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Logistic Algebras

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Abstract

We introduce a three-generator algebra where the eigenvalue of one generator is given by the logistic map. We also present the Casimir operator of this algebra, construct matrix representations that can be finite as well as infinite and obtain the Hamiltonian of the XX-model in a magnetic field from the lowest dimensional representation of the algebra.

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The relevance of symmetries in physical systems is well established. Algebraic structures can play an important role in the solution of physical models: in a few cases, for completely solved systems [1], symmetries can be very important since for these models they are in general connected to their invariants.

For systems that present some kind of non-regular deterministic behaviour (in special cases these systems may be even chaotic [2]) the known algebraic structures are not very useful since the construction of regular invariants, an usual procedure in solving physical models, is in opposition to the non-regular behaviour of these systems.

Within this framework it may be relevant to generalise the known algebraic structures and, in particular, to develop a class of algebras that may present a non-regular behaviour.

We start considering the algebra generated by $J_0, J_\pm$ described by the relations

$$J_+ J_i = J_{i+1} J_+ \quad , \quad i = 0, 1, 2, \cdots \tag{1.a}$$

$$J_i J_- = J_- J_{i+1} \quad , \tag{1.b}$$

$$J_+ J_- - J_- J_+ = a(J_0 - J_1) \ , \tag{1.c}$$

where $J_- = J_+^\dagger \ \ , \ \ J_i^\dagger = J_i$ and a is a real constant. Moreover,

$$J_i \equiv r J_{i-1}(1 - J_{i-1}) \tag{2}$$

with r a real parameter. We could have chosen $J_{i+1} \equiv a_0 + a_1 J_i + \cdots + a_n J_i^n + \cdots$ but we are particularly interested in eqn. (2) where J_i is given by the logistic map [2]. In fact, considering the general quadratic expansion ($n = 2$) for J_{i+1} most results we are going to discuss for the logistic map (eqn. (2)) are unchanged.

The hermitian operator J_0 can be diagonalised. Consider the state $|n_0\rangle$ with the highest eigenvalue of J_0

$$J_0 |n_0\rangle = n_0 |n_0\rangle \tag{3}$$

Since n_0 is the highest J_0 eigenvalue we must have

$$J_+ |n_0\rangle = 0 \ . \tag{4}$$

Using the algebraic relations in eqn. (1) we obtain*

$$J_0|n_m\rangle = n_m|n_m\rangle \quad ,$$
$$J_-|n_m\rangle = N_m|n_{m+1}\rangle \quad ,$$
$$J_+|n_{m+1}\rangle = N_m|n_m\rangle \quad , \tag{5}$$

where

$$N_m = \sqrt{a(n_0 - n_{m+1})} \tag{6}$$

and

$$n_{m+1} = rn_m(1 - n_m) \ . \tag{7}$$

Of course, since the eigenvalues of J_0 are given by the logistic map, eqn. (7), their values as m increases can have an irregular behaviour depending on the values of r and n_0, and the dimension of the representation.

Let us now consider the operator

$$C = J_+J_- + aJ_1 = J_-J_+ + aJ_0 \tag{8}$$

Using the algebraic relations (1) it is easy to see that

$$[C, J_0] = [C, J_\pm] = 0 \quad , \tag{9}$$

i.e., C is the Casimir operator of the algebra. In fact, we arrive easily at

$$C|n_m\rangle = c_0|n_m\rangle \quad , \tag{10}$$

with $c_0 = an_0$ independent of m. The value n_0 characterises the inequivalent representations of the algebra.

As far as matrix representations are concerned we give three examples that can be easily verified to satisfy eqns. (1-7)

Example 1. Two-Dimensional Representation

$$J_0 = \begin{pmatrix} n_0 & 0 \\ 0 & n_1 \end{pmatrix} \quad ; \quad J_+ = \begin{pmatrix} 0 & N_0 \\ 0 & 0 \end{pmatrix} \quad ; \quad J_- = J_+^\dagger \tag{11}$$

*In order to preserve the standard notation for the logistic map, the states generated by the lowering operator J_-, are labeled by $|n_m\rangle$ with **INCREASING** values of m.

For this case the allowed values of r and n_0 are determined by the equation $n_0 - n_2 = 0$ such that $n_0 - n_1 \neq 0$. There are two non-trivial solutions for these two equations

$$n_0^{\pm} = \frac{r + 1 \pm \sqrt{r^2 - 2r - 3}}{2r} \,. \tag{12}$$

The solution n_0^+ gives $n_0^+ > n_1^+$ implying $a > 0$, while $n_0^- < n_1^-$ give $a < 0$. For both cases $r \geq 3$.

Example 2. Three-Dimensional Representation

$$J_0 = \begin{pmatrix} n_0 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & n_2 \end{pmatrix} \;;\; J_+ = \begin{pmatrix} 0 & N_0 & 0 \\ 0 & 0 & N_1 \\ 0 & 0 & 0 \end{pmatrix} \;;\; J_- = J_+^{\dagger} \tag{13}$$

The allowed values are computed from $n_0 - n_3 = 0$, $n_0 - n_2 \neq 0$, $n_0 - n_1 \neq 0$. The solution can be obtained numerically and there are four non-trivial solutions to this case.

Example 3. Infinite Dimensional case

$$J_0 = \begin{pmatrix} n_0 & 0 & 0 & 0 & \cdots \\ 0 & n_1 & 0 & 0 & \cdots \\ 0 & 0 & n_2 & 0 & \cdots \\ 0 & 0 & 0 & n_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \;;\; J_+ = \begin{pmatrix} 0 & N_0 & 0 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & \cdots \\ 0 & 0 & 0 & N_2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \;;\; J_- = J_+^{\dagger} \tag{14}$$

The allowed values of r and n_0 can be computed for instance for $a < 0$ from $n_m > n_0$ for all values of $m \geq 1$. In figure I we show the half-leaf region, with the allowed values of r and n_0. In this case n_m was computed up to $m = 200$. The diagram with possible solutions has as superior boundary a composition of several curves. For $1 \leq r \leq 3$ we see the stable fixed point curve, $(r-1)/r$, of the logistic map. For $3 < r < 3.4$ we have the lowest branch attractor two-cycle logistic map curve, $(r + 1 - \sqrt{r^2 - 2r - 3})/(2r)$ and so on. The points that are below these curves are the allowed values for $a < 0$ infinite-dimensional representation. The peculiar structure of equations (1a,b) allows the algebra to have highest weights with finite and infinite representations.

In order to indicate a possible application of the algebra (eqn. (1)) for low-dimensional representions, where the irregularity aspect does not fully emerge, we construct a spin-1/2 nearest-neighbour quantum spin chain Hamiltonian invariant under this algebra.

The matrices $(\Delta J_0)_R, (\Delta J_\pm)_R$

$$(\Delta J_0)_R = \begin{pmatrix} n_1 & 0 & 0 & 0 \\ 0 & n_0 & 0 & 0 \\ 0 & 0 & n_0 & 0 \\ 0 & 0 & 0 & n_1 \end{pmatrix} \; ; \; (\Delta J_+)_R = \begin{pmatrix} 0 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & b_8 \\ 0 & 0 & 0 & b_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ; \; (\Delta J_-)_R = (\Delta J_+)^\dagger_R \tag{15}$$

with

$$\begin{aligned} b_2 &= i(-b_8^2 + N_0^2)^{1/2} \quad , \quad (b_8 - \text{arbitrary}), \\ b_3 &= i b_8 \ , \\ b_{12} &= (-b_8^2 + N_0^2)^{1/2} \ , \end{aligned} \tag{16}$$

are 4×4 matrices satisfying eqns. (1-7). These matrices are reducible representations since they can be constructed as

$$\begin{aligned} (\Delta J_0)_R &= a_1(J_0 \otimes 1\!\!1 + 1\!\!1 \otimes J_0) + a_2 1\!\!1 \otimes 1\!\!1 + a_3 J_0 \otimes J_0 \ , \\ (\Delta J_+)_R &= \alpha_1 J_+ \otimes 1\!\!1 + \alpha_2 1\!\!1 \otimes J_+ + \alpha_3 J_+ \otimes J_0 + \alpha_4 J_0 \otimes J_+ \ , \end{aligned} \tag{17}$$

with a similar equation for $(\Delta J)_R$, where

$$\begin{aligned} a_1 &= \frac{n_0 + n_1}{n_0 - n_1} \; ; \; a_2 = \frac{-n_1(n_0+n_1)}{n_0 - n_1} \; ; \; \alpha_1 = \frac{b_8[n_1(i-1) - n_0(i+1)]}{(i+1)(n_1 - n_0)} \; ; \\ \alpha_2 &= \frac{(n_0 + i n_i)\sqrt{-b_8^2 + N_0^2}}{n_1 - n_0} \; ; \; \alpha_3 = \frac{2b_8}{(1+t)(n_1 - n_2)} \; ; \; \alpha_4 = \frac{(i+1)\sqrt{-b_8^2 + N_0^2}}{n_0 - n_1} \ . \end{aligned} \tag{18}$$

Eqns. (17) represent a reduced form of a universal coproduct of the algebra (1).

Let $h \equiv \Delta C - \gamma 1\!\!1 \otimes 1\!\!1$, where ΔC is the result of the Casimir, eqn. (8), for the matrices in eqn. (15) and γ a constant. Consider the Hamiltonian

$$H = J \sum_{n=1}^{N-1} 1\!\!1 \otimes \cdots \otimes h_{n,n+1} \otimes 1\!\!1 \otimes \cdots \otimes 1\!\!1 \tag{19}$$

where J is a constant and $h_{n,n+1}$ is h acting in the $(n, n+1)$ slot of $(\mathbb{C})^{\otimes N}$. Then, we obtain

$$H = \frac{1}{2}\sum_{n=1}^{N-1}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \frac{1}{2}(\alpha+\alpha^{-1})(\sigma_n^z+\sigma_{n+1}^z)\right) + \frac{(\alpha-\alpha^{-1})}{4}(\sigma_1^z - \sigma_N^z) \quad (20)$$

with

$$\alpha = \frac{\sqrt{-b_8^2 + N_0^2}}{b_8} \quad (21)$$

This is the XX-Hamiltonian in an external magnetic field with a non-trivial boundary condition. We have just proven that this Hamiltonain is invariant under algebra described in eqn. (1). This result indicates that the lowest dimensional representation of the algebra (1) is connected to $slq(2)$ for $q^4 = 1$ [3].

We present now some comments. Firstly, instead of having eqn. (1c) we could have chosen a more general algebra

$$J_+J_- - J_-J_+ = \sum_{i=0} a_{2i}(J_{2i} - J_{2i+1}) \ .$$

In this case the Casimir becomes

$$C = J_+J_- + \sum_{i=0} a_{2i}J_{2i+1} \ .$$

The only difficulty with this general approach seems to be the increasing algebraic complication in the equations determining the allowed values of r and n_0. There are also other possible ways of generalising the algebra; for instance we could think to construct higher rank Logistic algebras or to find out possible non-trivial deformations of eqns. (1-7). It would be also interesting to find the bialgebra structure of the algebra, eqn. (1), and the completely integrable strucuture connected to it as indicated by the example worked at the end of this letter. It would be tempting to develop an example where the irregularity present in the algebra is manifest, for instance by considering a Lipkin-Meshkov-Glick [4] like model where now the generators belong to the algebra in (1).

Finally, we conjecture that the points in fig. I below the lower branch of the two-cycle curve $(r+1-\sqrt{r^2-2r-3})/2r$ for $3 < r < 3.4$ correspond to the XX-model as the upper boundary points. The reason is that the "asymptotic" states of the infinite dimensional representation $|n_0^{\pm}\rangle$ (see eqn. (12)) occur with infinite degeneracy, being then the relevant ones. Moreover, since $n_1^- \equiv r n_0^-(1-n_0) = n_0^+$ we can reconstruct with $|n_0^\pm\rangle$ the two-dimensional representation of the logistic algebra (1). It would be interesting to find out what are the statistical models on and below the higher-cycles curves in fig. I. Furthermore, the model associated to the allowed solutions $a < r < 4$ in fig. I, where a is the Feigenbaum point, is a potentially soluble model with a close relation to chaos.

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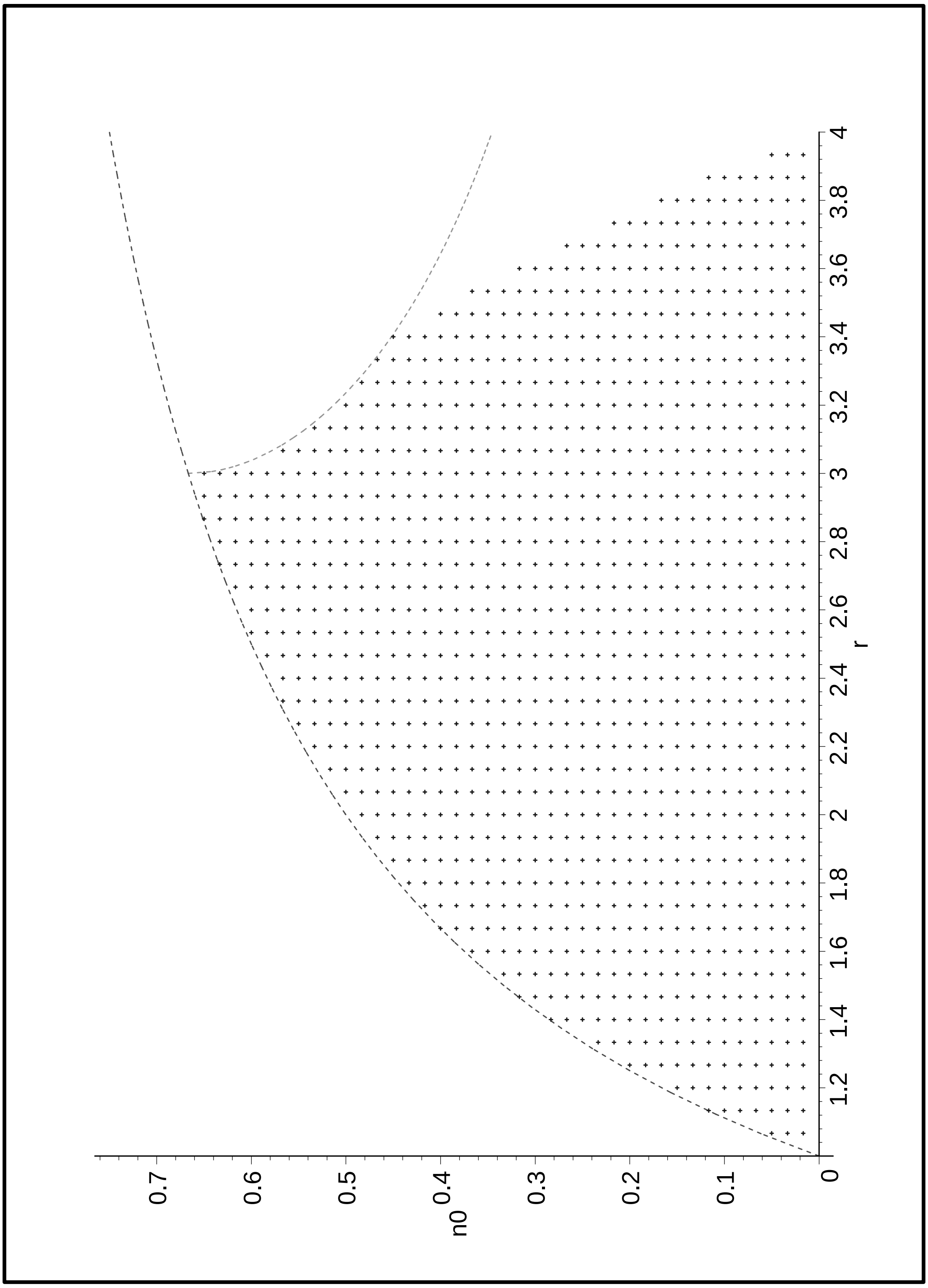

[Fig. I:] The points in this diagram are allowed values of r and n_0 for the $a < 0$ infinite dimension representation. The superior boundary of this diagram is a composition of several curves. For $1 \leq r \leq 3$ we see the stable fixed point curve $(r-1)/r$, of the logistic map. For $3 < r < 3.4$ we have the lowest branch attractor two-cycle logistic map curve, $(r+1-\sqrt{r^2-2r-3})/2r$ and so on.