# THE KERR SOLUTION OBTAINED BY THE EUCLIDON METHOD

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Using the so-called Euclidon method propounded by Ts.I. Gutsunaev, V.A. Chernyaev and S.L. Elsgolts [1], we show how to reobtain the Kerr and Schwarzschild solutions from a particular Lewis static solution.

## 1. Introduction

We are interested in this paper in constructing new axially symmetric stationary exact solutions of the Einstein equations in vacuo. This problem was investigated from various points of view: for more details and references, see the introduction to the article by Ts.I. Gutsunaev, V.A. Chernyaev and S.L. Elsgolts (GCE) [1]. GCE, using a technique of variation of constants, proposed a method for generating, in principle, new vacuum solutions from any axisymmetric solution. They start from a Euclidon solution that we can call a *matrix solution*, and vary the four constants appearing there: three of them appear in another arbitrarily chosen solution, called a *seed solution*; the fourth constant, to be simply called a *potential*, becomes a function $U$, which is determined from the seed solution by a set of two first-order partial differential equations, ((9) in [1]), in general, difficult to be solved. By such a general method, they generate a new solution which could be called a *daughter solution*. However, they do not exhibit a specific example of such a solution in their article.

Ts.I. Gutsunaev, A.A. Shaideman, and S.L. Elsgolts, in another article [2], with a purpose similar to GCE, apply the variation of constants method to a soliton-like matrix solution ((13) in [2]). As an example, they find again the Kerr solution. The complexity of the calculations is in general a technical obstacle in obtaining a daughter solution from an arbitrary seed solution. Besides, the interpretation of daughter solutions, when successfully obtained from a given seed solution, is in general of no physical interest because quite often new generated solutions do not exhibit good asymptotic behaviour.

The choice of an axisymmetric static seed solution permits one to considerably simplify the problem. In such a case, two compatibility equations, (23) and (24) in the present paper, can be constructed, the first one obeyed by the $U$ potential, the other one linking the seed solution to the daughter solution.

In this paper, we test the Euclidon method proposed by GCE using a Euclidon-like matrix solution. Then, we show how to obtain in this way the Kerr solution from a static Lewis-like seed solution. In Sec. 2, using the Weyl coordinates, we recall the Euclidon method used by GCE [1] with a slightly different presentation. Then, we apply this method to the case of a Lewis-like static seed solution. We show that the whole problem essentially reduces to determining the $U$ potential from two simple first-order partial differential equations, (20) and (21). In Sec. 3, using prolate spheroidal coordinates, we show how to generate the Kerr solution.

## 2. The Euclidon method

The Papapetrou-Lewis axisymmetric stationary metric in Weyl coordinates can be expressed as

$$ds^2 = f(dt - \omega d\phi)^2 - \frac{1}{f}[e^{2\gamma}(dr^2 + dz^2) + r^2 d\phi^2], \quad (1)$$

where $f$, $\omega$, $\gamma$ are functions of $r$ and $z$.

With the help of a twist potential $\Phi$, defined by

$$\frac{\partial \omega}{\partial r} = -\frac{r}{f^2}\frac{\partial \Phi}{\partial z}, \quad (2)$$

$$\frac{\partial \omega}{\partial z} = \frac{r}{f^2}\frac{\partial \Phi}{\partial r}, \quad (3)$$

we express the Einstein vacuum equations,

$$f\Delta f = (\vec{\nabla} f)^2 - (\vec{\nabla}\Phi)^2, \quad (4)$$

$$f\Delta \Phi = 2\vec{\nabla} f \cdot \vec{\nabla}\Phi, \quad (5)$$

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where the Laplacian and gradient operators are defined as follows:

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad (6)$$

$$\vec{\nabla} \equiv \frac{\partial}{\partial r}\vec{u}_r + \frac{\partial}{\partial z}\vec{u}_z. \quad (7)$$

The GCE method [1] rests on the choice of a special solution of the vacuum field equations, which we call the matrix solution, namely,

$$f_E = \frac{1}{c_1}(z - z_1 + R\tanh U_0), \quad (8)$$

$$\Phi_E = \frac{1}{c_1}\frac{R}{\cosh U_0} + c_2, \quad (9)$$

$$\omega_E = \frac{c_1(z-z_1)\coth U_0}{(z-z_1)\cosh U_0 + R\sinh U_0} + c_3, \quad (10)$$

where

$$R = [(z-z_1)^2 + r^2]^{1/2}, \quad (11)$$

and $c_1$, $c_2$, $c_3$ and $U_0$ are constants.

All the curvature tensor components of this solution vanish. That is why this solution is called a euclidon by GCE.

Using the variation of constants method, GCE suppose that the constants become functions of $r$ and $z$: if

$$c_1 = f_0(r,z), \qquad c_2 = \omega_0(r,z), \qquad c_3 = \Phi_0(r,z), \quad (12)$$

where $f_0$, $\omega_0$ and $\Phi_0$ form an arbitrary solution of the vacuum field equations called a seed solution by GCE, and

$$U_0 = U(r,z) \quad (13)$$

is a potential which satisfies a set of two partial differential equations ((9) in [1]), constructed from the solution (12), then, the so generated functions, constructed from (8)-(10) with (12)-(13) and denoted $\tilde{f}$, $\tilde{\Phi}$ and $\tilde{\omega}$, constitute a new solution of (2)-(5), to be called a daughter solution:

$$\tilde{f} = \frac{z - z_1 + R\tanh U}{f_0}, \quad (14)$$

$$\tilde{\Phi} = \frac{R}{f_0\cosh U} + \omega_0, \quad (15)$$

$$\tilde{\omega} = \frac{f_0(z-z_1)\coth U}{(z-z_1)\cosh U + R\sinh U} + \Phi_0. \quad (16)$$

Only $U$ remains to be determinated.

The complexity of the set of partial differential equations, allowing one in principle to determine the $U$ function, is in practice an obstacle to its integration. However, such a difficulty vanishes if a static solution is chosen as a seed solution:

$$f_0 = f_0(r,z), \qquad \Phi_0 = 0, \qquad \omega_0 = 0. \quad (17)$$

In this case, choosing

$$f_0 = \mathrm{e}^{\chi}, \quad (18)$$

the Einstein equations (4), (5) reduce to

$$\Delta\chi = 0. \quad (19)$$

For such an harmonic function $\chi$, the solution (17) is said to be Lewis-type [3]. Hence, the system (9) in [1] permitting one to determine the $U$ function reduces to

$$U_{,r} = a_1\chi_{,r} + a_2\chi_{,z}, \quad (20)$$

$$U_{,z} = -a_2\chi_{,r} + a_1\chi_{,z}, \quad (21)$$

where

$$a_1 = \frac{z-z_1}{R}, \qquad a_2 = \frac{r}{R}, \qquad a_1^2 + a_2^2 = 1. \quad (22)$$

Besides, (19)–(21) lead to a canonical elliptic equation for $U$ [5],

$$r[r^2 + (z-z_1)^2](U_{,rr} + U_{,zz}) - [r^2 - (z-z_1)^2]U_{,r} - 2r(z-z_1)U_{,z} = 0, \quad (23)$$

and, from (14)–(15),

$$f_0^2(\tilde{f}^2 + \tilde{\Phi}^2) - 2f_0(z-z_1)\tilde{f} - r^2 = 0. \quad (24)$$

## 3. The Kerr solution

We can present the Lewis static seed solution (17) as a euclidon-like solution,

$$f_0(\lambda,\mu) = z_1(\lambda+1)(\mu+1), \quad \Phi_0 = 0, \qquad \omega_0 = 0, \quad (25)$$

where $\lambda$ and $\mu$ are the prolate spheroidal coordinates linked to the Weyl coordinates $r$ and $z$ by the relations

$$\lambda = \frac{1}{2z_1}\{[(z+z_1)^2 + r^2]^{1/2} + [(z-z_1)^2 + r^2]^{1/2}\}, \quad (26)$$

$$\mu = \frac{1}{2z_1}\{[(z+z_1)^2 + r^2]^{1/2} - [(z-z_1)^2 + r^2]^{1/2}\}, \quad (27)$$

with

$$\lambda \geq 1, \qquad |\mu| \leq 1. \quad (28)$$

$f_0$ and $\Phi_0$ given by (25) are solutions of Einstein's equations (4)–(5). From (18) and (25) we find

$$\chi = \ln[z_1(\lambda+1)(\mu+1)], \quad (29)$$

where $\chi$ obeys (19).

The system (20)-(21), determining the potential $U(\lambda,\mu)$, now takes the form

$$U_{,\lambda} = \frac{\lambda\mu - 1}{\lambda - \mu}\chi_{,\lambda} + \frac{1-\mu^2}{\lambda-\mu}\chi_{,\mu}, \quad (30)$$

$$U_{,\mu} = -\frac{\lambda^2 - 1}{\lambda - \mu}\chi_{,\lambda} + \frac{\lambda\mu - 1}{\lambda-\mu}\chi_{,\mu}. \quad (31)$$

Hence, we find by integration

$$U = \ln\left[\frac{1+\lambda}{a_0(1+\mu)}\right], \tag{32}$$

where $a_0$ is a constant. On the other hand, (23) becomes, in prolate spheroidal coordinates,

$$(\lambda-\mu)[(\lambda^2-1)U_{,\lambda\lambda}+(1-\mu^2)U_{,\mu\mu}] \\ -2(\lambda\mu-1)(U_{,\lambda}+U_{,\mu})=0, \tag{33}$$

and it can be easily checked that (32) is a solution to (33). Eqs. (14)-(15), giving $\tilde{f}$ and $\tilde{\Phi}$, now become

$$\tilde{f}(\lambda,\mu) = \frac{\lambda\mu-1+(\lambda-\mu)\tanh U}{(\lambda+1)(\mu+1)}, \tag{34}$$

$$\tilde{\Phi}(\lambda,\mu) = \frac{\lambda-\mu}{(\lambda+1)(\mu+1)\cosh U} \tag{35}$$

where $U$ is given by (32). Hence, we immediately find

$$\tilde{f} = \frac{\lambda^2-1+a_0^2(\mu^2-1)}{(\lambda+1)^2+a_0^2(\mu+1)^2}, \tag{36}$$

$$\tilde{\Phi} = \frac{2a_0(\lambda-\mu)}{(\lambda+1)^2+a_0^2(\mu+1)^2}. \tag{37}$$

To interpret this solution, we propose to determine the corresponding solution of the Ernst equation [4]:

$$(\xi\xi^*-1)\Delta\xi = 2\xi^*\vec{\nabla}\xi\cdot\vec{\nabla}\xi, \tag{38}$$

where

$$\xi = P(\lambda,\mu)+iQ(\lambda,\mu), \tag{39}$$

and

$$P = \frac{1-\tilde{f}^2-\tilde{\Phi}^2}{(1-\tilde{f})^2+\tilde{\Phi}^2}, \tag{40}$$

$$Q = \frac{2\tilde{\Phi}}{(1-\tilde{f})^2+\tilde{\Phi}^2}. \tag{41}$$

Then, we find

$$\xi = \frac{\lambda+a_0^2\mu}{1+a_0^2}+i\frac{a_0(\lambda-\mu)}{1+a_0^2}. \tag{42}$$

This solution (42) can be easily transformed into the Kerr solution with the help of the unitary transformation $\mathrm{e}^{i\alpha}$, defined by

$$a_0 = -\tan\alpha. \tag{43}$$

Then, from (42)

$$\xi_K = \mathrm{e}^{i\alpha}\xi = p\lambda - iq\mu, \tag{44}$$

with

$$p = (1+a_0^2)^{-1/2}, \qquad q = a_0(1+a_0^2)^{-1/2}, \tag{45}$$

(36) or

$$q/p = a_0, \qquad p^2+q^2=1, \tag{46}$$

that is to say, we find again the Kerr solution.

It can be easily seen that (25) and (36)-(37) satisfy Eq. (24) by using the relations inverse to (26)-(27),

$$z-z_1 = z_1(\lambda\mu-1), \tag{47}$$

$$r^2 = z_1^2(\lambda^2-1)(1-\mu^2). \tag{48}$$

Finally, let us note that to find again the Schwarzschild solution, it suffices to use the asymptotic behaviour of the $U$ potential (32) in (34)-(35), and there is no need for the explicit solutions (36)-(37).

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## References

[1] Ts.I. Gutsunaev, V.A. Chernyaev and S.L. Elsgolts, *Grav. & Cosmol.* **5**, 335 (1999).

[2] Ts.I. Gutsunaev, A.A. Shaideman and S.L. Elsgolts, *Grav. & Cosmol.* **6**, 254 (2000).

[3] J. Gariel, G. Marcilhacy and N.O. Santos, "On the Lewis axisymmetric metric", *preprint* (2001).

[4] F.J. Ernst, *Phys. Rev.* **167**, 1175 (1968).

[5] F.G. Tricomi, "Equazioni a derivate parziali", Cremonese, Roma (1957).