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# A MELLIN TRANSFORM TECHNIQUE FOR THE HEAT KERNEL EXPANSION

by

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ABSTRACT

It is shown, for a wide class of operators, that the solution to the associated heat equation may be obtained as a series. This is accomplished using the inverse Mellin transform of the Kernel of the s-th power of the operator, together with the analytic properties of the Kernel in the complex s-plane.


Key-words: Heat Kernel; Asymptotic expansion.

A renewed interest on the study of the solutions of the Heat equation associated to elliptic operators, in the form of an asymptotic expansion, was observed these last years in several situations. Examples of such situations are found in the regularization of operator determinants associated with Grassmann variables in the path integral approach to Quantum Field Theory, in the calculation of non-Abelian anomalies[1,8], or in studies of Field Theories in curved spaces[2]. For instance, in 2-dimensional QCD[1,3] we may write the Generating Functional, after integration over the fermionic fields, as,

$$Z = \int \mathcal{D}A(x)\,\det\not{D}\,\exp\left[-\frac{1}{4}\int d^2x\, G_{\mu\nu a}G^{\mu\nu a}\right] \quad , \tag{1}$$

(apart from gauge fixing terms) where A(x) is the gauge field, $G_{\mu\nu a}$ is the gauge field strength tensor and $D=i(\partial+A(x))$. The determinant of $\not{D}$ appearing in (1) comes from the integration over the fermionic fields; it is a divergent quantity and must be regularized. One of the most popular ways is the proper time regularization method[4]. The regularized determinant is given in terms of the proper-time regularization parameter $\varepsilon$, through the diagonal part of a function[1] $F(\varepsilon,x,y)$ which obeys the "heat equation", associated to the operator $\not{D}^2$ (see eq. 2). The regularized determinant $\det\not{D}(\varepsilon)\,(\varepsilon\to0)$ must be known and then the behaviour of $F(\varepsilon,x,y)$ as $\varepsilon\to0$ must be found.

Beyond the particular example we have just given, we are led in general to the study of the solutions of the "heat equation",

$$\frac{d}{dt}\,F(t,x,y) = H\,F(t,x,y) \quad , \tag{2}$$

where t is a ("time") parameter and x and y are points of a D-dimensional compact manifold without boundary; the operator H acts on the x variable. In the case of $QCD_2$, t is the proper-time regularization parameter $\varepsilon$, and H is the operator $\not{D}^2$. For more generality, H may be taken as an order m pseudo-differeential operator[5]. Particularly important is the asymptotic behaviour of the diagonal part of F(t,x,y) as $t \to 0$. This is usually done by means of the de Witt *anzatz*[2],

$$F(t,x,y) = F_0(t,x,y) \sum_{\ell=0}^{\infty} t^{\ell} a_{\ell}(x,y) \quad ,$$

where $F_0$ is the solution of the "free heat equation".

In this paper we propose a new simple Mellin transform method to obtain an asymptotic expansion to the solution of the Heat equation F(t;x,y), the so-called *Heat Kernel*. This is done using the rigorous results of Seeley[6] on the analytic structure of the Kernel K(s;x,y) associated to the s-th power of the operator H of (eq.2), $H^s$, in the complex s-plane. Our expansion may be seen as an alternative to the de Witt *anzatz* in the case where the residues of the diagonal part of K(s;x,y) can be calculated. We remark that Mellin transform methods for obtaining asymptotic behaviours have been used in other contexts. Indeed, one of us used such methods to demonstrate theorems on the asymptotic behaviour of Feynman amplitudes[9].

To proceed we note that it may be shown that the Green function of $H^s$, Z(s,x,y), is related to the Seeley's Kernel of $H^s$ by Z(s,x,y)= K(-s,x,y), and to the solution of the heat equation (2) by a Mellin transform. So,

$$K(s,x,y) = \frac{1}{\Gamma(-s)} \int_0^{\infty} dt \; t^{-s-1} \, F(t,x,y). \qquad (3)$$

Conversely the inverse Mellin transform gives:

$$F(t,x,y) = \int_{-\infty}^{+\infty} \frac{d\,\mathrm{Im}\,s}{2i\pi}\, t^{s}\, \Gamma(-s) K(s,x,y) \quad , \tag{4}$$

provided K(s,x,y) can be extended to the whole complex s-plane. The s-integration in (4) goes parallel to the imaginary axis and Re(s) must belong to the analiticity domain of K(s,x,y,). One of Seeley's results[6] is that within the approximation made to construct the power operator $H^s$, it has a continuous Kernel for $Re(s)<-D/\underline{m}$. The diagonal elements K(s,x,y) extend to meromorphic functions of s, having as only singularities simple poles located at $s=(j-D)/\underline{m}$, $j=0,1,2,\ldots$; their residues can in principle be calculated[6] from the symbol (generalization of the characteristic polynomial) of $H^s$, using the formula,

$$\mathrm{Res}\, K(s=(j-D)|\bar{m}) = \frac{im}{(2\pi)^{D+1}} \iint_{|\xi|=1,\Gamma} \lambda^{\frac{j-D}{\underline{m}}}\, b_{-\underline{m}-j}(\lambda,\xi)\, d\lambda\, d\xi \,, \tag{4a}$$

where $\Gamma$ is a curve coming from $\infty$, going along the ray of minimal growth to a small circle around the origin, then going back to $\infty$. The quantities $b_{-\underline{m}-j}(\lambda\ \xi)$ are obtained from the coefficients of the symbol. $|\varepsilon|=1$ means that the set of variables $\{\xi\}$ is constrained to be at the surface of the D-dimensional unit sphere. The off-diagonal elements K(s,x,y) $x\neq y$ extend to entire functions of s.

For the diagonal elements, the general analytic structure of the integrand in (4) is displayed in figure 1. The inverse Mellin transform (4) is unambigously defined if we take the integration along a line $C_0$ in the "initial" analyticity domain of K(s,x,y), $Re(s)<-D/\bar{m}$. Then we may obtain an expansion in t by displacing the integration contour to the right, picking up successively the contributions from the poles. We note that double poles may exist in the integrand of (4) located at the real positive values of s. The double poles, if they are present, may be treated writing $\Gamma(-s)K(s,x,y)\approx\phi(s)/(s-\ell)^2$, for $s\approx\ell$, and performing an integration by parts. Then if the Kernel K has a good behaviour at infinite s, together with the vanishing[6] of the residues of K at positive integer s we are left with the remaining contributions from the residues at the poles. The diagonal elements of the solution of the "heat equation" (2) are thus expressible as the following series:

$$F(t,x,x) = -\sum_{\ell=0}^{\infty} t^{\ell} \left.\frac{d\phi}{ds}\right|_{s=\ell} - \sum_{j} t^{\frac{j-D}{\bar{m}}} \Gamma\left(\frac{D-j}{\bar{m}}\right) R_j(x) \quad , \qquad (5)$$

where the sum over j=0,1,2,... excludes the terms such that $(j-D)/\bar{m}= 0,1,2,\ldots$, and $R_j$ is the residue of K(s,x,x) at $s=(j-D)/\bar{m}$.

This is the result we would like to present here. In the following we illustrate our method with two simple examples:

## THE LAPLACIAN IN A RIEMANNIAN METRIC

We consider the operator $H=-\nabla^2+P$, where $\nabla^2$ is the Laplacian in a Riemannian manifold, and P the projection on the constants. In this case, K(s;x,x) has poles at the values $s=j-\frac{D}{2}$, j=0,1,2,... and if the dimension D is even, these poles are in finite number[6], located at $s=j-\frac{D}{2}$, $j=0,1,\ldots,\frac{D}{2}-1$. The residue at $s=-\frac{P}{2}$ can be calculated in geodesic coordinates using formula (4a), and we can obtain the leading term to the expansion (5) in differential form(*)

(*) We remember the definition of Seeley's Kernel, as such that if H acts on a manifold M, then $H^s f(x)= \int_M dy\, K(s,x,y)\, f(y)$.

$$t^{-\frac{D}{2}} \Gamma(\tfrac{D}{2}) \frac{(2\pi)^{-D}}{2} |S^{D-1}| dv ,$$

where $|S^{D-1}|$ is the surface area of the unit sphere in $\mathbb{R}^D$, and dv the volume element in the manifold.

The Euclidean Laplacian

Of course we do not know the exact Kernel in the general case, but we indeed know it in at least a particular one, and it may be instructive to see what happens in this case. Let us take H to be the Euclidean Laplacian operator, $H=\partial^2$, in D dimensions. The Green function of $H^k$, for real integer positive k is given in ref. 7. Starting from this we perform the extension from positive integers k to complex s-values, obtaining the exact Kernel of $H^s$ as a meromorphic function of s, for any dimensionality D,

$$K_L(s,x,y) = (-1)^{-s} \frac{e^{i\pi\frac{D}{2}} \Gamma(\frac{D}{2}+s)(P+i0)^{-\frac{D}{2}-s}}{4^{-s}\Gamma(-s)\pi^{\frac{D}{2}}}, \mathrm{Re}(s) < \frac{D}{2} , \qquad (6)$$

where P is the quadratic form $-\sum_{i=1}^{D}(x_i-y_i)^2 \equiv -(x-y)^2$. We note that the original restriction[7] $(-D/2) < k < 0$ for integer k implies after performing the analytic continuation, the "initial" domain of analyticity $(-D/2) < \mathrm{Re}(s) < 0$ for K(s,x,y). In this domain the inverse Mellin transform (6) is unambigously defined. Then starting from an integral along a line $\mathbb{C}_0$ parallel to the imaginary axis in the region $(-D/2) < \mathrm{Re}(s) < 0$ (figure 2) we sum up the contributions from the poles, obtaining the result,

$$F(x,y,t) = (-1)^D (4\pi t)^{\frac{D}{2}} \exp[-(x-y)^2/4t] \qquad (7)$$

which is the well known solution to the "free heat equation".

In the case of the Laplacian L, the analytic structure in s of the exact Kernel $K_L(s,x,y)$ of $L^s$ multiplied by $\Gamma(-s)$ is shown in figure 2. It is rather different from that corresponding to the Kernel of the power $H^s$ of *any* pseudo-differential operator H, as obtained from ref. 6 (figure 1). This is simply due to the fact that $\mathbb{R}^D$ does not belong to the class of spaces treated by Seeley in his work.

## CONCLUSION

The expansion (5) for the Heat Kernel, which is the result we would like to present here, is rather different from the de Witt's *anzatz* currently used. It could give new results when applied to more realistic examples. However, it is not our purpose in this short note to make physical applications. These will be the subject of a forthcoming paper. Our work must be understood as a new method to obtain an asymptotic expansion to the Heat Kernel, which could lead to new results in physical situations.

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FIGURE CAPTIONS

Fig. 1 - Poles of $\Gamma(-s)K(s,x,x)$ for a general pseudo-differential operator H; K is the approximate Kernel of $H^s$ as given by ref. 5.

Fig. 2 - Poles of $\Gamma(-s)K(s,x,y)$ in the case where H is the Laplacian. $K(s,x,y)$ is the exact Kernel of $H^s$.

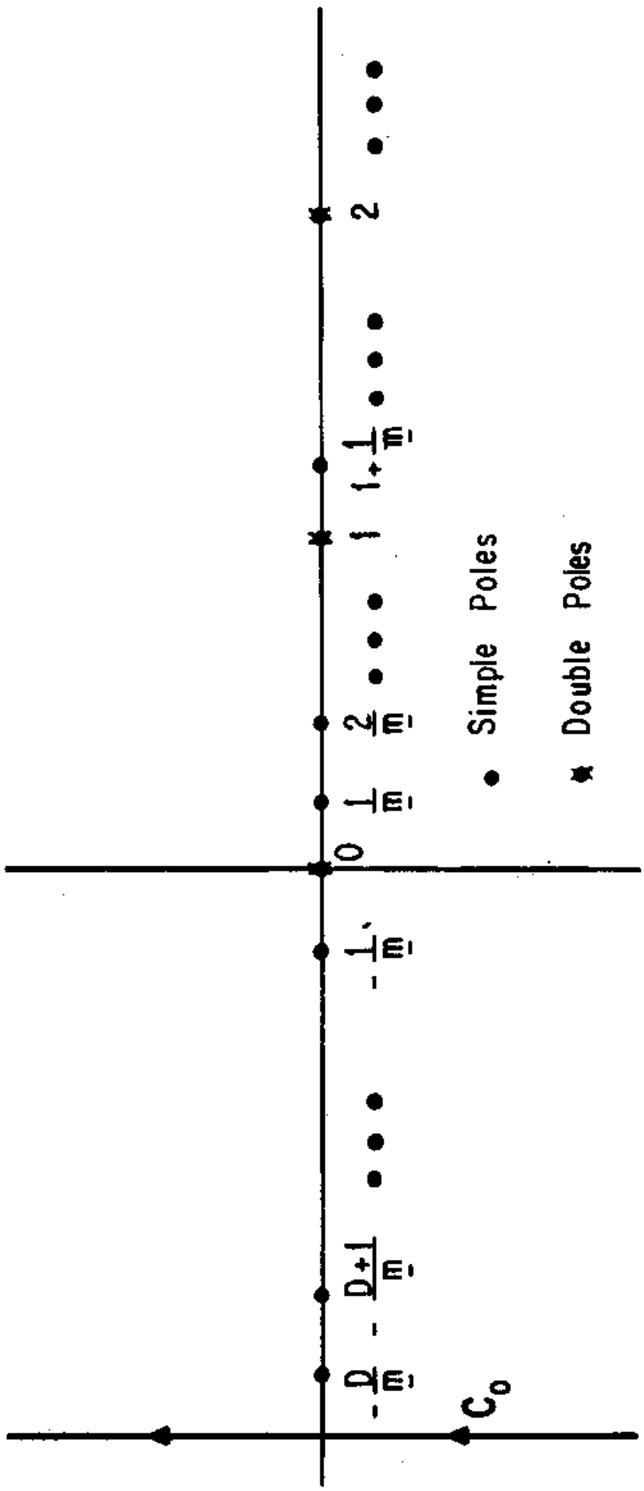
$-\frac{D}{m}$
$-\frac{D+1}{m}$
$-\frac{1}{m}$
0
$\frac{1}{m}$
$\frac{2}{m}$
1
$1+\frac{1}{m}$
2
$C_0$
Simple Poles
Double Poles

FIG.1

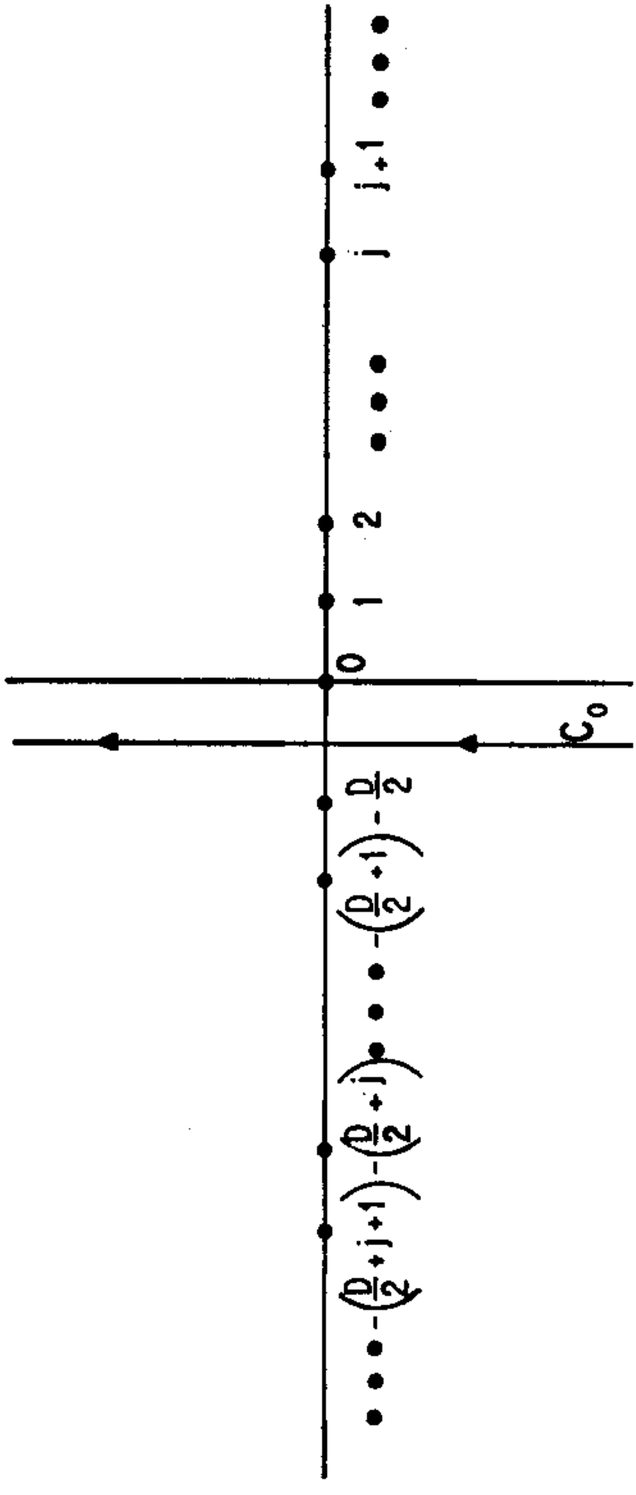
0
1
2
j
j+1
$C_0$
$-\frac{D}{2}$
$-\left(\frac{D}{2}+1\right)$
$-\left(\frac{D}{2}+j\right)$
$-\left(\frac{D}{2}+j+1\right)$

FIG. 2

REFERENCES

[1]A.M. Polyakov, P.B. Wilyman; Phys. Lett. 131B, 121 (1983).
O. Alvarez; Nucl. Phys. B238, 61 (1984).
R.E. Gamboa Saravi, F.A. Schaposnik, J.E. Solomin; Phys. Rev. D30, 1353 (1984).
L.C.L. Botelho, M.A. do Rego Monteiro, Phys. Rev. D30, 2242 (1984).
A.P. Balachandran, G. Marmo, Y.P. Nair, C.G. Traherm; Phys. Rev. D25 (10) 2713 (1982).

[2]See B. de Witt; "The Dynamical Theory of Groups and Fields", Gordon and Breach, New York (1965). N.D. Birrel and P.C.W. Daries; "Quantum Fields in Curved Space"; Cambridge University Press (1982). G. Cognola, S. Zerbini; "Some Physical Applications of the Heat Kernel Expansion"; Preprint Dipartimento di Fisica - Università di Trento - Ist. Naz. di Fisica Nucleare - Sezione di Padova (Italia) 1987.

[3]R.E. Gamboa Saravi, F.A. Schaposnik, J.E. Solomin; Nucl. Phys. B185, 239 (1981).

[4]S.W. Hawking; Comm. Math. Phys. 55, 133 (1977).
K.D. Rothe and B. Schroer, in: Field Theoretical Methods in Particle Physics, edited by Werner Rühl, NATO Advanced Study Institutes Series B, vol. 55 (Plenum, New York, 1980), p. 249.

[5]L. Hörmander; Comm. Pure Appl. Math. 18m 501 (1965).
J.J. Kohn, L. Miremberg; Comm. Pure Appl. Math. 18, 269 (1965).

[6]R.T. Seeley; Amer. Math. Soc. Proc. Symp. Pure Math., 10, 288 (1967).

[7]I.M. Gellfand, G.E. Chirlov; *Les Distributions*, vol. I, Dunod, Paris, 1962, pp. 273-277.

[8]K. Fujikawa; Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D21, 2848 (1980).

[9]M. Bergère, C. de Calan, A.P.C. Malbouisson; Comm. Math. Phys. 62, 137 (1978).
C. de Calan, A.P.C. Malbouisson; Ann. Inst. Henri Poincaré 32, 91 (1980).