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Stochastic Quantization of Scalar Fields in de Sitter Spacetime

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Abstract

We consider the stochastic quantization method for scalar fields defined in a curved manifold. The two-point function associated to a massive self-interacting scalar field is evaluated, up to the first order level in the coupling constant λ, for the case of de Sitter Euclidean metric. Its value for the asymptotic limit of the Markov parameter $\tau \to \infty$ is exhibited. We discuss in detail the covariant stochastic regularization to render the one-loop two-point function finite in the de Sitter Euclidean metric.

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1 Introduction

The program of stochastic quantization [1] and the stochastic regularization was carried out for generic fields defined in flat, Euclidean manifolds. For reviews of this program, see the Refs. [2] [3] [4] [5]. In the development of this program some authors applied this method to linearized Euclidean gravity [6] [7] and also non-linearized gravity. We may observe, as it was remarked earlier [8], that the study of a situation which lies between these two extremes is missing. A consistent logical step is to discuss an intermediate situation between fields in flat spacetime and quantum gravity, i.e., the semiclassical theory [9] [10].

The stochastic quantization method was used recently to study self-interacting scalar fields in manifolds which can be analytically continued to the Euclidean situation, i.e., the Einstein and the Rindler space [8]. First, these authors solved a Langevin equation for the mode coefficients of the field, then they exhibit the two-point function at the one-loop level. It was shown that it diverges and they used a covariant stochastic regularization to render it finite. It was shown that, indeed, the two-point function is regularized. It is important to remark that this procedure of analytically extend the real manifold to a complex one in the above situations renders the action a real quantity, allowing the implementation of the stochastic quantization in a straightforward way.

Our aim in this article is to discuss the stochastic quantization of scalar fields defined in a curved manifold, being more specific, the de Sitter spacetime. For a pedagogical material discussing the classical geometry of the de Sitter space and the quantum field theory see, for instance, the Refs. [11] [12] [13] [14] [15]. Note that the stochastic quantization is quite different from the other quantization methods, therefore it can reveal new structural elements of a theory which so far have gone unnoticed. For example, a quite important point in a regularization procedure is that it must preserve all the symmetries of the unregularized Lagrangian. Many authors have stressed that a priori we can not expect that a regularization independent proof of the renormalization of theories in a curved background exists. The presence of the Markov parameter as an extra dimension lead us to a regularization scheme, which preserves all the symmetries of the theory under study. Since the stochastic regularization is not an action regularization, it may be a way to construct such proof. As a starting point of this program, for instance, in the $\lambda\varphi^4$ theory, we should calculate the two-point function up to the first order level in the coupling constant λ and apply the continuum stochastic regularization. Our results are to be compared with the usual ones in the literature.

The organization of the paper is the following: in section II we discuss the stochastic quantization for the $(\lambda\varphi^4)_d$ scalar theory in a d-dimensional Euclidean manifold. In section III we use the stochastic quantization and the stochastic regularization to obtain the two-point Schwinger function in the one-loop approximation in the Euclidean de Sitter manifold. Conclusions are given in the section IV. In this paper we use $\hbar = c = k_B = G = 1$.

2 Stochastic quantization for the $(\lambda\varphi^4)_d$ scalar theory: the Euclidean case

In this section, we give a brief survey for the case of self-interacting scalar fields, implementing the stochastic quantization and the continuum stochastic regularization theory up to the one-loop level. Let us consider a neutral scalar field with a $(\lambda\varphi^4)$ self-interaction. The Euclidean action that usually describes a free scalar field is

$$S_0[\varphi\,] = \int d^d x\, \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2} m_0^2\, \varphi^2(x)\right), \tag{1}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^d x\, \frac{\lambda}{4!}\, \varphi^4(x). \tag{2}$$

The simplest starting point of the stochastic quantization to obtain the Euclidean field theory is a Markovian Langevin equation. Assume a flat Euclidean d-dimensional manifold, where we are choosing periodic boundary conditions for a scalar field and also a random noise. In other words, they are defined in a d-torus $\Omega \equiv T^{\,d}$. To implement the stochastic quantization we supplement the scalar field $\varphi(x)$ and the random noise $\eta(x)$ with an extra coordinate τ, the Markov parameter, such that $\varphi(x) \to \varphi(\tau, x)$ and $\eta(x) \to \eta(\tau, x)$. Therefore, the fields and the random noise are defined in a domain: $T^{\,d} \times R^{\,(+)}$. Let us consider that this dynamical system is out of equilibrium, being described by the following equation of evolution

$$\frac{\partial}{\partial\tau}\varphi(\tau, x) = -\frac{\delta\, S_0}{\delta\, \varphi(x)}\Big|_{\varphi(x)=\varphi(\tau,\, x)} + \eta(\tau, x), \tag{3}$$

where τ is a Markov parameter, $\eta(\tau, x)$ is a random noise field and $S_0[\varphi\,]$ is the usual free Euclidean action defined in Eq. (1). For a free scalar field, the Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau, x) = -(-\Delta + m_0^2\,)\varphi(\tau, x) + \eta(\tau, x), \tag{4}$$

where Δ is the d-dimensional Laplace operator. The Eq. (4) describes a Ornstein-Uhlenbeck process. Assuming a Gaussian noise distribution we have that the random noise field satisfies

$$\langle\, \eta(\tau, x)\, \rangle_\eta = 0 \tag{5}$$

and

$$\langle\, \eta(\tau, x)\, \eta(\tau', x')\, \rangle_\eta\, = 2\delta(\tau - \tau')\, \delta^d(x - x'), \tag{6}$$

where $\langle\, ...\rangle_\eta$ means stochastic averages. The above equation for the two-point correlation function defines a delta-correlated random process. In a generic way, the stochastic average for any functional of φ given by $F[\varphi\,]$ is defined by

$$\langle\, F[\varphi\,]\, \rangle_\eta = \frac{\int [d\eta] F[\varphi\,] \exp\Big[-\frac{1}{4}\int d^d x \int d\tau\, \eta^2(\tau, x)\Big]}{\int [d\eta] \exp\Big[-\frac{1}{4}\int d^d x \int d\tau\, \eta^2(\tau, x)\Big]}. \tag{7}$$

Let us define the retarded Green function for the diffusion problem that we call $G(\tau - \tau', x - x')$. The retarded Green function satisfies $G(\tau - \tau', x - x') = 0$ if $\tau - \tau' < 0$ and otherwise also

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right] G(\tau - \tau', x - x') = \delta^d(x - x')\delta(\tau - \tau'). \tag{8}$$

Using the retarded Green function and the initial condition $\varphi(\tau, x)|_{\tau=0} = 0$, the solution for Eq. (4) reads

$$\varphi(\tau, x) = \int_0^\tau d\tau' \int_\Omega d^dx'\, G(\tau - \tau', x - x')\eta(\tau', x'). \tag{9}$$

Let us define the Fourier transforms for the field and the noise given by $\varphi(\tau, k)$ and $\eta(\tau, k)$. We have respectively

$$\varphi(\tau, k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dx\, e^{-ikx}\, \varphi(\tau, x), \tag{10}$$

and

$$\eta(\tau, k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^dx\, e^{-ikx}\, \eta(\tau, x). \tag{11}$$

Substituting Eq. (10) in Eq. (1), the free action for the scalar field in the $(d + 1)$-dimensional space writing in terms of the Fourier coefficients reads

$$S_0[\varphi(k)]\,|_{\varphi(k)=\varphi(\tau,\, k)} = \frac{1}{2} \int d^dk\, \varphi(\tau, k)(k^2 + m_0^2)\varphi(\tau, k). \tag{12}$$

Substituting Eq. (10) and Eq. (11) in Eq. (4) we have that each Fourier coefficient satisfies a Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi(\tau, k) = -(k^2 + m_0^2)\varphi(\tau, k) + \eta(\tau, k). \tag{13}$$

In the Langevin equation the particle is subject to a fluctuating force (representing a stochastic environment), where its average properties are presumed to be known and also the friction force. Note that the "friction coefficient" in the Eq. (13) is given by $(k^2 + m_0^2)$.

The solution for Eq. (13) reads

$$\varphi(\tau, k) = \exp\Big(-(k^2 + m_0^2)\tau\Big)\, \varphi(0, k) + \int_0^\tau d\tau' \exp\Big(-(k^2 + m_0^2)(\tau - \tau')\Big)\, \eta(\tau', k). \tag{14}$$

Using the Eq. (5) and Eq. (6), we get that the Fourier coefficients for the random noise satisfy

$$\langle\, \eta(\tau, k)\, \rangle_\eta = 0 \tag{15}$$

and

$$\langle\, \eta(\tau, k)\eta(\tau', k')\, \rangle_\eta = 2\delta(\tau - \tau')\delta^d(k + k'). \tag{16}$$

It is possible to show that $\langle\, \varphi(\tau, k)\varphi(\tau', k')\, \rangle_\eta|_{\tau=\tau'} \equiv D(k, k'; \tau, \tau')$ is given by:

$$D(k; \tau, \tau) = (2\pi)^d \delta^d(k + k') \frac{1}{(k^2 + m_0^2)} \Big(1 - \exp\Big(-2\tau(k^2 + m_0^2)\Big)\Big) \tag{17}$$

where we assume $\tau = \tau'$.

Now let us analyze the stochastic quantization for the $(\lambda\varphi^4)_d$ self-interaction scalar theory. In this case the Langevin equation reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -(-\Delta + m_0^2)\varphi(\tau,x) - \frac{\lambda}{3!}\varphi^3(\tau,x) + \eta(\tau,x). \tag{18}$$

The two-point correlation function associated with the random field is given by the Eq. (6), while the other connected correlation functions vanish, i.e.,

$$\langle\, \eta(\tau_1,x_1)\eta(\tau_2,x_2)...\eta(\tau_{2k-1},x_{2k-1})\,\rangle_\eta = 0, \tag{19}$$

and also

$$\langle\eta(\tau_1,x_1)...\eta(\tau_{2k},x_{2k})\,\rangle_\eta = \sum \langle\eta(\tau_1,x_1)\eta(\tau_2,x_2)\,\rangle_\eta \langle\, \eta(\tau_k,x_k)\eta(\tau_l,x_l)\,\rangle_\eta ..., \tag{20}$$

where the sum is to be taken over all the different ways in which the $2k$ labels can be divided into k parts, i.e., into k pairs. Performing Gaussian averages over the white random noise, it is possible to prove the important formulae

$$\lim_{\tau\to\infty}\langle\, \varphi(\tau_1,x_1)\varphi(\tau_2,x_2)...\varphi(\tau_n,x_n)\,\rangle_\eta = \frac{\int[d\varphi]\varphi(x_1)\varphi(x_2)...\varphi(x_n)\, e^{-S(\varphi)}}{\int[d\varphi]\, e^{-S(\varphi)}}, \tag{21}$$

where $S[\varphi] = S_0[\varphi] + S_I[\varphi]$ is the d-dimensional action. This result leads us to consider the Euclidean path integral measure a stationary distribution of a stochastic process. Note that the solution of the Langevin equation needs a given initial condition. As for example

$$\varphi(\tau,x)|_{\tau=0} = \varphi_0(x). \tag{22}$$

Let us use the Langevin equation to perturbatively solve the interacting field theory. One way to handle the Eq. (18) is with the method of Green's functions. We defined the retarded Green function for the diffusion problem in the Eq. (8). Let us assume that the coupling constant is a small quantity. Therefore to solve the Langevin equation in the case of a interacting theory we use a perturbative series in λ. Therefore we can write

$$\varphi(\tau,x) = \varphi^{(0)}(\tau,x) + \lambda\varphi^{(1)}(\tau,x) + \lambda^2\varphi^{(2)}(\tau,x) + ... \tag{23}$$

Substituting the Eq. (23) in the Eq. (18), and if we equate terms of equal power in λ, the resulting equations are

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(0)}(\tau,x) = \eta(\tau,x), \tag{24}$$

$$\left[\frac{\partial}{\partial\tau} + (-\Delta_x + m_0^2)\right]\varphi^{(1)}(\tau,x) = -\frac{1}{3!}\left(\varphi^{(0)}(\tau,x)\right)^3, \tag{25}$$

and so on. Using the retarded Green function and assuming that $\varphi^{(q)}(\tau,x)|_{\tau=0} = 0,\ \forall\, q$, the solution to the first equation given by Eq. (24) can be written formally as

$$\varphi^{(0)}(\tau,x) = \int_0^\tau d\tau' \int_\Omega d^dx'\, G(\tau-\tau',x-x')\eta(\tau',x'). \tag{26}$$

The second equation given by Eq. (25) can also be solved using the above result. We obtain

$$\varphi^{(1)}(\tau,x) = -\frac{1}{3!}\int_0^\tau d\tau_1\int_\Omega d^dx_1\,G(\tau-\tau_1,x-x_1)$$
$$\left(\int_0^{\tau_1} d\tau'\int_\Omega d^dx'\,G(\tau_1-\tau',x_1-x')\eta(\tau',x')\right)^3. \qquad (27)$$

We have seen that we can generate all the tree diagrams with the noise field contributions. We can also consider the n-point correlation function $\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)...\varphi(\tau_n,x_n)\,\rangle_\eta$. Substituting the above results in the n-point correlation function, and taking the random averages over the white noise field using the Wick-decomposition property defined by Eq. (20) we generate the stochastic diagrams. Each of these stochastic diagrams has the form of a Feynman diagram, apart from the fact that we have to take into account that we are joining together two white random noise fields many times. Besides, the rules to obtain the algebraic values of the stochastic diagrams are similar to the usual Feynman rules.

As simple examples let us show how to derive the two-point function in the zeroth order $\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(0)}$, and also the first order correction to the scalar two-point-function given by $\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(1)}$. Using the Eq. (5), Eq. (6) and also Eq. (9) we have

$$\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(0)} = 2\int_0^{min(\tau_1,\tau_2)} d\tau'\int_\Omega d^dx'\,G(\tau_1-\tau',x_1-x')\,G(\tau_2-\tau',x_2-x'). \quad (28)$$

For the first order correction we get:

$$\langle\,\varphi(X_1)\varphi(X_2)\,\rangle_\eta^{(1)} =$$
$$= -\frac{\lambda}{3!}\langle\int dX_3\int dX_4\Big(G(X_1-X_4)G(X_2-X_3)+G(X_1-X_3)G(X_2-X_4)\Big)$$
$$\eta(X_3)\left(\int dX_5\,G(X_4-X_5)\eta(X_5)\right)^3\rangle_\eta. \qquad (29)$$

where, for simplicity, we have introduced a compact notation:

$$\int_0^\tau d\tau\int_\Omega d^dx \equiv \int dX, \qquad (30)$$

and also $\varphi(\tau,x)\equiv\varphi(X)$ and finally $\eta(\tau,x)\equiv\eta(X)$.

The process can be repeated and therefore the stochastic quantization can be used as an alternative approach to describe scalar quantum fields. Therefore, the two-point function up to the first order level in the coupling constant λ is given by

$$\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(1)} = (a)+(b)+(c), \qquad (31)$$

where (a) is the zero order two-point function and (b) and (c) are given, respectively, by:

$$(b) = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\int d^dk\int_0^{\tau_1} d\tau\,G(k_1;\tau_1-\tau)D(k;\tau,\tau)D(k_2;\tau_2,\tau), \qquad (32)$$

$$(c) = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\int d^dk\int_0^{\tau_2} d\tau\,G(k_2;\tau_2-\tau)D(k;\tau,\tau)D(k_1;\tau_1,\tau). \qquad (33)$$

These are the contributions in first order. A simple computation shows that we recover the correct equilibrium result at equal asymptotic Markov parameters ($\tau_1 = \tau_2 \to \infty$):

$$(b)|_{\tau_1=\tau_2\to\infty} = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\frac{1}{(k_2^2+m_0^2)}\frac{1}{(k_1^2+k_2^2+2m_0^2)}\int d^dk\frac{1}{(k^2+m_0^2)}. \tag{34}$$

Obtaining the Schwinger functions in the asymptotic limit does not guarantee that we gain a finite physical theory. The next step is to implement a suitable regularization scheme. A crucial point to find a satisfactory regularization scheme is to use one that preserves the symmetries of the original model. The presence of the Markov parameter as an extra dimension lead us to a new regularization scheme, the stochastic regularization method, which preserves all the symmetries of the theory under study. Therefore, let us implement a continuum regularization procedure [16] [17] [18] [19] [20]. We begin with a regularized Markovian Parisi-Wu Langevin system:

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\frac{\delta S_0}{\delta\varphi(x)}\Big|_{\varphi(x)=\varphi(\tau,x)} + \int d^dy\,R_{xy}(\Delta)\,\eta(\tau,y). \tag{35}$$

Since we are still assuming a Gaussian noise distribution we have that the random noise field still satisfies the relations given by Eqs. (5) and (6). The regulator $R(\Delta)$ that multiplies the noise is a function of the Laplacian:

$$\Delta_{xy} = \int d^dz\,(\partial_\mu)_{xz}(\partial_\mu)_{zy}, \tag{36}$$

where

$$(\partial_\mu)_{xy} = \partial_\mu^x\delta^d(x-y). \tag{37}$$

In this paper we will be working with a heat kernel regulator with the form:

$$R(\Delta;\Lambda) = \exp\Big(\frac{\Delta}{\Lambda^2}\Big), \tag{38}$$

where Λ is a parameter introduced to regularize the theory. The basic restrictions on the form of this heat kernel regulator are:

$$R(\Delta;\Lambda)|_{\Lambda\to\infty} = 1, \tag{39}$$

or

$$R_{xy}(\Delta;\Lambda)|_{\Lambda\to\infty} = \delta^d(x-y), \tag{40}$$

which guarantees that the regularized process given by Eq. (35) reduces to the formal process given by Eq. (3) in the formal regulator limit $\Lambda\to\infty$.

With this modification in the Langevin equation, it is possible to show that all the contributions to the n-point function at all orders in the coupling constant λ are finite. For instance, the contribution to the two-point function at the one-loop level given by Eq. (32) is rewritten as:

$$(b)|_{\tau_1=\tau_2\to\infty} = -\frac{\lambda}{2}\,\delta^d(k_1+k_2)\frac{R_{k_2}^2}{(k_2^2+m_0^2)}\frac{1}{(k_1^2+k_2^2+2m_0^2)}\int d^dk\frac{R_k^2}{(k^2+m_0^2)}, \tag{41}$$

where R_k is the Fourier transform of the regulator, i.e.,

$$R_k(\Lambda) = R(\Delta;\Lambda)|_{\Delta=-k^2}. \tag{42}$$

Now we use this method to discuss the quantization of scalar theories with self-interaction in a curved spacetime with event horizon. Being more specific, we are interested to investigate the $\lambda\varphi^4$ theory in de Sitter Euclidean manifold.

3 Stochastic quantization for the $(\lambda\varphi^4)_d$ scalar theory: the de Sitter case

The aim of this section is to implement the stochastic quantization and the stochastic regularization for the self-interacting $\lambda\varphi^4$ theory in the one-loop level in the de Sitter spacetime. Let us consider a M^4 manifold that admit a non-vanishing timelike Killing vector field X. If one can always introduce coordinates $t = x^0, x^1, x^2, x^3$ locally such that $X = \frac{\partial}{\partial t}$ and the components of the metric tensor are independent of t, M^4 is stationary. If further the distribution $X^{\perp}$ of 3-planes orthogonal to X is integrable, then M^4 is static. Each integral curve of the Killing vector vector field $X = \frac{\partial}{\partial t}$ is a world line of an possible observer. Since $X = \frac{\partial}{\partial t}$ generates isometries, the 3-planes $X^{\perp}$ are invariant under these isometries. For static manifold, it is possible to perform a Wick rotation, i.e., analytically extend the pseudo-Riemannian manifold to the Riemannian domain without problem. Therefore for static spacetime the implementation of the stochastic quantization is straightforward.

In the previous section, we have been working in an Euclidean space $R^{\,d} \times R^{\,(+)}$, where $R^{\,d}$ is the usual Euclidean space and $R^{\,(+)}$ is the Markov sector. Now let us generalize this to a more complicated case, i.e., let us work in de Sitter manifold M. In other words, we will consider a classical field theory defined in a $M \times R^{\,(+)}$ manifold coupled with a heat reservoir.

In general, we may write the mode decompositions as:

$$\varphi(\tau, x) = \int d\tilde{\mu}(k)\varphi_k(\tau)u_k(x), \tag{43}$$

and

$$\eta(\tau, x) = \int d\tilde{\mu}(k)\eta_k(\tau)u_k(x), \tag{44}$$

where the measure $\tilde{\mu}(k)$ depends on the metric we are interested in. For instance, in the flat case, we have that in a four dimensional space $d\tilde{\mu}(k) = d^4k$ and the modes $u_k(x)$ are given by:

$$u_k(x) = \frac{1}{(2\pi)^2}e^{ikx}. \tag{45}$$

Four-dimensional de Sitter space is most easily represented as the hyperboloid [21] [22]

$$z_0^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 = -\alpha^2, \tag{46}$$

embedded in five-dimensional Minkowski space with metric

$$ds^2 = dz_0^2 - dz_1^2 - dz_2^2 - dz_3^2 - dz_4^2 \tag{47}$$

¿From the form of Eq. (46), we see that the symmetry group of de Sitter space is the ten parameter group $SO(1, 4)$ of homogeneous Lorentz transformations in the five-dimensional embedding space known as the de Sitter group. Just as the Poincaré group plays a central role in the quantization of fields in Minkowski space, so the de Sitter group of symmetries on de Sitter space is fundamental to the discussion of its quantization. The de Sitter space-time $S_{1,3}$ is of constant curvature. Its Ricci curvature is equal to $\frac{n(n-1)}{\alpha^2}$.

Let us introduce in $S_{1,3}$ the coordinates $x^\beta = (t, \xi^{\,i})$, where $\beta, \delta, \gamma = 0,1,2,3$ and $i,j = 1,2,3$. We are following the Refs. [23] [24]. We have:

$$z^0 = \alpha \tan t; \quad -\pi/2 < t < \pi/2 \tag{48}$$

$$z^a = \frac{\alpha}{\cos t} k^a(\xi^1, \xi^2, \xi^3); \quad a,b = 1,2,3,4, \tag{49}$$

ξ^1, ξ^2, ξ^3 being coordinates on the sphere $k_1^2 + k_2^2 + k_3^2 + k_4^2 = 1$. The infinitely remote "future" ("past") corresponds to the value $t = \pi/2$ ($t = -\pi/2$) of the time-like coordinate t. The three-dimensional spheres $z^0 = const.$ are hypersurfaces of equal time.

If we denote:

$$dk_1^2 + dk_2^2 + dk_3^2 + dk_4^2 = \omega_{ij}(\xi^1, \xi^2, \xi^3)\, d\xi^i\, d\xi^j, \tag{50}$$

where $\omega_{ij} = \frac{\partial k_a}{\partial \xi_i}\frac{\partial k_a}{\partial \xi_j}$ is the metric tensor of S_3, the interval of the de Sitter space-time is written as

$$ds^2 = \frac{\alpha^2}{\cos^2 t}\Big(dt^2 - \omega_{ij}(\xi^1, \xi^2, \xi^3)\, d\xi^i\, d\xi^j\Big). \tag{51}$$

With an usual Wick rotation, we end up in the Euclidean de Sitter space, written in the coordinates above.

The modes for the conformally coupled scalar field equation for the Euclidean de Sitter space are given by

$$u^{\pm}_{ps\sigma}(x) = N\sqrt{s+1}\,\Phi_{s\sigma}\,[k(\xi)] f^{\pm}_p(t)\,\cosh t, \tag{52}$$

with $s = 0,1,2,...$, $\sigma = 1,...,(s+1)^2$. The functions $\Phi_{s\sigma}$ are basic orthonormal harmonic polynomials in k of degree s. They are labeled by the index σ. The $f^{\pm}_p(t)$ are expressed through a hypergeometric function

$$f^{\pm}_p(t) = \frac{1}{(ip)!}\sqrt{\Gamma(ip+\mu)\Gamma(ip-\mu+1)}\, e^{\pm ip\,t} F\Big(\mu, 1-\mu; ip+1; \frac{1 \pm \tanh t}{2}\Big), \tag{53}$$

with $\mu = \frac{1}{2}(1 - \sqrt{1-4m^2})$ and $m = m_0\alpha$. The measure for the mode decomposition for Eqs. (43) and (44) is given by

$$\int d\tilde{\mu}_{ps\sigma} = \frac{1}{2\pi}\int dp \sum_{s,\,\sigma}. \tag{54}$$

For fields defined in spaces with a general line element given by

$$ds^2 = g_{00}dt^2 + h_{ij}dx^i dx^j, \tag{55}$$

is possible to prove that the the Parisi-Wu Langevin equation for scalar fields in Euclidean de Sitter manifold reads

$$\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\frac{g_{00}}{\sqrt{g}}\,\frac{\delta\, S_0}{\delta\,\varphi(x)}\Big|_{\varphi(x)=\varphi(\tau,\,x)} + \eta(\tau,x), \tag{56}$$

where $g = \det g_{\mu\nu}$. The classical Euclidean action S_0 that appears in the above equation is given by

$$S_0 = \int d^4x\,\sqrt{g}\,\mathcal{L}, \tag{57}$$

where the Lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}\, g_{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi + \frac{1}{2}\,(m_0^2 + \xi R)\varphi^2. \tag{58}$$

Note that we introduce a coupling between the scalar field and the gravitational field represented by the term $\xi R\varphi^2$, where ξ is a numerical factor and R is the Ricci scalar curvature. Assuming a conformally coupled field, we have $\xi = \frac{1}{4}\frac{(n-2)}{(n-1)}$. The random noise field $\eta(\tau, x)$ obeys the Gaussian statistical law:

$$\langle\, \eta(\tau, x)\, \rangle_\eta = 0 \tag{59}$$

and

$$\langle\, \eta(\tau, x)\, \eta(\tau', x')\, \rangle_\eta \;=\; \frac{2}{\sqrt{g(x)}}\, \delta^4(x - x')\, \delta(\tau - \tau'). \tag{60}$$

Substituting Eq. (57) and Eq. (58) in the Langevin equation given by Eq. (56), we get

$$\frac{\partial}{\partial\tau}\varphi(\tau, x) = -g_{00}\Big(-\Delta + m_0^2 + \frac{2}{\alpha^2}\Big)\varphi(\tau, x) + \eta(\tau, x). \tag{61}$$

Denoting the covariant derivative by ∇, we can define the four-dimensional Laplace-Beltrami operator Δ by

$$\begin{aligned}\Delta &= g^{-1/2}\partial_\mu(g^{1/2} g_{\mu\nu}\partial_\nu)\\ &= g_{\mu\nu}\nabla_\mu\nabla_\nu.\end{aligned} \tag{62}$$

To proceed, as in the flat situation, let us introduce the retarded Green function for the diffusion problem $G(\tau - \tau', x - x')$, which obeys

$$\left[\frac{\partial}{\partial\tau} + g_{00}\Big(-\Delta_x + m_0^2 + \frac{2}{\alpha^2}\Big)\right] G(\tau - \tau', x - x') = \frac{1}{\sqrt{g}}\, \delta^4(x - x')\delta(\tau - \tau'), \tag{63}$$

if $\tau - \tau' > 0$, and $G(\tau - \tau', x, x') = 0$ if $\tau - \tau' < 0$.

Using the retarded Green function and the initial condition $\varphi(\tau, x)|_{\tau=0} = 0$, a formal solution for Eq. (61) reads

$$\varphi(\tau, x) = \int_0^\tau d\tau' \int_\Omega d^4x'\, \sqrt{g(x')}\, G(\tau - \tau', x - x')\eta(\tau', x'). \tag{64}$$

Inserting the modes given by Eq. (52) in Eq. (61) we have that each mode coefficient satisfy the Langevin equation given by

$$\frac{\partial}{\partial\tau}\varphi_q(\tau) = -(q^2 + 1)\varphi_q(\tau) + \eta_q(\tau), \tag{65}$$

where $q^2 = p^2 + \kappa^2$ and $\kappa^2 = s(s + 2)$.

The solution for Eq. (65), with the initial condition $\varphi_q(\tau)|_{\tau=0} = 0$, reads:

$$\varphi_q(\tau) = \int_0^\tau d\tau' G_q(\tau, \tau')\eta_q(\tau'), \tag{66}$$

where

$$G_q(\tau, \tau') = \exp\Big(-(q^2 + 1)(\tau - \tau')\Big)\, \theta(\tau - \tau') \tag{67}$$

is the retarded Green function for the diffusion problem.

Using the relations given by Eq. (59) and Eq. (60), we get that the mode coefficients for the random noise satisfies

$$\langle\, \eta_q(\tau)\,\rangle_\eta = 0 \tag{68}$$

and

$$\langle\, \eta_q(\tau)\eta_{q'}(\tau')\,\rangle_\eta = 2\delta(\tau-\tau')\delta^4(q,q'), \tag{69}$$

where $\delta^4(q,q') = \delta(p+p')\delta_{ss'}\delta_{\sigma\sigma'}$.

The two-point function $D_q(\tau,\tau')$ can be calculated in a similar way as in the Euclidean flat case. We have

$$D_q(\tau,\tau') = \frac{1}{(2\pi)}\delta^4(q,q')\frac{1}{(q^2+1)}\left(e^{-(q^2+1)|\tau-\tau'|} - e^{-(q^2+1)(\tau+\tau')}\right), \tag{70}$$

or, in the coordinate representation space:

$$\begin{aligned} D(\tau,\tau';x,x') &= \int d\tilde{\mu}(q)\, u_q^+(x)\, u_q^-(x') D_q(\tau,\tau') = \\ &\int d\tilde{\mu}(q)\, u_q^+(x)\, u_q^-(x')\frac{1}{(q^2+1)}\left(e^{-(q^2+1)|\tau-\tau'|} - e^{-(q^2+1)(\tau+\tau')}\right), \end{aligned} \tag{71}$$

where $\int d\tilde{\mu}(q)$ is given by Eq. (54) and, for simplicity, the index q denotes the set of indices $ps\sigma$ that label the modes. After a simple calculation [25] [26] it is easy to prove that in the equilibrium limit ($\tau' = \tau \to \infty$) we reach the usual result presented in the literature.

Now, let us apply this method for the case of a self-interacting theory with an interaction action given by

$$S_I[\varphi] = \int d^4x\, \sqrt{g(x)}\frac{\lambda}{4!}\,\varphi^4(x). \tag{72}$$

In the same way as in the Euclidean flat case, we can solve the equation using a perturbative series in λ. The two-point function up to the one loop level is given by

$$\langle\, \varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(1)} = (a)+(b)+(c), \tag{73}$$

where (a) is the zero order two-point function given by Eq. (70) and (b) and (c) are given in the momentum space respectively by

$$(b) = -\frac{\lambda}{2}\,\delta^4(q,k)\int d\tilde{\mu}(p)\, u_p^+\, u_p^- \int_0^{\tau_1} d\tau\, G_q(\tau_1-\tau)D_p(\tau,\tau)D_k(\tau_2,\tau), \tag{74}$$

$$(c) = -\frac{\lambda}{2}\,\delta^4(q,k)\int d\tilde{\mu}(p)\, u_p^+\, u_p^- \int_0^{\tau_2} d\tau\, G_k(\tau_2-\tau)D_p(\tau,\tau)D_q(\tau_1,\tau). \tag{75}$$

These are the contributions in first order. Substituting the expressions for $G_q(\tau-\tau')$ and $D_q(\tau,\tau')$ defined in Eqs. (67) and (70), respectively, one can easily show that in the asymptotic limit ($\tau_1 = \tau_2 \to \infty$) the term (b), for example, is written as

$$(b)|_{\tau_1=\tau_2\to\infty} = -\frac{\lambda}{2}\,\delta^4(q,k)\,\frac{1}{(k^2+1)}\,\frac{1}{(q^2+k^2+2)}\,I, \tag{76}$$

where the quantity I is defined as

$$I = \int dp \sum_{s=0}^{\infty} \sum_{\sigma=1}^{(s+1)^2} u^+_{ps\sigma}(x_1) u^-_{ps\sigma}(x_1) \, \frac{1}{p^2 + (s+1)^2}. \tag{77}$$

Inserting the expression for the modes given by Eq. (52) and using a summation theorem for harmonic polynomials, the expression above becomes

$$I = \frac{N^2}{2\pi^2} \cosh^2 t_1 \sum_{s=0}^{\infty} (s+1)^3 \int dp \; f^+_p(t_1)\, f^-_p(t_1) \, \frac{1}{p^2 + (s+1)^2}. \tag{78}$$

Recalling that the functions $f^{\pm}_p(t)$ are, in fact, hypergeometric functions, the integral in the expression above can be performed as long as we choose the appropriate contour in the p complex plane. These functions are regular in the simple pole $p_0 = -i(s+1)$, and the integration yields

$$I = \frac{N^2}{2\pi} \cosh^2 t_1 \sum_{s=0}^{\infty} (s+1)^2 \, f^+_p(t_1)\, f^-_p(t_1) \Big|_{p=-i(s+1)}. \tag{79}$$

The series in this equation is clearly divergent, so we need a procedure to regularize it and obtain a finite quantity for the two-point function. This can be done through the covariant stochastic regularization procedure. We first introduce a regularized Langevin equation, which is a generalization of Eq. (56):

$$\frac{\partial}{\partial \tau} \varphi(\tau, x) = -\frac{g_{00}}{\sqrt{g}} \frac{\delta S_0}{\delta \varphi(x)} \Big|_{\varphi(x)=\varphi(\tau,x)} + \int d^4y \, \sqrt{g}\, R_{xy}(\Delta) \eta(y), \tag{80}$$

where the regulator is a function of the Laplacian. Using the mode decomposition given by Eqs. (43) and (44), the expression above becomes

$$\frac{\partial}{\partial \tau} \varphi_q(\tau) = -(q^2+1)\varphi_q(\tau) + R_q \eta_q(\tau), \tag{81}$$

where $R_q = R_{xy}(\Delta)|_{\Delta=-(p^2+(s+1)^2)}$ and $R_{xy}(\Delta) = \int \, d\tilde{\mu}(q)\, u^+_q(x) u^-_q(y)\, R_q$. Then, it is easy to show that the two-point correlation function for the free field in momentum space is given by

$$D_q(\tau, \tau') = \frac{1}{(2\pi)} \delta^4(q, q') \frac{R^2_q}{(q^2+1)} \left(e^{-(q^2+1)|\tau - \tau'|} - e^{-(q^2+1)(\tau+\tau')} \right), \tag{82}$$

and the first order contribution to the two-point correlation function is given by

$$(b)_\Lambda|_{\tau_1=\tau_2\to\infty} = -\frac{\lambda}{2}\, \delta^4(q,k)\, \frac{R^2_k}{(k^2+1)} \, \frac{1}{(q^2+k^2+2)}\, I(\Lambda). \tag{83}$$

The term $I(\Lambda)$ is given by

$$I(\Lambda) = \int dp \sum_{s=0}^{\infty} \sum_{\sigma=1}^{(s+1)^2} u^+_{ps\sigma}(x_1) u^-_{ps\sigma}(x_1) \, \frac{R^2_p}{p^2 + (s+1)^2}, \tag{84}$$

which is almost the same expression as the one given by Eq. (77), with the difference that now it is regularized. Performing the same calculations, one easily obtains

$$I(\Lambda) = \frac{N^2}{2\pi^2}\cosh^2 t_1 \int dp \sum_{s=0}^{\infty} (s+1)^3 \, \frac{e^{-2(p^2+(s+1)^2)/\Lambda^2}}{p^2+(s+1)^2} \, f_p^+(t_1) f_p^-(t_1). \tag{85}$$

The behavior of the exponential function guarantees that the integral in the above equation is finite. Although we are not able to present such integral in terms of known functions, to our purpose it is enough that the integral is finite as we commented. Therefore, we have obtained the regularized two-point Schwinger function at the one-loop level for a massive conformally coupled scalar field in de Sitter space. A similar treatment can be done to find a regularized four-point Schwinger function also at the one-loop level.

A quite important point is that this regularization procedure preserves all the symmetries of the unregularized Lagrangian, since it is not an action regularization. The next step would be to isolate the parts that go to infinity in the limit $\Lambda \to \infty$ and remove them with a suitable redefinition of the constants of the theory, i.e., carry out the renormalization program. A natural question now would be if we can actually renormalize all the n-point functions at all orders at the coupling constant λ. Birrel [27] has given arguments that a priori we cannot expect that a regularization independent proof of the renormalizability of the $\lambda\varphi^4$ theory in a curved background exists. One attempt of general proof of renormalizability of $\lambda\varphi^4$ theory defined in a spacetime which can be analytically continued to Euclidean situation was given by Bunch [28]. Using the Epstein-Glaser method, Brunetti and Fredenhagen [29] presented a perturbative construction of this theory on a smooth globally hyperbolic curved spacetime. Our derivation shows that the stochastic regularization may be an attempt in a direction of such regularization independent proof, even though we are still restricted to the same situation studied by Bunch.

4 Conclusions and perspectives

The picture that emerges from these discussions is that the implementation of the stochastic quantization in curved background is related to the following fact. For static manifold, it is possible to perform a Wick rotation, i.e., analytically extend the pseudo-Riemannian manifold to the Riemannian domain without problem. Nevertheless, for non-static curved manifolds we have to extend the formalism beyond the Euclidean signature, i.e., to formulate the stochastic quantization in pseudo-Riemannian manifold, not in the Riemannian space (as in the original Euclidean space) as was originally formulated. Of course, this situation is a special case of ordinary Euclidean formulation for systems with complex actions [30]. It was also shown that, using a non-Markovian Langevin equation with a colored random noise, the convergence problem can be solved. It was proved that it is possible to obtain convergence toward equilibrium even with an imaginary Chern-Simons coefficient. The same method was applied to the self-interacting scalar model [31]. We conclude saying that several alternative methods have been proposes to deal with interesting physical systems where the Euclidean action is complex. These methods do not suggest any systematic analytic treatment to solve the particular difficulties that arise in each situation.

The possibility of using the stochastic quantization in a generic curved incomplete manifold is not free of problems. A natural question that arises in a general situation, or, specifically, when we are working in Rindler space [32] [33] [34] [35], is what happens to the noise field correlation function near an event horizon. It is not difficult to see that, whenever we have $g = \det g_{\mu\nu} = 0$, this correlation function diverges, and, therefore, all n-point correlation functions $\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2)...\varphi(\tau_n, x_n)\,\rangle_\eta$ will have meaningless values, in virtue of the solution of the Langevin equation. Someone may implement a brick wall-like model [36] [37] in order to account for these effects; in other words, one imposes a boundary condition on solutions of the Langevin equation at a point near the cosmological event horizon [38]. On the other hand, in the limit $g = \det g_{\mu\nu} \to \infty$, all the correlation functions vanish. The crucial point of our article was to circumvent the problem above discussed using a coordinate system where the event horizon is absent. A natural extension of this paper is to study the stochastic quantization using a different coordinatization of the de Sitter space hyperboloid where the event horizon explicitly appears.

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