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The Thermal Coupling Constant and the Gap Equation in the $\lambda\varphi^4_D$ Model

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Abstract

By the concurrent use of two different resummation methods, the composite operator formalism and the Dyson-Schwinger equation, we re-examinate the behavior at finite temperature of the $O(N)$-symmetric $\lambda\varphi^4$ model in a generic D-dimensional Euclidean space. In the cases $D = 3$ and $D = 4$, an analysis of the thermal behavior of the renormalized squared mass and coupling constant are done for all temperatures. It results that the thermal renormalized squared mass is positive and increases monotonically with the temperature. The behavior of the thermal coupling constant is quite different in odd or even dimensional space. In $D = 3$, the thermal coupling constant decreases up to a minimum value diferent from zero and then grows up monotonically as the temperature increases. In the case $D = 4$, it is found that the thermal renormalized coupling constant tends in the high temperature limit to a constant asymptotic value. Also for general D-dimensional Euclidean space, we are able to obtain a formula for the critical temperature of the second order phase transition. This formula agrees with previous known values at $D = 3$ and $D = 4$.

Key-words: HOT QFT; Phase transition.

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1 Introduction

Considerable progress has been done during the last two decades in our understanding of finite temperature field theory. A clear account on the foundations of this presently well established branch of theoretical physics is done in Refs.[1]. One of the outstanding questions for the subject is the possible existence of a deconfinement phase transition in QCD. It is expected that at sufficiently high temperatures quarks and gluons are not bounded inside hadrons, but have a behavior somehow analogous to the plasma phase, the so called quark-gluon plasma [2]. In connection to this expected phenomenon it naturally arises questions concerning the nature of the transition from the low temperature colorless hadron gas to the high temperature quark-gluon plasma [3]. In particular, if in fact it is a phase transition, the question of its order is still open. In pure Yang-Mills theory it is a point of consensus that there is a phase transition, but if quarks are included, the answer seems to be more difficult. Also if the theory involves quarks and gluons, another phase transition may occur and its order depends on the number of quark flavours: the chiral symmetry phase transition. The chiral symmetry is spontaneously broken at zero temperature and it is restored at some finite temperature. For two massless flavours it is a second order one, while for three flavours it is a first order phase transition. For more than three flavours no clear conclusion has been obtained about its order, nor even if there is a chiral phase transition. Another important point to be noticed in the the case of the chiral symmetry phase transition in QCD, is that the order parameter is composite in the fields and perturbative methods fail. Non-perturbative methods are mandatories.

In any case, "real" QCD is very difficult to deal with and no definite theoretical results for many questions have been obtained up to the present moment. So, simpler models still remain very useful "laboratories" to try to get some insights about what happens in the real world, particularly if long range, non-perturbative effects are involved, as it should be the case for both phase transitions mentioned above. For instance, it is well known that the low energy dynamics of QCD may be quite well described by the $O(4)$ model, since there is an isomorphism between the $O(4)$ and the $SU(2)XSU(2)$ symmetry groups, the latter being the group of two flavors of massless quarks [4] (in the $O(N)$ model when the vacuum of the model exhibits spontaneous symmetry breaking it is known as the linear σ model).

In a previous paper two of the authors derived expressions for the renormalized thermal mass and coupling constant in the $\lambda\varphi^4$ model in a D-dimensional flat spacetime at the one-loop approximation [5]. The main results are that the thermal squared mass increases with the temperature, while the thermal coupling constant decreases with the temperature. These authors conjectured that in $D = 3$ it could exist a temperature above which the renormalized coupling constant becomes negative. In this regime the system would develop a first order phase transition. As we will see latter, in the framework of the vector N-component model at a non-perturbative level, the answer seems to be negative. In other words, in this context, there is no temperature at which the coupling constant changes of sign, and no first order phase transition induced by the thermal renormalized coupling constant seems to be possible in $D = 3$. On the other side, in a slightly different context, another result that deserves our attention was obtained by Ananos and Svaiter [6]. These authors studying the $(\lambda\varphi^4 + \sigma\varphi^6)_3$ model at finite temperature up the two-loop approximation, exhibit the existence of the tricritical phenomenon when both, the thermal renormalized squared mass and quartic coupling constant become zero for some temperature.

The purpose of this paper is to get an extension of the above mentioned result [5] to all orders of perturbation theory. This may be done considering the $O(N)$-symmetric model in the framework of the large-N expansion at the leading order in $\frac{1}{N}$, and using resummation methods: the composite operator formalism (CJT) [7] and also the Dyson-Schwinger equation, both adapted to finite temperature. It is well known that these resummation methods can solve the problem of the breakdown of perturbation theory in some massless field theories at finite temperature. For instance, in the N-component scalar $\lambda\varphi^4$ model it is possible, for large N, to sum a class of Feynman diagrams, the ring diagrams, by the use of the recurrent Dyson-Schwinger equation. This allows to solve at the leading order in $\frac{1}{N}$, the problem of infrared divergences [8]. An alternative method that takes into account leading and subleading contributions from multiloops diagrams was developed by Cornwall, Jackiw and Toumboullis (CJT formalism) [7]. In this approach one considers a generalization of the effective action $\Gamma(\varphi)$ which depends not only on the expectation value of the field, but also on the expectation value of the time ordered product of two fields. In this formalism we naturally sum a large class of diagrams, and the gap equation is easily obtained using a variational technique. This formalism has been further developed by Pettini and more recently by Amelino-Camelia and Pi [9]. It may also be mentioned that recently discussions on the pure scalar model have been done by many authors [10]. Of special interest are Ref.[11] and Ref.[12]. In this last article Eylal et al. [12] discussed the phase structure of the $O(N)$ model in a generic D-dimensional euclidean space, and also the sign of the second coefficient of the large-N renormalization group β function. It may be noticed that Kessler and Neuberger [13] using lattice regularization obtained quite surprisingly that for $D = 3$ the sign of this cofficient is arbitrary. The authors in Ref.[12] raised the possibility that this ambiguity could be an artifact of the large N-expansion.

In the next sections we apply the above mentioned resummation methods to get non-perturbative results for the renormalized thermal squared mass and coupling constant for the vector massive $\lambda\varphi^4$ model defined in a generic D-dimensional Euclidean space. The outline of the paper is the following. In section II we briefly discus the CJT formalism. In section III the thermal gap equation is obtained. In section IV using the gap equation we sum all the daisy and super-daisy diagrams to obtain the thermal renormalized coupling constant. Conclusions are given in section V. In this paper we use $\frac{h}{2\pi} = c = k_B = 1$.

2 The Cornwall, Jackiw and Tomboulis (CJT) formalism

In this section we will briefly discuss the effective action formalism for composite operators extended to finite temperature in a D-dimensional Euclidean space [14]. Let us consider the vacuum persistence amplitude $Z(J,K)$ in the presence of local and non-local sources $J(x)$ and $K(x,y)$ respectively, where $J(x)$ couples to $\Phi(x)$ and $K(x,y)$ to $\frac{1}{2}\Phi(x)\Phi(y)$. This object is a generalized generating functional of the Green's functions of the model i.e.,

$$Z(J,K) = \int D\Phi \exp\left\{-\left[I(\Phi) + \int d^Dx\,(\Phi(x)J(x)) + \frac{1}{2}\int d^Dx \int d^Dy\,(\Phi(x)K(x,y)\Phi(y))\right]\right\}, \tag{1}$$

where $D\Phi$ is an appropriate functional measure. Introducing the interaction Lagrange density L_{int}, the classical Euclidean action $I(\Phi)$ is given by

$$I(\Phi) = \int d^Dx \int d^Dy\,\left(\Phi(x)D_0^{-1}(x-y)\Phi(y)\right) + \int d^Dx L_{int}, \tag{2}$$

where $D_0(x-y)$ is the free propagator i.e.,

$$D_0^{-1}(x-y) = -(\Box + m^2)\delta^D(x-y). \tag{3}$$

The generalized effective action $\Gamma(\varphi, G)$ is defined by a double Legendre transform of the generalized generating functional of the connected correlation functions $W(\varphi, G) = \ln Z(\varphi, G)$ (the Helmholtz free energy),

$$\frac{\delta W(J,K)}{\delta J(x)} = \varphi(x), \tag{4}$$

$$\frac{\delta W(J,K)}{\delta K(x)} = \frac{1}{2}\left(\varphi(x)\varphi(y) + G(x,y)\right), \tag{5}$$

and

$$\Gamma(\varphi, G) = W(J,K) - \int d^D x \,(\varphi(x)J(x)) - \frac{1}{2}\int d^D x \int d^D y \,(\varphi(x)\varphi(y) + G(x,y))\, K(x,y). \tag{6}$$

In the absence of sources, the generalized effective action $\Gamma(\varphi, G)$ satisfies

$$\frac{\delta \Gamma(\varphi, G)}{\delta \varphi(x)} = J(x) - \int d^D y \,(K(x,y)\varphi(y)) = 0 \tag{7}$$

and

$$\frac{\delta \Gamma(\varphi, G)}{\delta G(x,y)} = -\frac{1}{2} K(x,y) = 0. \tag{8}$$

In eq.(4) and eq.(5), the quantity $\varphi(x)$ is the normalized vacuum expectation value of the field,

$$\varphi(x) = < 0|\Phi(x)|0 > \tag{9}$$

and $G(x,y)$ is the two-point function. If we assume that translation invariance is not spontaneously broken we have $\varphi(x) \equiv \varphi$ and the propagator depends only on the distance in Euclidean space, $G(x,y) \equiv G(x-y)$. Consequently it is possible to define the generalized effective potential $V(\varphi, G)$ as a straightforward generalization of the usual definition,

$$\Gamma(\varphi, G) = -V(\varphi, G) \int d^D x. \tag{10}$$

The stability conditions given by eqs. (7) and (8) become in terms of the effective potential,

$$\frac{\partial V(\varphi,G)}{\partial\varphi}=0 \quad , \quad \frac{\partial V(\varphi,G)}{\partial G(k)}=0, \tag{11}$$

where $G(k)$ is the Fourier transform of $G(x,y)$

$$G(x,y)=G(x-y)=\frac{1}{(2\pi)^D}\int d^Dk e^{-ik(x-y)}G(k)=\frac{1}{(2\pi)^D}\int d^Dk e^{-ik(x-y)}\frac{1}{k^2+M^2}, \tag{12}$$

and $M^2=m^2+\frac{\lambda}{2}\varphi^2$.

Let us consider the model described by the Euclidean Lagrange density in the form

$$\mathcal{L}=\frac{1}{2}\partial_\mu\Phi\partial_\mu\Phi+\frac{1}{2}m^2\Phi^2+\frac{\lambda}{4!}\Phi^4, \tag{13}$$

whose effective potential is given by

$$\begin{aligned} V(\varphi,M)=\frac{1}{2}m^2\varphi^2+\frac{\lambda}{4!}\varphi^4+\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}\ln(k^2+M^2)- \\ \frac{1}{2}(M^2-m^2-\frac{\lambda}{2}\varphi^2)G(x,x)+\frac{\lambda}{8}G(x,x)G(x,x). \end{aligned} \tag{14}$$

Combining the above equation with the stationary requirements given by eq.(11) we have

$$\frac{\partial V(\varphi,M)}{\partial\varphi}=\varphi\left(m^2+\frac{\lambda}{6}\varphi^2+\frac{\lambda}{2}G(x,x)\right)=0\ , \tag{15}$$

$$\frac{\partial V(\varphi,M)}{\partial M^2}=-\frac{1}{2}\frac{\partial G(x,x)}{\partial M^2}\left(M^2-m^2-\frac{\lambda}{2}\varphi^2-\frac{\lambda}{2}G(x,x)\right)=0. \tag{16}$$

The effective potential is obtained by evaluating $V(\varphi,M)$ from eq.(16). It is composed by the classical, the one-loop and two-loop contributions, i.e.,

$$V(\varphi,M(\varphi))=V_0+V_I+V_{II}, \tag{17}$$

where the classical contribution is

$$V_0(\varphi)=\frac{1}{2}m^2\varphi^2+\frac{\lambda}{4!}\varphi^4, \tag{18}$$

the one loop contribution is

$$V_I(\varphi, M(\varphi)) = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D}\ln(k^2 + M^2(\varphi)), \tag{19}$$

and finally the two-loop contribution is

$$V_{II}(\varphi, M(\varphi)) = -\frac{\lambda}{8}G(x,x)G(x,x). \tag{20}$$

In the above expressions we have,

$$M^2(\varphi) = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{2}G(x,x), \tag{21}$$

and $G(x,x)$ is given by

$$G(x,x) = \int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2 + M^2(\varphi)}. \tag{22}$$

To study finite temperature effects there are two different formalisms, the real time formalism and the imaginary time (Matsubara) formalism. In the Matsubara formalism the Euclidean time τ is restricted to the interval $0 \leq \tau \leq \beta = \frac{1}{T}$ and in the functional integral the field $\Phi(\tau, \mathbf{x})$ satisfies periodic boundary conditions in Euclidean time,

$$\Phi(0, \mathbf{x}) = \Phi(\beta, \mathbf{x}). \tag{23}$$

All the Feynman rules are the same as in the zero temperature case, except that the momentum-space integral over the zero-th component is replaced by a sum over discrete frequencies. For boson fields we have to perform the replacement

$$\int \frac{d^D p}{(2\pi)^D} f(p) \rightarrow \frac{1}{\beta}\sum_n \int \frac{d^{D-1}p}{(2\pi)^{D-1}} f(\frac{2n\pi}{\beta}, \mathbf{p}). \tag{24}$$

The thermal gap equation may be get from the zero temperature gap equation,

$$M^2(\varphi) = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{2}\int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2 + M^2(\varphi)}, \tag{25}$$

after performing the replacement given by eq.(24). In the next section we use these result to analyse the thermal behavior of the squared mass and coupling constant. We remark that the previous analysis may also be done using a simpler but *ad-hoc* procedure [16], replacing the thermal mass obtained in the one-loop approximation by the Dyson-Schwinger equation. Nevertheless, the CJT formalism provides a more elegant way and and a consistent basis for this study, since the gap equation is derived from a stationary requirement. Indeed, recently Campbell-Smith analysed the composite operator formalism in $(QED)_3$ and proved that it reproduces the usual gap equation derived using the *ad-hoc* Dyson-Schwinger approach [17].

3 The thermal gap equation in the $\lambda\varphi^4_D$ model

Let us suppose that our system is in equilibrium with a thermal bath. At the one-loop approximation the thermal mass and coupling constant for the $\lambda\varphi^4$ model in a D-dimensional Euclidean space have been obtained in a previous work [5] and are given by

$$m^2(\beta) = m_0^2 + \frac{\lambda_0}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left(\frac{m_0}{\beta n}\right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1}(m_0 n\beta) \tag{26}$$

and

$$\lambda(\beta) = \lambda_0 - \frac{3}{2}\frac{\lambda_0^2}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left(\frac{m_0}{\beta n}\right)^{\frac{D}{2}-2} K_{\frac{D}{2}-2}(m_0 n\beta), \tag{27}$$

where $K_\nu(z)$ is the modified Bessel function and m_0^2 and λ_0 are the zero temperature renormalized squared mass and coupling constant respectively [15]. It is possible to improve the above results studying the gap equation for the temperature dependent squared mass. One way to get this improvement is to take the $O(N)$ model, as it has been done by Dolan and Jackiw. In this case the Feynman diagrams are classified according to its topology and each of these classes is associated to a given power of $\frac{1}{N}$. In the large N

limit, only some classes of diagrams, those associated to the smallest power of $\frac{1}{N}$ give the leading contributions. These contributions, which include those from all daisy and super-daisy diagrams may be summed up, resulting that in the large N limit the results are exact. This approach was further developed by Weldon and also by Eboli et al. and more recently by Drummond et al. [16]. We can obtain from the thermal gap equation resulting from the combination of eq.(24) and eq.(25), an expression to replace the one-loop result above. For simplicity we will first analyse the thermal corrections in the the disordered phase, which corresponds to absence of spontaneous symmetry breaking. In this case we adopt the notation $m^2(\beta)$ for the squared thermal mass and we suppress the quadratic term in the field. We follow a procedure analogous to that Malbouisson and Svaiter have used in [5]. We use a mix between dimensional and zeta function analytical regularizations to evaluate formally the integral over the continuous momenta and the summation over the Matsubara frequencies. We get a sum of a polar (temperature independent) term plus a thermal analytic correction. The pole is suppressed by the renormalization procedure. Then after some technical manipulations, the gap equation may be rewritten in the form,

$$m^2(\beta) = m_0^2 + \frac{\lambda_0}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left(\frac{m(\beta)}{\beta n} \right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1}\left(m(\beta) n \beta \right). \tag{28}$$

To solve the above equation, and consequently to go beyond perturbation theory, let us take an integral representation of the Bessel function [18] given by

$$K_\nu(z) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\nu + \frac{1}{2})} (\frac{1}{2} z)^\nu \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt, \tag{29}$$

which is valid for $Re(\nu) > -\frac{1}{2}$ and $|arg(z)| < \frac{\pi}{2}$. This integral representation attend for our purposes if we restrict ourselves to $D > 1$. Substituting eq.(29) in eq.(28) and defining

$$F(D) = \frac{1}{2^{D-1}} \frac{1}{\pi^{\frac{D-1}{2}}} \frac{1}{\Gamma(\frac{D-1}{2})}, \tag{30}$$

the gap equation becomes

$$m^2(\beta) = m_0^2 + \lambda_0 F(D)\,(m(\beta))^{D-2} \int_1^\infty dt(t^2-1)^{\frac{D-3}{2}} \frac{1}{e^{m(\beta)\beta t}-1}. \tag{31}$$

Defining a new variable $\tau = m(\beta)\beta t$ it is easy to show that

$$m^2(\beta) = m_0^2 + \lambda_0 F(D)\,(m(\beta))^{D-2} \int_{m(\beta)\beta}^\infty d\tau \left((\frac{\tau}{m(\beta)\beta})^2 - 1 \right)^{\frac{D-3}{2}} \frac{1}{e^\tau - 1}. \tag{32}$$

When D is odd, the power $\frac{D-3}{2} = p$ is an integer and the use of the Newton binomial theorem will give a very direct way for evaluating $m^2(\beta)$. When D is even (the most interesting case) the expansion of $\left((\frac{\tau}{m(\beta)\beta})^2 - 1 \right)^{\frac{D-3}{2}}$ yields a infinite power series, and the expression for the thermal squared mass becomes

$$m^2(\beta) = m_0^2 + \lambda_0 \beta^{2-D} \sum_{k=0}^\infty f(D,k)\,(m(\beta)\beta)^{2k} \int_{m(\beta)\beta}^\infty d\tau \frac{\tau^{D-3-2k}}{e^\tau - 1}, \tag{33}$$

where

$$f(D,k) = F(D)(-1)^k C^k_{\frac{D-3}{2}}, \tag{34}$$

and

$$C_p^0 = 1, C_p^1 = \frac{p}{1!}, ..C_p^k = \frac{p(p-1)..(p-k+1)}{k!}, \tag{35}$$

are a generalization of the binomial coeficients. Note that for small values of k the integral that appear in eq.(33) is a Debye integral of the type

$$I_1(x,n) = \int_x^\infty d\tau \frac{1}{e^\tau - 1}\tau^n = \sum_{q=1}^\infty e^{-qx}(\frac{x^n}{q} + \frac{nx^n}{q^2} + ...\frac{n!}{q^{n+1}}), \tag{36}$$

which is valid for $x > 0$ and $n \geq 1$ [18]. For k satisfying $k > \frac{D-4}{2}$, which corresponds to $n < 1$ in the preceeding equation, it is necessary to generalize the Debye integral (the case $n = 0$ is trivial). Let us investigate the case $n < 0$. This generalization has been done by Svaiter and Svaiter [19] and the result reads,

$$I_2(x,n) = \int_x^\infty d\tau \frac{1}{e^\tau - 1}\frac{1}{\tau^n} = -\sum_{q=0, q\neq n}^\infty \frac{B_q}{q!}\frac{x^{q-n}}{q-n} - \frac{1}{(n!)} B_n \ln x + \gamma_{\frac{n-1}{2}}, \tag{37}$$

(for odd n), $Re(x) > 0$, $2\pi > |x| > 0$ and $\gamma_{\frac{n-1}{2}}$ being a constant. The quantites B_n are the Bernoulli numbers. Note that this generalization can be used only for high-temperatures i.e. $m(\beta)\beta < 2\pi$. Thus, in the high temperature regime, if we define

$$I(x, D-3-2k) = \begin{cases} I_1(x, D-3-2k), & for \quad x > 0, \quad k \leq \frac{D-4}{2} \\ I_2(x, D-3-2k), & for \quad 0 < x < \pi, \quad k > \frac{D-4}{2}, \end{cases} \tag{38}$$

we may write

$$m^2(\beta) = m_0^2 + \lambda_0 \beta^{2-D} \sum_{k=0}^{\infty} f(D,k) \left(m(\beta)\beta\right)^{2k} I\left(m(\beta)\beta, D-3-2k\right). \tag{39}$$

The above equation gives a non-perturbative expression for the thermal squared mass in the high temperature regime in the case of even dimensional Euclidean space. In the odd dimensional case the summation in k finishes at $\frac{D-3}{2}$.

For any dimension, it is possible to perform a numerical analysis of the behavior of the renormalized squared mass for all temperatures using eq.(28). It is found that in both cases $D = 3$, and $D = 4$, the thermal squared mass appears as a positive monotonically increasing function of the temperature. For a negative squared mass m_0^2, the model exhibits spontaneous breaking of the $O(N)$ symmetry to $O(N-1)$. Since the thermal correction to the squared mass is positive, the symmetry is restored at sufficiently high temperature. The critical temperature β_c^{-1} is defined as the value of the temperature for which $M^2(\beta)$ vanishes. In the neighborhood of the critical temperature we can use without loss of generality, eq.(28) instead of eq.(25), since the normalized expectation value of the field (the order parameter) vanishes. Using the limiting formula for small arguments of the Bessel function it is not difficult to show that the critical temperature in a generic D-dimensional Euclidean space ($D > 2$) is given by:

$$(\beta_c^{-1})^{D-2} = -\frac{m_0^2}{\lambda g(D)}, \tag{40}$$

where $g(D) = \frac{1}{4\pi^{\frac{D}{2}}}\Gamma(\frac{D}{2}-1)\zeta(D-2)$. In $D=4$ the above result reproduces the known value of the critical temperature. For $D=3$ the zeta function in $g(D)$ has a pole and a renormalization procedure implies that the quantity β^{D-2} is proportional to the the regular part of the analytic extension of the zeta function in the neighborhood of the pole. It is not difficult to show that in this case the critical temperature is given by $\frac{8\pi m^2}{\lambda_0} = \beta_c^{-1} ln(\mu^2\beta_c^2)$. This result is in agreement with the estimate of Einhorn et al. [20], which has been recently confirmed by numerical simulation [21].

4 The thermal coupling constant for the Vector $\lambda\varphi^4_D$ model

In this section we investigate the behavior of the renormalized thermal coupling constant of the vector N-component $\lambda\varphi^4_D$ model. Again, withouth loss of generality, let us suppose that we are in the symmetric phase i.e $m_0^2 > 0$. To go beyond perturbation theory, we take the leading order in $\frac{1}{N}$, in which case we know that the contributions come only from some classes of diagrams (the chains of elementary four-point bubbles) and that it is it is possible to perform summations over them. Proceeding in that way, we get for the thermal renormalized coupling constant an expression of the form,

$$\lambda(\beta) = \frac{\lambda_0}{1-\lambda_0 L\left(m^2(\beta),\beta\right)}, \tag{41}$$

where

$$L\left(m^2(\beta),\beta\right) = -\frac{3}{2}\frac{1}{(2\pi)^{D/2}}\sum_{n=1}^{\infty}\left(\frac{m(\beta)}{\beta n}\right)^{\frac{D}{2}-2} K_{\frac{D}{2}-2}\left(m(\beta)n\beta\right), \tag{42}$$

and λ_0 is the zero-temperature renormalized coupling constant. For simplicity we have suppresed the factor $\frac{1}{N}$ everywhere. Incidentally, before going into more details of the non-perturbative finite temperature case, some comments are in order. We remark that at zero temperature and $D = 4$ it is known that the theory has tachyons, which are related to the occurence of a Landau pole in $\lambda(m^2)$. The conventional way to circunvect this problem is to interpret the model as an effective theory (introducing a ultraviolet cut-off) which restricts the energy to a region far below the tachyon mass. This is a necessary procedure, to take into account the widelly spread conjecture that the $\lambda\varphi^4$ model has a trivial continuum limit at $D = 4$ and is meaningful only as an effective theory. Coming back to finite temperature, firstly let us investigate the thermal behavior of the coupling constant in a Euclidean space satisfying $D > 3$. Proceeding in a manner analogous as we have done in the previous section, we use again an integral representation of the Bessel function given by eq.(29), which leads to the result,

$$L(m^2(\beta), \beta) = G(D)\,(m(\beta))^{D-4} \int_1^\infty dt(t^2-1)^{\frac{D-5}{2}} \frac{1}{e^{m(\beta)\beta t}-1} \tag{43}$$

where

$$G(D) = -\frac{3}{2}\frac{1}{(2\sqrt{\pi})^{D-1}}\frac{1}{\Gamma(\frac{D-3}{2})}. \tag{44}$$

Note that there are no poles in the Gamma function and $G(D)$ never vanishes. Defining

$$g(D,k) = G(D)(-1)^k C^k_{\frac{D-5}{2}} \tag{45}$$

it is not dificult to show that

$$L\left(m^2(\beta), \beta\right) = \beta^{-D}\sum_{k=0}^{\infty} g(D,k)\,(m(\beta)\beta)^{2k+2}\int_{m(\beta)\beta}^{\infty} d\tau \frac{\tau^{D-5-2k}}{e^\tau - 1}. \tag{46}$$

A straigthforward calculation gives us the following expression

$$L\left(m^2(\beta), \beta\right) = \beta^{-D}\sum_{k=0}^{\infty} g(D,k)\,(m(\beta)\beta)^{2k+2}\, I\left(m(\beta)\beta, D-5-2k\right), \tag{47}$$

where again $I(x,n)$ is defined in eq.(38). Then, substituting eq.(47) in the eq.(41) we get the high temperature thermal coupling constant for $D > 3$. In the case $D \leq 3$, the integral representation of the Bessel function gived by eq.(29) can not be used. Consequently, we take another integral representation of the Bessel function i.e.

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty dt\ e^{-t-\frac{z^2}{4t}}\ t^{-(\nu+1)} \tag{48}$$

which is valid for $|arg(z)| < \frac{\pi}{2}$ and $Re(z^2) > 0$. A straighforward calculation gives

$$L\left(m^2(\beta), \beta\right) = Q(D) m(\beta)^{D-4} \int_0^\infty dt\ e^{-t}\ t^{-\frac{D}{2}+1}\left(\Theta_3(\pi, e^{-\frac{m^2\beta^2}{4t}}) - 1\right), \tag{49}$$

where the theta function $\Theta_3(z,q)$ is defined by [18]:

$$\Theta_3(z,q) = 1 + 2\sum_{n=1}^\infty q^{n^2}\cos(2nz). \tag{50}$$

and $Q(D) = -\frac{3}{2}\frac{1}{(2\sqrt{\pi})^D}$. Substituting $L\left(m^2(\beta), \beta\right)$ in eq.(41) we have a closed expression for the thermal renormalized coupling constant in the case $D \leq 3$. It is interesting to note that a long time ago Braden [22] has employed also the theta function $\Theta_3(z,q)$, to investigate the finite temperature $\lambda\varphi^4$ model. We would like to stress that the behavior of the thermal renormalized coupling constant is quite different from the monotonically increasing in temperature behavior obtained for the squared mass. Indeed, the kind of the thermal behavior of the coupling constant depends on the Euclidean dimension. For $D = 3$ the coupling constant (as a function of the temperature) decrease until some minimum value and then start to increase. This thermal behavior of the coupling constant is ploted in fig.(1). For $D = 4$ the thermal renormalized coupling constant tends to a constant value in the high temperature limit. See fig.(2). Our result is consistent with the work of Fendley [23].

It must be noticed that the thermal behaviour of the coupling constant is very sensitive to the thermal behaviour of the mass. As an ilustration of this fact we exhibit in fig.(3) the general aspect of the coupling constant as function of the temperature for the same model we have treated here, but subjected to Wick ordering [24]. In this case all tadpoles are suppressed and the thermal behaviour of the coupling constant does not depend at all on the mass thermal behaviour. We see that in this situation the coupling constant is a monotonic decreasing function of the temperature. The absence of Wick ordering deeply changes this behaviour. We show in fig.(4) for $D = 3$ in the same scale the plots for $\lambda(T)$ with and without Wick ordering, respectivelly the lower and the upper curves. In the region of temperatures where $\lambda_W(T)$ goes practically to zero, $\lambda(T)$ is practically constant at a value slightly lower than the common zero-temperature coupling constant λ_0. The growth of $\lambda(T)$ with the temperature presented in fig.(1) is in a much smaller scale for $\lambda(T)$ than in fig.(4). In fact this growth is "microscopic" in a scale where the Wick ordered coupling constant $\lambda_W(T)$ presents asymptotic thermal freedom.

5 Conclusions

We have done in this paper an analysis of the vector $\lambda\varphi^4$ model in a flat D-dimensional Euclidean space in equilibrium with a thermal bath. The form of the thermal corrections to the mass and coupling constant have been discussed using resummation methods. We have formulated the problem in a general framework, but our results are at leading order in the $\frac{1}{N}$ expansion. We have chosen this way of working, in order to get answers as much as possible of a non-perturbative character. In what concerns the thermal mass behavior, we have shown that the thermal renormalized squared mass is a monotonic increasing function of the temperature for any Euclidean dimension. We have been able to obtain a general formula for the critical temperature of the second order phase transition valid for any Euclidean dimension $D > 2$, provided the necessary renormalization procedure is done to circunvect singularities of the zeta function. The values obtained for the critical temperatures in $D = 3$ and $D = 4$ agree with previous results.

The behavior of the thermal coupling constant depends on the dimensionality of the Euclidean space. In $D = 3$ the renormalized coupling constant decreases until some positive minimum value and then starts to increase slightly as a function of the temperature. See fig.(1). This result seems to indicate that at a non-perturbative level (for large N) in the framework of the vector N-component model, the answer to the question raised in Ref.([5]) is negative: there is no first order phase transition induced by the thermal coupling constant in $D = 3$. In $D = 4$ the thermal renormalized coupling constant in the high temperature limit tends to a constant value, $(\lambda_0 - (3\frac{\sqrt{6}}{8\pi})\lambda_0^{\frac{3}{2}}$, which coincides exactly with the result obtained by Fendley [23]).

A natural extension of this work should be to go beyond the $\frac{1}{N}$ leading order results using renormalization group methods. Another possible direction is to introduce an abelian gauge field coupled to the N-component scalar field. In $D = 3$ a topological Chern-Simons term may be added, and also a φ^6 term. In this case we have shown in a previous work using perturbative and semi-classical techniques that the topological mass makes appear a richer phase structure introducing the possibility of first or second order phase transitions depending on the value of the topological mass [25]. The use of resummation methods to investigate the thermal behavior of the physical quantities could generalize these previous results to the N-component vector model. These will be subjects of future investigation.

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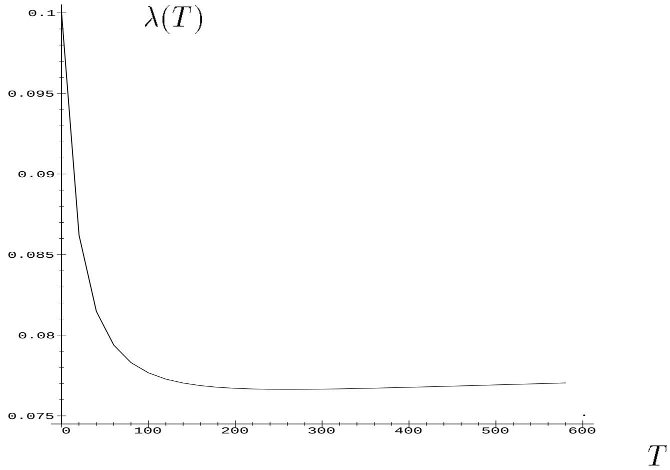

Figure 1: Coupling constant thermal behavior obtained from eqs. (28),(41) and (42) in dimension $D = 3$.

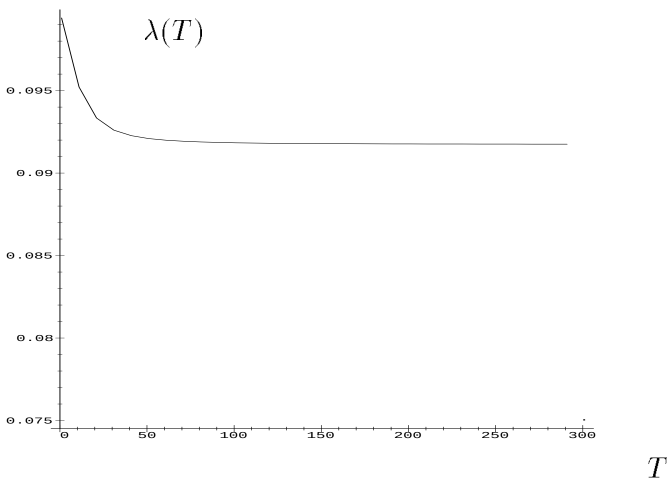

Figure 2: Coupling constant thermal behavior obtained from eqs. (28),(41) and (42) in dimension $D = 4$.

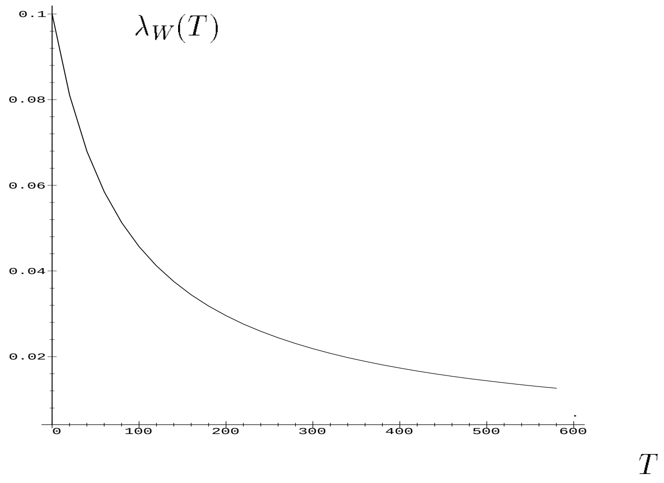

Figure 3: General aspect of coupling constant thermal behavior obtained from eqs. (28),(41) and (42) for the Wick ordered model.

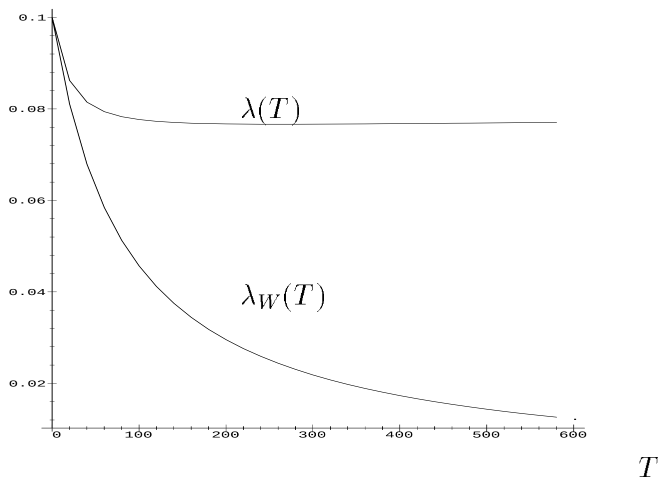

Figure 4: Compared thermal behaviors of the coupling constant for the Wick-ordered model and non Wick-ordered model.