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Conformal symmetry, anomaly and effective action for metric-scalar gravity with torsion

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ABSTRACT

We consider some aspects of conformal symmetry in a metric-scalar-torsion system. It is shown that, for some special choice of the action, torsion acts as a compensating field and the full theory is conformally equivalent to General Relativity on classical level. Due to the introduction of torsion, this equivalence can be provided for the positively-defined gravitational and scalar actions. One-loop divergences arising from the scalar loop are calculated and both the consequent anomaly and the anomaly-induced effective action are derived.

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1 Introduction

The studies in the framework of gravity with torsion have a long history and many interesting achievements. In particular, the relation between torsion and local conformal transformation (see, for example, [1, 2] for a recent reviews with broad lists of references) is a topic of special relevance. Torsion appears naturally in the effective action of string and, below the Planck scale, should be treated, along with the metric, as part of the gravitational background for the quantized matter fields. In this paper, we are going to investigate some details of conformal symmetry that come up in some metric-scalar-torsion systems. First, we briefly consider, following earlier papers [3, 4, 5], a similar torsionless system and then investigate the theory with torsion; proceeding further, we derive the one-loop divergences, conformal anomaly and the anomaly-induced effective action.

Since we are going to consider the issues related to conformal transformation and conformal symmetry in gravity, it is worthy to make some general comments. The conformal transformations in gravity may be used for two different purposes. First one can, by means of the conformal transformation, change the so-called conformal frame in the scalar-metric theory. Indeed, this choice is not arbitrary since the masses of the fields must be introduced. There may be various reasons to choose one or another frame as physical; one may find a detailed discussion on this issue in [1]. All theories for which this consideration usually applies do not have conformal symmetry, even before the masses are introduced. Second, one can consider the theory with conformal symmetry.

In the present paper, we are going to discuss only the second case, that is, massless theories with unbroken conformal symmetry. Then, the change of the conformal frame is nothing but the invariant (at least, at the classical level) choice of the dynamical variables. It is worth mentioning that the theories with conformal symmetry have prominent significance in many areas of modern theoretical physics. The most important, at least for the sake of applications, is that at the quantum level this symmetry is violated by anomaly. As examples of the application of the trace anomaly, one may recall the derivation of the effective equations for strings in background fields (which are nothing but conditions for the anomaly cancellation) [6], the study of the renormalization group flows (see, for example, [7] for the recent developments and list of references), the derivation of the black hole evaporation in the semiclassical approach [8], first inflationary model [9] and some others (we recommend [11] for a general review on conformal anomaly). Recently, the application of anomalies for $n = 4$ has been improved by the use of the anomaly-induced effective action obtained long ago [12, 13]. In particular, this led to a better understanding of the anomaly-generated inflation [14] and allowed to perform a systematic classification of the vacuum states in the semiclassical approach to the black hole evaporation [15]. Anomaly-induced effective action has been used to develop the quantum theory for the conformal factor [16] and for the consequent study of the back reaction of gravity to the matter fields [17].

The generalization of the anomaly and the anomaly-induced action for the case of completely antisymmetric torsion has been given in [18]. Here we investigate a non-trivial conformal properties of the metric-scalar-torsion system, which modify the Noether identity corresponding to the conformal symmetry. This modification is due to the fact that, in the model under discussion, torsion *does transform*, while its antisymmetric part considered before [18] does not. As a result of this new feature, the conformal anomaly is not the anomaly of the Energy-Momentum Tensor trace, but rather the trace of some modified quantity. However, using a suitable parametrization for the fields one can, as it will be shown below, reduce the calculation of this new anomaly, and obtain the expressions for both the anomaly and the anomaly-induced effective action using corresponding results from [12, 13, 18].

For the sake of generality, we present all classical formulae in an n-dimensional space-time, for $n \neq 2$ (see the Appendix of [5] for the discussion of the special $n = 2$-case in the torsion-free theory). The divergences and anomaly are all evaluated around $n = 4$.

2 Brief review of the torsionless theory.

Our purpose here is to show that conformally invariant second-derivative n-dimensional metric-dilaton model is conformally equivalent to General Relativity. This equivalence has been originally demonstrated and discussed in [3] for the four-dimensional space and free scalar field, and later generalized in [4, 5] for the interacting scalar field in an arbitrary $n \neq 2$ dimensions. In this section, we review the result of [3, 4] presenting calculations in a slightly different manner. Our starting point will be the Einstein-Hilbert action with cosmological constant. For further convenience we take this action with negative sign.

$$S_{EH}[g_{\mu\nu}] = \int d^n x \sqrt{-\hat{g}}\, \left\{ \frac{1}{G}\, \hat{R} + \Lambda \right\} \cdot \tag{1}$$

The above action depends on the metric $\hat{g}_{\mu\nu}$ and we now set $\hat{g}_{\mu\nu} = g_{\mu\nu} \cdot e^{2\sigma(x)}$. In order to describe the local conformal transformation in the theory, one needs some relations between geometric quantities of the original and transformed metrics:

$$\sqrt{-\hat{g}} = \sqrt{-g}\; e^{n\sigma}\,, \qquad \hat{R} = e^{-2\sigma} \left[R - 2(n-1)(\Box\sigma) - (n-2)(n-1)(\nabla\sigma)^2 \right] \cdot \tag{2}$$

Substituting (2) into (1), after integration by parts, we arrive at:

$$S_{EH}[g_{\mu\nu}] = \int d^n x \sqrt{-g}\, \left\{ \frac{(n-1)\,(n-2)}{G}\, e^{(n-2)\sigma}\, (\nabla\sigma)^2 + \frac{e^{(n-2)\sigma}}{G}\, R + \Lambda e^{n\sigma} \right\}, \tag{3}$$

where $(\nabla\sigma)^2 = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma$. If one denotes

$$\varphi = e^{\frac{n-2}{2}\sigma} \;\cdot\; \sqrt{\frac{8}{G} \cdot \frac{n-1}{n-2}}\,, \tag{4}$$

the action (1) becomes

$$S = \int d^n x \sqrt{-g} \left\{ \frac{1}{2}\, (\nabla\varphi)^2 + \frac{n-2}{8(n-1)} R\varphi^2 + \Lambda \left(\frac{G}{8} \cdot \frac{n-2}{n-1} \right)^{\frac{n}{n-2}} \cdot \varphi^{\frac{2n}{n-2}} \right\} \cdot \tag{5}$$

And so, the General Relativity with cosmological constant is equivalent to the metric-dilaton theory described by the action of eq. (5). One has to notice that the latter exhibits an extra local conformal symmetry, which compensates an extra (with respect to (1)) scalar degree of freedom. Moreover, (5) is a particular case of a family of similar actions, linked to each other by the reparametrization of the scalar or (and) the conformal transformation of the metric [5]. The symmetry transformation which leaves the action (5) stable,

$$g'_{\mu\nu} = g_{\mu\nu} \cdot e^{2\rho(x)}\,, \qquad \varphi' = \varphi \cdot e^{(1-\frac{n}{2})\rho(x)}, \tag{6}$$

degenerates at $n \to 2$ and that is why this limit cannot be trivially achieved [5]. If we start from the positively defined gravitational action (1), the sign of the scalar action (5) should be negative, indicating to the well known instability of the conformal mode of General Relativity (see [19, 20] for the recent account of this problem and further references).

3 Conformal invariance in metric-scalar-torsion theory

In this section, we are going to build up the conformally symmetric action with additional torsion field. Our notations are similar to the ones accepted in [21] for the $n = 4$ case. Torsion is defined as

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} - \tilde{\Gamma}^{\alpha}_{\gamma\beta} = T^{\alpha}_{\beta\gamma}\,,$$

where $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ is the non-symmetric affine connection in the space provided by independent metric and torsion fields. The covariant derivative, $\widetilde{\nabla}_{\alpha}$, satisfies metricity condition, $\widetilde{\nabla}_{\alpha} g_{\mu\nu} = 0$. Torsion tensor may be decomposed into three irreducible parts:

$$\tilde{\Gamma}_{\alpha,\beta\gamma} - \tilde{\Gamma}_{\alpha,\gamma\beta} = T_{\alpha,\beta\gamma} = \frac{1}{n-1}\left(T_{\beta} g_{\alpha\gamma} - T_{\gamma} g_{\alpha\beta}\right) - \frac{1}{3!(n-3)!}\varepsilon_{\alpha\beta\gamma\mu_1\dots\mu_{n-3}} S^{\mu_1\dots\mu_{n-3}} + q_{\alpha\beta\lambda}\,. \tag{7}$$

Here $T_{\alpha} = T^{\beta}{}_{\alpha\beta}$ is the trace of the torsion tensor $T^{\beta}{}_{\alpha\gamma}$. The tensor $S^{\mu_1\dots\mu_{n-3}}$ is completely antisymmetric and, in the case of a purely antisymmetric torsion tensor, $T_{\alpha\beta\gamma}$ is its dual. $\varepsilon_{\alpha\beta\gamma\mu_1\dots\mu_{n-3}}$ is the maximal antisymmetric tensor density in the n-dimensional space-time. The sign of the S-dependent term corresponds to an even n. We notice that, in four dimensions, the axial tensor $S_{\mu_1\dots\mu_{n-3}}$ reduces to the usual S_{μ} – axial vector [21]. In n dimensions, the number of distinct components of the $S_{\mu_1\dots\mu_{n-3}}$ tensor is $\frac{n^3-3n^2+2n}{6}$. The tensor $q^{\beta}{}_{\alpha\gamma}$ satisfies, as in the $n = 4$ case, the two constraints:

$$q^{\beta}{}_{\alpha\beta} = 0 \qquad \text{and} \qquad q_{\alpha\beta\gamma} \cdot \varepsilon^{\alpha\beta\gamma\mu_1\dots\mu_{n-3}} = 0$$

and has $\frac{n^3-4n}{3}$ distinct components. We shall denote, as previously, Riemannian covariant derivative and scalar curvature by ∇_α and R respectively, and keep the notation with tilde for the geometric quantities with torsion.

The purpose of the present work is to describe and discuss conformal symmetry in the metric-scalar-torsion theory. It is well-known that torsion does not interact minimally with scalar fields, but one can formulate such an interaction in a non-minimal way (see [21] for the introduction). Moreover, this interaction between scalar field and torsion is necessary element of the renormalizable quantum field theory in curved space-time with torsion [22]. One may construct a general non-minimal action for the scalar field coupled to metric and torsion as below:

$$S = \int d^n x \sqrt{-g}\, \left\{ \frac{1}{2}\, g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi + \frac{1}{2}\, \xi_i P_i \varphi^2 - \frac{\lambda}{4!}\, \varphi^{\frac{2n}{n-2}} \right\} . \tag{8}$$

Here, the non-minimal sector is described by five structures:

$$P_1 = R\,, \quad P_2 = \nabla_\alpha T^\alpha\,, \quad P_3 = T_\alpha\, T^\alpha\,, \quad P_4 = S_{\mu_1 \dots \mu_{n-3}}\, S^{\mu_1 \dots \mu_{n-3}}\,, \quad P_5 = q_{\alpha\beta\gamma}\, q^{\alpha\beta\gamma}\,; \tag{9}$$

in the torsionless case the only $\xi_1 R$ term is present. ξ_i are the non-minimal parameters which are typical for the theory in external field. Renormalization of these parameters is necessary to remove corresponding divergences which really take place in the interacting theory [22]. Furthermore, the renormalizable theory always includes some vacuum action. This action must incorporate all structures that may show up in the counterterms. The general expression for the vacuum action in the case of a metric-torsion background has been obtained in [23]; it contains 168 terms. In fact, one can always reduce this number, because not all the terms with the allowed dimension really appear as counterterms. The discussion of the vacuum renormalization for the external gravitational field with torsion has been previously given in [18] for a purely antisymmetric torsion. In the present article, we will be interested in the special case of the (8) action, which possesses an interesting conformal symmetry. The appropriate expression for the corresponding vacuum action will be given in Section 4, after we derive vacuum counterterms.

Using the decomposition (7) given above, the equations of motion for the torsion tensor can be split into three independent equations written for the components T_α, S_α, $q_{\alpha\beta\gamma}$;

they yield:

$$T_\alpha = \frac{\xi_2}{\xi_3} \cdot \frac{\nabla_\alpha \varphi}{\varphi}, \qquad S_{\mu_1 \dots \mu_{n-3}} = q_{\alpha\beta\gamma} = 0\,. \tag{10}$$

Replacing these expressions back into the action (8), we obtain the on-shell action

$$S = \int d^n x \sqrt{-g}\, \left\{ \frac{1}{2}(1 - \frac{\xi_2^2}{\xi_3})\, g^{\mu\nu} \partial_\mu \varphi\, \partial_\nu \varphi + \frac{1}{2}\, \xi_1 \varphi^2 R - \frac{\lambda}{4!} \varphi^{\frac{2n}{n-2}} \right\}, \tag{11}$$

that can be immediately reduced to (5), by an obvious change of variables, whenever

$$\xi_1 = \frac{1}{4}\left(1 - \frac{\xi_2^2}{\xi_3}\right) \frac{n-2}{n-1}\,. \tag{12}$$

Therefore, we notice that the version of the Brans-Dicke theory with torsion (8) is conformally equivalent to General Relativity (1) provided that the new condition (12) is satisfied and no sources for the components T_α, $S_{\mu_1 \dots \mu_{n-3}}$ and $q_{\alpha\beta\gamma}$ of the torsion tensor are included. In fact, the introduction of external conformally covariant sources for $S_{\mu_1 \dots \mu_{n-3}}$, $q^\alpha{}_{\beta\gamma}$ or to the transverse component of T_α does not spoil the conformal symmetry.

One has to remark, that the theory with torsion provides, for $\frac{\xi_2^2}{\xi_3} - 1 > 0$, the equivalence of the positively defined scalar action (8) to the action (1) with the negative sign. The negative sign in (1) signifies, in turn, the positively defined gravitational action. Without torsion one can achieve positivity in the gravitational action only by the expense of the negative kinetic energy for the scalar action in (5). Thus, the introduction of torsion may lead to some theoretical advantage.

The equation of motion for T_α may be regarded as a constraint that fixes the conformal transformation for this vector to be consistent with the one for the metric and scalar. Then, instead of (6), one has

$$g'_{\mu\nu} = g_{\mu\nu} \cdot e^{2\rho(x)}\,, \qquad \varphi' = \varphi \cdot e^{(1-\frac{n}{2})\rho(x)}\,, \qquad T'_\alpha = T_\alpha + (1 - \frac{n}{2}) \cdot \frac{\xi_2}{\xi_3} \cdot \partial_\alpha \rho(x) \tag{13}$$

Now, in order to be sure about the number of degrees of freedom in this theory, let us compute the remaining field equations for the whole set of fields. From this instant we consider free theory and put the coupling constant $\lambda = 0$, because scalar self-interaction does not lead to the change of the qualitative results.

The dynamical equations for the theory encompass eqs. (10) along with

$$\begin{aligned} &\frac{1}{2}\xi_1 \varphi^2 \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) - \frac{1}{4} g_{\alpha\beta} (\nabla\varphi)^2 + \frac{1}{2} \nabla_\alpha \varphi \nabla_\beta \varphi \\ + \;& \frac{1}{2}\xi_2 \varphi^2 \left(T_\alpha T_\beta - \frac{1}{2} g_{\alpha\beta} T_\rho T^\rho \right) + \frac{1}{2}\xi_3 \varphi^2 \left(\nabla_\alpha T_\beta - \frac{1}{2} g_{\alpha\beta} \nabla_\lambda T^\lambda \right) = 0 \end{aligned} \tag{14}$$

and

$$\left(\Box - \xi_1 R - \xi_2 \nabla_\lambda T^\lambda - \xi_3 T_\lambda T^\lambda\right) \varphi = 0\,. \tag{15}$$

In case of the on-shell torsion (10), equations (14) reduce to:

$$\frac{1}{8}\frac{n-2}{n-1}\phi^2 \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{4}g_{\mu\nu}(\nabla\phi)^2 + \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi = 0\,, \tag{16}$$

$$\Box\phi - \frac{1}{4}\left(\frac{n-2}{n-1}\right) R\,\phi = 0\,. \tag{17}$$

Taking the trace of the first equation, it can be readily noticed that this equation is exactly the same as the one for scalar field. This indeed justifies our procedure for replacing (10) into the action. It is easy to check, by direct inspection, that even off-shell, the theory with torsion (8), satisfying the relation (12), may be conformally invariant whenever we define the transformation law for the torsion trace according to (13): and also postulate that the other pieces of the torsion, S_μ and $q^\alpha{}_{\beta\gamma}$, do not transform. The quantities $\sqrt{-g}$ and R transform as in (2). One may introduce into the action other conformal invariant terms depending on the torsion. For instance:

$$S = -\frac{1}{4}\int d^n x\,\sqrt{-g}\;\varphi^{\frac{2\cdot(n-4)}{n-2}}\,T_{\alpha\beta}\,T^{\alpha\beta}, \tag{18}$$

where $T_{\alpha\beta} = \partial_\alpha T_\beta - \partial_\beta T_\alpha$. This term reduces to the usual vector action when $n \to 4$. It is not difficult to propose other conformally invariant terms containing other components of the torsion tensor in (7).

One can better understand the equivalence between GR and conformal metric-scalar-torsion theory (8), (12) after presenting an alternative form for the symmetric action. It proves a useful way to divide the torsion trace T_μ into longitudinal and transverse parts:

$$T_\mu = T_\mu^\perp + \frac{\xi_2}{\xi_3}\,\partial_\mu T\,, \tag{19}$$

where $\nabla^\mu T_\mu^\perp = 0$ and T is the scalar component of the torsion trace, T_μ. The coefficient $\frac{\xi_2}{\xi_3}$ has been introduced for the sake of convenience. Under the conformal transformation (13), the transverse part is inert and the scalar component transforms as $T' = T - \sigma$. Now, we can see that the conformal invariant components, $S, T^\perp$ and q, appear in the action (8) in the combination

$$\mathcal{M}^2 = \xi_3\,(T_\mu^\perp)^2 + \xi_4\,S_{\mu_1\mu_2\ldots\mu_{n-3}}S^{\mu_1\mu_2\ldots\mu_{n-3}} + \xi_5\,q^2_{\mu\nu\lambda}\,, \tag{20}$$

which has the conformal transformation, $\mathcal{M}^{2'} \to e^{(2-n)\rho(x)} \cdot \mathcal{M}^2$ similar to that of the square of the scalar field. In fact, the active compensating role of the torsion trace, T_μ, is accommodated in the scalar mode T. Other components of the torsion tensor, including $T_\mu^\perp$, appear only in the form (20) and allow to create some kind of *conformally covariant* "mass". On the mass-shell, this "mass" disappears because all its constituents, $T_\mu^\perp, S_{\mu_1 \dots \mu_{n-3}}$ and $q_{\mu\nu\lambda}$, vanish.

The next observation is that all torsion-dependent terms may be unified in the expression

$$P = -\frac{n-2}{4(n-1)}\,\frac{\xi_2^2}{\xi_3}\,R + \xi_2\,(\nabla_\mu T^\mu) + \xi_3\,T_\mu T^\mu + \xi_4\,S^2_{\mu_1\mu_2\dots\mu_{n-3}} + \xi_5\,q^2_{\mu\nu\lambda}$$
$$= -\frac{n-2}{4(n-1)}\,\frac{\xi_2^2}{\xi_3}\left[R - \frac{4(n-1)}{n-2}\,(\nabla T)^2 - \frac{4(n-1)}{n-2}\,\Box T\right] + \mathcal{M}^2\,. \qquad (21)$$

It is easy to check that the transformation law for this P is the same as the one for $\mathcal{M}^2$. Using new definitions, the invariant action becomes

$$S_{inv} = \int d^n x \sqrt{-g}\,\left\{\frac{1}{2}\,g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \frac{n-2}{8(n-1)}\,R\varphi^2 + \frac{1}{2}\,P\,\varphi^2\right\}\,. \qquad (22)$$

Let us make a change of variables in the last action:

$$T = \ln\psi\,,$$

with obvious transformation law, $\psi' = \psi \cdot e^{-\sigma}$, for the new scalar ψ. After some small algebra, one can cast the action (22) in the form of a 2-component conformally invariant sigma-model:

$$S_{inv} = \frac{1}{2}\int d^n x \sqrt{-g}\,\left\{-\,\varphi\Delta_2\varphi\,+\,\mathcal{M}^2\varphi^2\,+\,\frac{\xi_2^2}{\xi_3}\,(\varphi^2\psi^{-1})\Delta_2\psi\right\}\,, \qquad (23)$$

where

$$\Delta_2 = \Box - \frac{n-2}{4(n-1)}\,R$$

is a second-derivative conformally covariant operator acting on scalars. It is easy to check that if one takes, for example, the conformally flat metric, $g_{\mu\nu} = \eta_{\mu\nu} \cdot \chi^2(x)$, one arrives at three-scalar non-linear sigma-model, but two of these three scalars are dependent. The last expression for the action (23) shows explicitly the conformal invariance of the action and also confirms the conformal covariance of the quantity P.

One may construct a trivial 2-scalar sigma-model conformally equivalent to GR, by making a substitution $\sigma \to \sigma_1 + \sigma_2$ in the action (3), and after regarding $\sigma_1(x)$ and $\sigma_2(x)$ as distinct fields. A detailed analysis shows that the theory (23) can not be reduced to such a 2-scalar sigma-model by a change of variables. In order to understand why this is so, we need one more representation for the metric-scalar-torsion action with local conformal symmetry.

Let us start, once again, from the action (8), (12) and perform only part of the transformations (13):

$$\varphi \to \varphi' = \varphi \cdot e^{(1-\frac{n}{2})\rho(x)}\,, \qquad T_\alpha \to T'_\alpha = T_\alpha + (1-\frac{n}{2}) \cdot \frac{\xi_2}{\xi_3} \cdot \partial_\alpha \rho(x)\,. \tag{24}$$

Of course, if we supplement (24) by the transformation of the metric, we arrive at (13) and the action does not change. On the other hand, the results of Section 2 suggest that (24) may lead to an alternative conformally equivalent description of the theory. Taking $\varphi \cdot e^{(1-\frac{n}{2})\rho(x)} = \frac{8(n-1)}{G\,(n-2)} \left(1 - \frac{\xi_2^2}{\xi_3}\right) = const$, we obtain, after some algebra, the following action:

$$S = \frac{1}{G} \int d^n x \sqrt{-g}\; R +$$

$$+ \frac{4(n-1)}{G(n-2)(1-\xi_2^2/\xi_3)} \int d^n x \sqrt{-g}\; \left\{ \xi_4 S^2_{\mu_1 \dots \mu_{n-3}} + \xi_5 q^2_{\mu\nu\lambda} + \xi_3 \left(T_\alpha - \frac{\xi_2}{\xi_3} \nabla_\alpha \ln \varphi \right)^2 \right\}\,. \tag{25}$$

This form of the action does not contain interaction between the curvature and the scalar field. At the same time, the latter field is present until we use the equations of motion (10) for torsion. Torsion trace looks here like a Lagrange multiplier, and only using its corresponding equation of motion (and also for other components of torsion), we can obtain the action of GR. It is clear that one can arrive at the same action (25) making the transformation of the metric as in (13) instead of (24).

To complete this part of our consideration, we mention that the direct generalization of the Einstein-Cartan theory including an extra scalar may be conformally equivalent to General Relativity, provided that the non-minimal parameter takes an appropriate value. To see this, one uses the relation

$$\tilde{R} = R - 2\nabla_\alpha T^\alpha - \frac{n}{n-1} T_\alpha T^\alpha + \frac{1}{2}\, q_{\alpha\beta\gamma} q^{\alpha\beta\gamma} + \frac{1}{4} \cdot \frac{1}{3!(n-3)!}\, S^{\alpha_1 \dots \alpha_{n-3}}\, S_{\alpha_1 \dots \alpha_{n-3}} \tag{26}$$

and replace it into the "minimal" action

$$S_{ECBD} = \int d^n x \sqrt{-g}\, \left\{ \frac{1}{2}\, g^{\mu\nu} \partial_\mu \varphi\, \partial_\nu \varphi + \frac{1}{2}\, \xi\, \tilde{R} \varphi^2 \right\} . \tag{27}$$

It is easy to see that the condition (12) is satisfied for the special value

$$\xi = \frac{n(n-2)}{8(n-1)} . \tag{28}$$

In particular, in the four-dimensional case, the symmetric version of the theory corresponds to $\xi = \frac{1}{3}$, contrary to the famous $\xi = \frac{1}{6}$ in the torsionless case. The effect of changing conformal value of ξ due to the non-trivial transformation of torsion has been discussed earlier in [24] (see also further references there).

In the next section we shall see how the non-trivial conformal transformation for torsion changes the Noether identity and the quantum conformal anomaly.

4 Divergences, anomaly and induced effective action

In four dimensions, by integrating over the free scalar field (even without self-interaction), one meets vacuum divergences and the resulting trace anomaly breaks conformal invariance. As it was already mentioned in the Introduction, the anomaly and its application is one of the most important aspects of the conformal theories. The anomaly is a consequence of the quantization procedure, and it appears due to the lack of a completely invariant regularization. In particular, trace anomaly is usually related to the one-loop divergences [25]. At the same time, one has to be careful, because the non-critical use of this relation may, in principle, lead to mistakes.

Consider the renormalization and anomaly for the conformal metric-scalar theory with torsion formulated above. The renormalizability of the theory requires the vacuum action to be introduced, which has to be (as it was already noticed in Section 3) of the form of the possible counterterms. The total four-dimensional action including the vacuum term can be presented as:

$$S_t = S_{inv} + S_{vac} \, , \tag{29}$$

where S_{inv} has been given in (22) and the form of the vacuum action will be established later.

Before going on to calculate the divergences and anomaly, one has to write a functional form for the conformal symmetry. It is easy to see that the Noether identity corresponding to (13) looks like

$$2g_{\mu\nu}\,\frac{\delta S_t}{\delta g_{\mu\nu}} + \frac{\xi_2}{\xi_3}\,\partial_\mu\,\frac{\delta S_t}{\delta T_\mu} - \varphi\,\frac{\delta S_t}{\delta\varphi} \,=\, 0\,. \tag{30}$$

Now, if we are considering both metric and torsion as external fields, and only the scalar as a quantum field, in the vacuum sector we meet the first two terms of (30). This means (due to the invariance of the vacuum divergences) that the vacuum action may be chosen in such a way that

$$-\sqrt{-g}\,\mathcal{T} \,=\, 2g_{\mu\nu}\,\frac{\delta S_{vac}}{\delta g_{\mu\nu}} + \frac{\xi_2}{\xi_3}\,\partial_\mu\,\frac{\delta S_{vac}}{\delta T_\mu} \,=\, 0\,. \tag{31}$$

The new form (31) of the conformal Noether identity indicates to a serious modification in the conformal anomaly. In the theory under discussion, the anomaly would mean $<\mathcal{T}>\neq 0$ instead of usual $<T^\mu_\mu>\neq 0$. Therefore, at first sight, we meet here some special case and one can not directly use the relation between the one-loop counterterms and the conformal anomaly derived in [25], because this relation does not take into account the non-trivial transformation law for the torsion field. It is reasonable to remind that, for the case of completely antisymmetric torsion discussed in [18], this problem did not show up just because S_μ is inert under conformal transformation.

Let us formulate a more general statement about conformal transformation and anomaly. When the classical metric background is enriched by the fields which do not transform, the Noether identity corresponding to the conformal symmetry remains the same as for the pure metric background. In this case one can safely use the standard relations [25] between divergences and conformal anomaly. However, if the new background field has a non-trivial transformation law, one has to care about possible modifications of the Noether identity and consequent change of anomaly.

In our case, the anomaly is modified, because it has a new functional form, $<\mathcal{T}>\neq 0$. One can derive this new anomaly using, for instance, the methods described in [25] or [21]. However, it is possible to find $<\mathcal{T}>$ in a more economic way, using some special decomposition of the background fields. As we shall see, the practical calculation of a new anomaly and even the anomaly-induced action may be reduced to the results known from [25, 12, 13] and especially [18], where the theory of the antisymmetric torsion was investigated.

Our purpose is to change the background variables in such a way that the transformation of torsion is absorbed by that of the metric. The crucial observation is that P, from (21), transforms[4] under (13) as $P' = P \cdot e^{-2\rho(x)}$. Therefore, the non-trivial transformation of torsion is completely absorbed by P. Since P only depends on the background fields, we can present it in any useful form. One can imagine, for instance, P to be of the form $P = g^{\mu\nu}\,\Pi_\mu\,\Pi_\nu$ where vector Π_μ doesn't transform, and then the calculation readily reduces to the case of an antisymmetric torsion [18]. In particular, we can now use standard results for the relation between divergences and anomaly [25].

In the framework of the Schwinger-DeWitt technique, we find the 1-loop counterterms in the form

$$\Gamma^{(1)}_{div} = -\frac{\mu^{n-4}}{(4\pi)^2\,(n-4)} \int d^n x \sqrt{-g}\, \left\{ \frac{1}{120}\, C^2 - \frac{1}{360}\, E + \frac{1}{180}\, \Box R + \frac{1}{6}\, \Box P + \frac{1}{2}\, P^2 \right\}\,, \qquad (32)$$

Here

$$C^2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \qquad \text{and} \qquad E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu\alpha\beta}R^{\mu\nu} + R^2$$

are the square of the Weyl tensor, and the scalar integrand of the Gauss-Bonnet term. The eq. (32) gives, as a by-product, the list of necessary terms in the vacuum action. Taking into account the arguments presented above, we can immediately cast the anomaly under the form

$$< \mathcal{T} > = -\frac{1}{(4\pi)^2} \left[\frac{1}{120}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta} - \frac{1}{360}\, E + \frac{1}{180}\, \Box R + \frac{1}{6}\, \Box P + \frac{1}{2}\, P^2 \right]\,. \qquad (33)$$

One may proceed and, following [12, 13, 18], derive the conformal non-invariant part of the effective action of the vacuum, which is responsible for the anomaly (33). Taking into account our previous treatment of the conformal transformation of torsion, we consider

[4] As a consequence, the action $\int \sqrt{-g} P\phi^2$ is conformal invariant. This fact has been originally discovered in [26].

it is hidden inside the quantity P of eq. (21), and again imagine P to be of the form $P = g^{\mu\nu}\,\Pi_\mu\,\Pi_\nu$. Then the equation for the effective action $\Gamma[g_{\mu\nu},\Pi_\alpha]$ is [5]

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\frac{\delta\,\Gamma}{\delta g_{\mu\nu}} =< \mathcal{T} > \ . \tag{34}$$

In order to find the solution for Γ, we can factor out the conformal piece of the metric $g_{\mu\nu} = \bar{g}_{\mu\nu}\cdot e^{2\sigma}$, where $\bar{g}_{\mu\nu}$ has fixed determinant and put $P = \bar{P}\cdot e^{-2\sigma(x)}$, that corresponds to $\bar{\Pi}_\alpha = \Pi_\alpha$. This transformation for artificial variable Π_α is identical to the one for the S_α pseudovector, that is the case considered in [18]. Then the result can be obtained directly from the effective action derived in [18], and we get

$$\Gamma \,=\, S_c[\bar{g}_{\mu\nu};\bar{P}] \,-\, \frac{1}{12}\cdot\frac{1}{270(4\pi)^2}\,\int d^4x\sqrt{-g(x)}\,R^2(x) + \frac{1}{(4\pi)^2}\,\int d^4x\sqrt{-\bar{g}}\,\Big\{\sigma\cdot[\frac{1}{120}\,\bar{C}^2 -$$

$$-\frac{1}{360}\,(\bar{E}-\frac{2}{3}\,\bar{\nabla}^2\bar{R}) + \frac{1}{2}\,\bar{P}^2] + \frac{1}{180}\,\sigma\bar{\Delta}\sigma - \frac{1}{6}\,(\bar{\nabla}_\mu\sigma)\,\bar{\nabla}^\mu\bar{P} + \frac{1}{6}\,\bar{P}(\bar{\nabla}_\mu\sigma)^2\Big\}\ , \tag{35}$$

where $S_c[\bar{g}_{\mu\nu};\bar{P}]$ is an unknown functional of the metric $\bar{g}_{\mu\nu}(x)$ and $\bar{P}$, which acts as an integration constant for any solution of (34).

Now, one has to rewrite (35) in terms of the original field variables, $g_{\mu\nu}, T^\alpha{}_{\beta\gamma}$. Here, we meet a small problem, because we only have, for the moment, the definition $\Pi_\alpha = \bar{\Pi}_\alpha$ for the artificial variable Π_α, but not for the torsion. Using the previous result (13), we can define

$$T^\alpha{}_{\beta\gamma} = \bar{T}^\alpha_{\beta\gamma} - \frac{1}{3}\cdot\Big[\,\delta^\alpha_\gamma\,\partial_\beta\sigma - \delta^\alpha_\beta\,\partial_\gamma\sigma\,\Big]\ , \tag{36}$$

so that $\bar{T}^\alpha_{\beta\gamma}$ is an arbitrary tensor. Also, we call $\bar{T}^\alpha = \bar{g}^{\alpha\beta}\,\bar{T}_\beta$ etc. Now, we can rewrite (35) in terms of metric and torsion components

$$\Gamma \,=\, S_c[\bar{g}_{\mu\nu};\bar{T}^\alpha_{\beta\gamma}] \,-\, \frac{1}{12}\cdot\frac{1}{270(4\pi)^2}\,\int d^4x\sqrt{-g(x)}\,R^2(x) +$$

$$+\frac{1}{(4\pi)^2}\,\int d^4x\sqrt{-\bar{g}}\,\Big\{+\frac{1}{180}\,\sigma\bar{\Delta}\sigma + \frac{1}{120}\,\bar{C}^2\,\sigma - \frac{1}{360}\,(\bar{E}-\frac{2}{3}\,\bar{\nabla}^2\bar{R})\,\sigma$$

$$+\frac{1}{72}\,\sigma\,\Big[-\frac{\xi_2^2}{\xi_3}\,\bar{R} + 6\xi_2\,(\bar{\nabla}_\mu\bar{T}^\mu) + 6\xi_3\,\bar{T}_\mu\bar{T}^\mu + 6\xi_4\,\bar{S}_\mu\bar{S}^\mu + 6\xi_5\,\bar{q}_{\mu\nu\lambda}\bar{q}^{\mu\nu\lambda}\Big]^2 +$$

[5] We remark that this equation is valid only for the "artificial" effective action $\Gamma[g_{\mu\nu},\Pi_\alpha]$, while the effective action in original variables $g_{\mu\nu}, T^\alpha_{\beta\gamma}$ would satisfy the modified equation (31). The standard form of the equation for the effective action is achieved through the special decomposition of the external fields.

$$\frac{1}{6}\left[(\bar{\nabla}^2\sigma + (\bar{\nabla}_\mu\sigma)^2\right] \cdot \left[-\frac{\xi_2^2}{\xi_3}\bar{R} + 6\xi_2\,(\bar{\nabla}_\mu\bar{T}^\mu) + 6\xi_3\,\bar{T}_\mu\bar{T}^\mu + 6\xi_4\,\bar{S}_\mu\bar{S}^\mu + 6\xi_5\,\bar{q}_{\mu\nu\lambda}\bar{q}^{\mu\nu\lambda}\right]\Bigg\}\,, \tag{37}$$

This effective action is nothing but the generalization of the similar expressions of [12, 13, 18] for the case of general metric-torsion background and conformal symmetry described in Section 3. The curvature dependence in the last two terms appears due to the non-trivial transformation law for torsion.

5 Conclusions.

We have considered the conformal properties of the second-derivative metric-torsion-dilaton gravity, on both classical and quantum level. The main results, achieved above, can be summarized as follows:

i) If the parameters ξ_i of the general metric-torsion-dilaton theory (8) satisfy the relation (12), there is conformal invariance of the new type, more general than the one considered in [26, 24] and different from the ones considered in [22, 21].

ii) In this case the vector trace of torsion plays the role of a compensating field, providing the classical conformal on-shell equivalence between this theory and General Relativity. The conformal invariant scalar-metric-torsion action can be presented in alternative forms (22), (23), (25), while (23) is some new nonlinear n-dimensional conformal invariant sigma-model. If the nonminmal parameters satisfy an additional relation $\xi_2^2 > \xi_3$, the equivalence holds between the positively defined General Relativity and scalar field theory. This indicates to the stabilization of the conformal mode in gravity with torsion and may indicate that the introduction of torsion can be some useful alternative to other approaches to this problem (see [19, 20]).

iii) The conformal Noether identity (30) indicates that the conformal anomaly with torsion is not the anomaly of the trace of the Energy-Momentum Tensor, but rather the anomaly of some modified quantity $\mathcal{T}$. But, using the appropriate decomposition of the background variables, the calculation of this new anomaly can be readily reduced to the use of the known results of [25] and [18], and the derivation of the anomaly-induced effective action can be done through the similar decomposition of variables and the use of the known results of [12, 13, 18]. As in other known cases, the induced action (37) breaks the conformal equivalence between metric-torsion-scalar action and (1) and therefore may lead to the nontrivial applications to inflationary cosmology.

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