have

$$\tilde\tau \to \tilde\tau + 1. \tag{19}$$

In order to define the extended duality group of the theory we combine the θ-periodicity of the partition function, which we refer to as $\boldsymbol{T}$, with the conjectured GNO duality, which we refer to as $\boldsymbol{S}$. Consider the action of the two generators $\boldsymbol{S}$ and $\boldsymbol{T}$ on these new electric and magnetic charge vectors, and on the complex coupling $\tilde\tau$:

$$\begin{aligned}\boldsymbol{S}: &\qquad \tilde\tau \mapsto -\frac{1}{\eta\tilde\tau}, &\qquad X \mapsto \begin{pmatrix} 0 & 1/\sqrt{\eta} \\ -\sqrt{\eta} & 0\end{pmatrix} X,\\ \boldsymbol{T}: &\qquad \tilde\tau \mapsto \tilde\tau + 1, &\qquad X \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix} X.\end{aligned} \tag{20}$$

Furthermore these transformation have an action on the gauge group of the theory

$$\begin{aligned}\boldsymbol{S}: &\qquad G \mapsto G^*,\\ \boldsymbol{T}: &\qquad G \mapsto G.\end{aligned} \tag{21}$$

It can easily be verified that these transformations are a symmetry of the universal mass formula (15):

$$M_{\boldsymbol{S}G}(\boldsymbol{S}X, \boldsymbol{S}\tilde\tau) = M_G(X,\tilde\tau), \qquad M_{\boldsymbol{T}G}(\boldsymbol{T}X, \boldsymbol{T}\tilde\tau) = M_G(X,\tilde\tau). \tag{22}$$

If G is simply-laced ($\eta = 1$ and $G \simeq G^*$) then $\boldsymbol{S}$ and $\boldsymbol{T}$ generate the modular group $\mathrm{SL}(2,\mathbb{Z})$, and we recover the standard extended duality conjecture. On the contrary, if G

[2] MACK would like to thank J. Labastida for providing a proof of the following result. The proof may also be obtained from the formulae in [17].

is non-simply-laced ($\eta \neq 1$) then $\boldsymbol{S}$ is not a modular transformation and it is helpful to consider some additional generators, which although redundant, help elucidate the action of the duality group. This is done by separating out the group of transformations which relate multiplets transforming under the same gauge group. This subgroup of the full duality group will be generated by $\boldsymbol{T}$, $\boldsymbol{STS}$ and $\boldsymbol{S}^2$, where

$$\boldsymbol{STS}: \qquad G \mapsto G, \qquad \tilde{\tau} \mapsto \frac{\tilde{\tau}}{1-\eta\tilde{\tau}}, \qquad X \mapsto \begin{pmatrix} -1 & 0 \\ -\eta & -1 \end{pmatrix} X. \tag{23}$$

Notice that although the transformation $\boldsymbol{S}^2$ acts trivially on the complex coupling, it reverses the sign of the electric and magnetic charges (it is the CP operator for the theory). $\boldsymbol{T}$, $\boldsymbol{STS}$, and $\boldsymbol{S}^2$ generate a subgroup of the modular group $\mathrm{SL}(2,\mathbb{Z})$ called $\Gamma_0\,(\eta)$.[3]

A general transformation in $\Gamma_0(\eta)$ has the form

$$\tilde{\tau} \mapsto \frac{a\tilde{\tau}+b}{c\tilde{\tau}+d}, \qquad X \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} X, \tag{24}$$

where a, b, c, d are integers such that $ad - bc = 1$ and $c = 0$ modulo η.

Now we identify the extended group of duality transformations. The transformation $\boldsymbol{S}$ generates $\mathbb{Z}_4$, however the $\mathbb{Z}_2$ subgroup generated by $\boldsymbol{S}^2$ is already a subgroup of $\Gamma_0(\eta)$. Hence, the full S-duality group $\mathcal{D}$ of the theory is the $\mathbb{Z}_2$ quotient of a semi-direct product:

$$\mathcal{D} = \left[\Gamma_0(\eta) \rtimes \mathbb{Z}_4\right]/\mathbb{Z}_2, \tag{25}$$

generated by $\boldsymbol{S}$ and $\boldsymbol{T}$. Notice that this is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ when $\eta = 1$. The universal mass formula (15) is now invariant under any transformation in $\boldsymbol{U} \in \mathcal{D}$:

$$M_{\boldsymbol{U}G}(\boldsymbol{U}X, \boldsymbol{U}\tilde{\tau}) = M_G(X, \tilde{\tau}). \tag{26}$$

Given the duality symmetry established above we can determine the spectrum of states. First of all, the spectrum of massive gauge bosons is

$$M_G\left(X_{\boldsymbol{\alpha}}, \tilde{\tau}\right), \qquad \boldsymbol{\alpha} \in \Phi'(g), \tag{27}$$

where

$$X_{\boldsymbol{\alpha}} = \begin{pmatrix} |\boldsymbol{\alpha}_{\mathrm{long}}|^{-1}\boldsymbol{\alpha} \\ 0 \end{pmatrix}. \tag{28}$$

[3] We use the terminology for subgroups of the modular group defined in [18]. For a discussion of the subgroups of $\mathrm{SL}(2,\mathbb{Z})$ in $N = 2$ theory see for example [19]. The appearance of $\Gamma_0(2)$ for the case $\mathrm{Sp(n)} \leftrightarrow \mathrm{SO}(2n+1)$ was first noticed in [5]. For a related discussion of the duality groups of theories with arbitrary gauge groups see [20].

The extended duality states that for each element $\boldsymbol{U} \in \mathcal{D}$ there is a reformulation of the theory with gauge group $\boldsymbol{U}G$ and with coupling constant $\boldsymbol{U}\tilde{\tau}$. The spectrum of the theory must contain the gauge bosons of each of the dual formulations, i.e. the spectrum of the theory must contain states of mass

$$M_{\boldsymbol{U}G}(X_{\tilde{\boldsymbol{\alpha}}}, \boldsymbol{U}\tilde{\tau}), \qquad \tilde{\boldsymbol{\alpha}} \in \Phi'(\boldsymbol{U}g), \tag{29}$$

for each $\boldsymbol{U} \in \mathcal{D}$. By using the symmetry of the mass formula (22) the spectrum is equivalently

$$M_G(\boldsymbol{U}^{-1}X_{\tilde{\boldsymbol{\alpha}}}, \tilde{\tau}), \qquad \tilde{\boldsymbol{\alpha}} \in \Phi'(\boldsymbol{U}g), \tag{30}$$

for each $\boldsymbol{U} \in \mathcal{D}$. To find the spectrum explicitly we note the states can be split into two sets. The first set is generated by $\boldsymbol{U} = \boldsymbol{A}$, where $\boldsymbol{A} \in \Gamma_0(\eta)$, i.e. have charge vectors

$$\boldsymbol{A}^{-1}X_{\boldsymbol{\alpha}} = \begin{pmatrix} p|\boldsymbol{\alpha}_{\rm long}|^{-1}\boldsymbol{\alpha} \\ q|\boldsymbol{\alpha}_{\rm long}|^{-1}\boldsymbol{\alpha} \end{pmatrix}, \qquad \boldsymbol{\alpha} \in \Phi'(g), \tag{31}$$

where q and p are co-prime integers and $q = 0$ modulo η, and hence have masses

$$\sqrt{\frac{4\pi}{{\rm Im}\tau}}\left|\left(p + q|\boldsymbol{\alpha}_{\rm long}|^{-2}\tau\right)\boldsymbol{v}\cdot\boldsymbol{\alpha}\right|, \qquad \boldsymbol{\alpha} \in \Phi'(g). \tag{32}$$

The second set of states is generated by $\boldsymbol{U} = \boldsymbol{S}\boldsymbol{A}$, where $\boldsymbol{A} \in \Gamma_0(\eta)$, i.e. have charge vectors

$$\boldsymbol{A}^{-1}\boldsymbol{S}^{-1}X_{\lambda\boldsymbol{\alpha}^*} = \begin{pmatrix} p|\boldsymbol{\alpha}_{\rm long}|\boldsymbol{\alpha}^* \\ q|\boldsymbol{\alpha}_{\rm long}|\boldsymbol{\alpha}^* \end{pmatrix}, \qquad \boldsymbol{\alpha} \in \Phi'(g), \tag{33}$$

where q and p are co-prime integers and $q \neq 0$ modulo η, and hence have masses

$$\sqrt{\frac{4\pi}{{\rm Im}\tau}}\left|\left(p|\boldsymbol{\alpha}_{\rm long}|^2 + q\tau\right)\boldsymbol{v}\cdot\boldsymbol{\alpha}^*\right|, \qquad \boldsymbol{\alpha} \in \Phi'(g). \tag{34}$$

For example, the spectrum of monopoles (6) is recovered by taking $(p, q) = (0, 1)$ in (34).

So the complete mass spectrum can be described as follows. States are associated to the co-prime pair of integers (p, q) familiar from the SU(2) theory. If $q = 0$ modulo η then the states have masses given by (32) and transform in representations of K, isomorphic to those of the gauge bosons of the G theory. On the contrary if $q \neq 0$ modulo η, then the states have masses given by (34) and transform in representations of K^*, isomorphic to the gauge bosons of the G^* theory. We have illustrated the lattice of states that arise for the case $\eta = 2$ in figure 1, and for $\eta = 3$ in figure 2. At each node (p, q) labelled by G, or G^*, there are a set of states with masses (32), or (34), respectively. Clearly when $\eta = 1$ the group is self-dual and picture is directly analogous to the $SL(2,\mathbb{Z})$ lattice of states in the SU(2) theory. We should emphasize that in the general case, unlike the SU(2) example, the lengths of vectors on the lattices do not encode the masses of the states; rather one must apply the mass formulae (32) and (34).

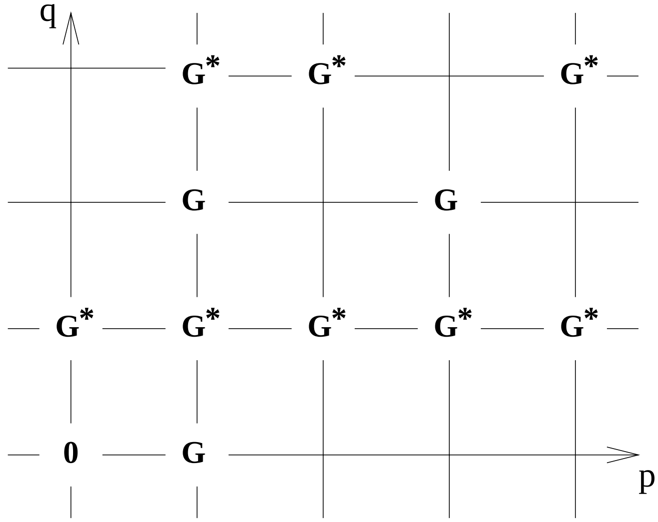

Figure 1. The lattice of states for $\eta = 2$.

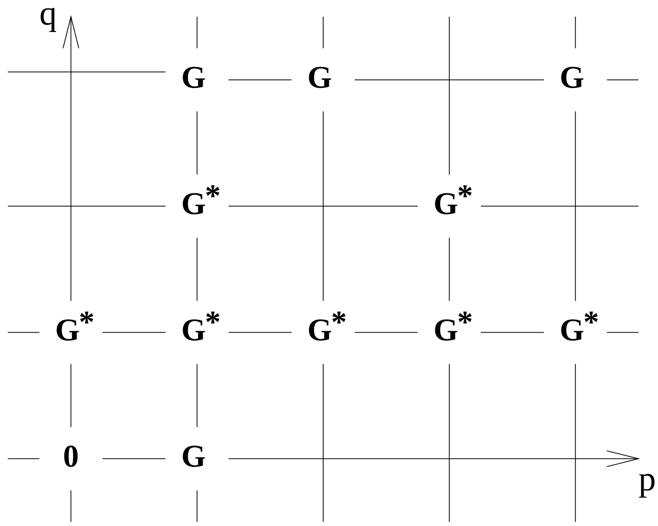

Figure 2. The lattice of states for $\eta = 3$.

We have seen that, starting from GNO duality, the required θ-periodicity of the spectrum dictates a unique S-duality group. Hence the GNO duality conjecture leads directly to a corresponding S-duality conjecture. As in the SU(2) theory, a strong test of this conjecture is the existence at the semi-classical level of the predicted spectrum of dyons. In particular, the conjecture predicts a tower of dyon states with the same set of magnetic charges as the monopoles themselves, by taking $(p,q)=(p,\eta)$ (with p and η co-prime) in (32), for short roots only, giving masses

$$\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\left|\left(p|\boldsymbol{\alpha}_{\mathrm{short}}|^2+\tau\right)\boldsymbol{v}\cdot\boldsymbol{\alpha}^*\right|,\qquad \boldsymbol{\alpha}\in\Phi'_{\mathrm{short}}(g),\qquad (p,\eta)\ \text{co-prime},\tag{35}$$

and $(p,q)=(p,1)$ in (34), for any roots, giving masses

$$\sqrt{\frac{4\pi}{\mathrm{Im}\tau}}\left|\left(p|\boldsymbol{\alpha}_{\mathrm{long}}|^2+\tau\right)\boldsymbol{v}\cdot\boldsymbol{\alpha}^*\right|,\qquad \boldsymbol{\alpha}\in\Phi'(g).\tag{36}$$

The states in (35) and (36) are the analogues of the Julia-Zee dyons in the SU(2) theory. These states should be obtained, in semi-classical limit, by quantizing the U(1) degree-of-freedom associated with electric charge rotations of a BPS monopole [21].

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