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Mechanisms for the Emission or Accretion of Matter and Gravitational Waves in Robinson-Tautman Spacetimes

by

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Abstract

We study the structure of Robinson-Trautman solutions of Einstein's equations for which the angular dependence of metric functions is non-analytic. We show that the non-analyticity produces new physical effects, in the sense that Einstein's equations demand the presence of an equatorial shell of matter that can modelled by neutrinos, strings and gravitational waves propagating radially on the shell. The presence of the shell allows us to characterize unambiguously the mass-energy loss or mass-energy accretion due to the emission or absorption of neutrinos, strings and gravitational waves. In the light of these models, we discuss the connection of the non-analiticity with processes that extract mass from the configuration, in the realm of Robinson-Trautman spacetimes. We also present a simple, though contrived model in which gravitational waves sent from the past null infinity collapse to form a Schwarzschild black-hole with infinitesimal mass.

1 Introduction

Robinson-Trautman (RT) metrics are the simplest known exact solutions of vaccum Einstein's equations which may be interpreted as representing an isolated gravitationally radiating system [1, 2]. By construction, RT spacetimes are assumed to admit a shearfree null congruence of geodesic [3] which is hypersurface-orthogonal. This family of null hypersurfaces foliates the spacetime and, in a coordinate system where they are labeled by $u =$ constant, the metric can be expressed [3]

$$ds^2 = \alpha^2(u,\theta)du^2 + 2edudr + r^2K^2(u,\theta)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

where r is an affine paramete defined along the congruence of the shear-free null geodesics. We use the angular coordinates (θ,ϕ) to label locally the points of the spacelike surfaces $u =$ const.,$r =$ const., and we assume that these two-dimensional manifolds are compact and orientable. We use the symbol $e = \pm 1$ to characterize the coordinate u as having the asymptotic nature of a retarded or advanced time coordinate, respectively. The geometry is non-stationary and axially symmetric, admitting the obvious Killing vector $\partial/\partial\phi$. Einstein equations in vacuum result in

$$\alpha^2(u,r,\theta) = \lambda(u,\theta) + \frac{B(u)}{r} + 2re\,\frac{K'(u,\theta)}{K(u,\theta)} \tag{2}$$

where $\lambda(u,\theta)$ is the Gaussian curvature of the surfaces ($u =$ const., $r =$const.) defined by

$$\lambda(u,\theta) = \frac{1}{K^2} - \frac{K_{\theta\theta}}{K^3} + \frac{K_\theta^2}{K^4} - \frac{K_\theta}{K^3}\,cotg\theta \tag{3}$$

and

$$3B(u)\,\frac{K'}{K} + B'(u) + e\,\frac{(\lambda_\theta\sin\theta)_\theta}{2K^2\sin\theta} = \theta \tag{4}$$

In the above, a prime denotes $\partial/\hat{o}u$ and subscript θ denotes $\partial/\partial\theta$. Equations (3) and (4) will be the basis of our analysis in this paper. Note that λ is determined from $K(u,\theta)$ through equation (3). Equation (4) is denoted the Robinson-Trautman (RT) equation and allows to evolve initial data $K(u,\theta)$ prescribed on a given null surface $u = u_0$ (except for the case $B(u) = 0$). If $B(u)$ is non-zero, it can be always reduced to a constant by a coordinate transformation. Yet in the physical situations discussed in this paper B will a determined function of u, and its derivative will be proportional to the flux of particles on a time-like shell of matter present in the model, as we will see, thus providing a more transparent physical interpretation when emission processes are analysed.

There is a respectable body of results on the existence of solutions of the RT equation and on their global structure [4–13]. The paper by Foster and Newman [4] is very illustrative of the above questions. They studied solutions of the RT equation linearized about the Schwarzschild values $\lambda = 1 = K$ and showed that, for analytic perturbations (small analytic inital data), the Robinson-Trautman vacuum solutions will tend towards the Schwarzschild solution will tend towards the Schwarzchild solution in the infinite future. The existence question for solutions of the full non-linear equation was first examined by Schmidt [5]. Showing that a solution exists locally in times for arbitrarily prescribed initial data of appropriate differenciability. Later Rendall [6] showed global existence for sufficiently small initial data with antipodal symmetry. The most general analysis of the existence and behaviour of solutions of the Robinson-Trautman equation (4) was given by Chrusciel [9, 10] and by Chrusciel and Singleton [11]. The established result is that the Robinson-Trauman spacetimes exist globally for all positive "u-times" and convergence asymptotically to a Schwarzschild metric, this global time extension being realized for arbitrary, sufficiently smooth initial data in the family of Robinson-Trautman spacetimes; the extension of these spacetimes across the Schwarzschild-like event horizon is not analytic. In the present paper we make an analysis of solutions of Robinson-Trautman equations (3)-(4) for which the initial data are continuous but not diffrerentiable, namely, the metric functions are of class C^0 in the coordinate θ. As we will show, under this assumption Einstein's equations demand the presence of an equatorial shell of matter, which can be appropriately modelled and can provide an unambigous definition of energy loss by the sources, and also allows to attach a meaning to the time-dependence of B. This shell model actually substitutes lines of sigularities in the metric functions through which a balance of energy emitted or absorbed by the source could be realized. In the case $B(u) = O$, the spacetime is more degenerate in the sense of the Petrov classification, and the metric functions will contain an artibtrary function of u, typical of free gravitational wave.

We organize the paper as follows. In section 2 we briefly examine the algebraic structure of Weyl tensor and the characterization of a gravitational wave zone based in the Petrov classification and Peeling properties. We exhibit the three distinct principal null directions of the Weyl tensor and show that only one is globally non-twisting, defining unambiguously the gravitational wave fronts of the spacetime. Although this Section is basically a review, it contains the necessary material to characterize the presence or not of gravitational waves in the spacetime solutions discussed in the following Sections. Sections 3 and 4 are the bulk of the paper, exhibiting RT solutions with non-analytic metric functions, and the consequent physical structures. Section 5 deals with gravitational waves non-analytic perturbations of the Minkowski spacetime. With the exception of Section 5, in the remaining of the paper we restrict ourselves to $e = 1$ and $m > 0$. Throughout the paper we use units such that $8\pi G = c = 1$.

2 The Structure of the Weyl Tensor and the Characterization of the Gravitational Wave Zone

In order to define distinct classes of perturbations of the solutions of the RT equations (3)-(4) and to characterize the possible radiative nature of the associated Robinson-Trautman spacetimes, we now proceed to discuss the algebraic structure of the Weyl tensor of the geometries (1)-(2). Let us introduce the semi-null tetrad basis determined by the 1-forms

$$\begin{aligned} O^0 &= du \ , \\ O^1 &= \alpha^2/2\ du + dr \ , \\ O^2 &= rK d\theta \ , \\ O^3 &= rK \sin\theta d\phi \end{aligned} \tag{5}$$

In this basis, the non-zero Weyl tensor components are given by

$$C_{2323} = -C_{0101} = 2C_{0212} = \frac{B(u)}{r^3} \tag{6.a}$$

$$C_{0303} = -C_{0202} = -\frac{A(u,\theta)}{r^2} - \frac{F(u,\theta)}{r} \quad , \quad C_{0323} = \frac{\lambda_\theta}{2Kr^2} \tag{6.b}$$

where the functions A and F are

$$\begin{aligned} A(u,\theta) &= \frac{1}{4K^2}\left(\lambda_\theta - 2\,\frac{\lambda_\theta K_\theta}{K} - \lambda_\theta cotg\theta\right) \\ F(u,\theta) &= \frac{1}{2K^2}\,\partial_u\left[\left(\frac{K_\theta}{K}\right)_\theta - \frac{K_\theta}{K}\ cotg\theta - \left(\frac{K_\theta}{K}\right)^2\right] \end{aligned} \tag{7}$$

Equations (6a,b) can be expressed as

$$C_{ABCD} = \frac{II_{ABCD}}{r^3} + \frac{III_{ABCD}}{r^2} + \frac{N_{ABCD}}{r} \tag{8}$$

where II_{ABCD}, III_{ABCD} and N_{ABCD} are of algebraic type II, type III and type N in the Petrov classification [15–18], respectively. The Robinson-Trautman spacetime is in general of Petrov type II; the principal null direction presenting multiplicity two is defined by the shear free geodesic [3] vector field (in the basis (5))

$$k^A = \delta^A_1 \tag{9.a}$$

while the remaining two distinct principal null are defined by

$$n^A = (\eta\eta^*, 1, (\eta^* + \eta^*)/2, \ -(\eta - \eta^*)/2) \ , \tag{9.b}$$

where

$$\eta = -\frac{2Q}{C} \pm \sqrt{\frac{4Q^2}{C^2} - \frac{3F}{C}}$$

and the coefficients (Q, C, F), given in terms of the components (6b) of the Weyl tensor, have the form

$$\begin{aligned} Q &= -\frac{\sqrt{2}}{4}\,\frac{\lambda_\theta}{2Kr^2} \\ C &= \frac{A(u,\theta)}{r^2} + \frac{F(u,\theta)}{r} \\ F &= \frac{B}{r^3} \end{aligned}$$

They satisfy $C_{ABCD}n^D = O(r^{-2})$. In other words, for r large the two principal null directions (9b) coincide with the repeated principal null direction k^A (9a), which is shear free and orthogonal to the surfaces $u =$ const. If the spacetime is such that N_{ABCD} is non-zero, then for large values of the distance parameter r the curvature tensor has the approximate asymptotic expression $C_{ABCD} = N_{ABCD}/r$, that is, it is of Petrov type N, with the degenerate principal null direction given by k. This is the curvature tensor of a gravitational wave spacetime, with propagation vector k. The wave fronts are the $u =$const. surfaces and the field looks like a plane wave at large distances. The nonvanishing of the scalars N_{ABCD} is then taken as an invariant criterion for the presence of grativational waves, and the asymptotic region (where the $O(1/r)$ term in (8) is dominant) defined as the wave zone. The above characterization is based on two pilars: (i) the Peeling Theorem (for the linearized Riemann tensor of retarded multipole fields, see Refs. [19, 20]; for the general Ref. [21]: for a review, including peeling properties of the Maxwell tensor, see Ref. [22]); (ii) the analysis of the spacetime of gravitational wave solutions of Einstein's equations, and their relation to electromagnetic wave in Maxwell's theory [1, 22, 23, 24]. It is worth remarking that the principal null directions (9b) are not surface orthogonal, except for large r where they coincide with (9a); they are actually twisting (to $O(r^{-2})$), and hence the definition of the gravitational wave fronts as u=const. is unambigous.

We are now able to make and invariant characterization of distinct classes of Robinson-Trautman spacetimes arising from solutions of the Robinson-Trautman equation (4), with or without gravitational waves. This will be the object of the next section.

3 Robinson-Trautman Spacetimes in a Non-Analytic Regime

According to the criteria of the previous Section, a Robinson-Trautman geometry (1) describes a spacetime with gravitational waves if and only if the quantity $F(u,\theta) = 1/2K^2\partial u\,[(K_\theta/K)_\theta) - K_\theta/K costg\theta - (K_\theta/K)^2]$ given in (7) is non-zero. In particular, if the metric function $K(u,\theta)$, solution of (3)-(4), is separable (what is equivalent to K being independent of u, modulo a coordinate transformation) gravitational waves are not present.

Let us then consider a particular solution of RT equation having the form $K = k(\theta)$ and $B = b(u)$, so that the associated Weyl curvature tensor has no Petrov type N sector [14]. The function $\lambda = g(\theta)$ is related to $k(\theta)$ by Eq. (3). An example is the solution $\lambda = 1 = K,\ B = -2m$ that corresponds to the Schwarzschild metric expressed in Eddingto-Finkelstein coordinates, with the only non-zero Weyl tensor tetrad components $C_{2323} = -C_{0101} = 2\ C_{0212} = 2m/r^3$. As expected, only type D terms are present. Let us now consider a class of solutions of the RT equation (3)-(4) of the form

$$\begin{aligned} \lambda &= g(\theta)(1+\varepsilon W(u,\theta)) \\ K &= k(\theta)(1+\varepsilon Y(u,\theta)) \\ B(u) &= b(u)+\varepsilon Z(u)\ , \end{aligned} \tag{10}$$

where ε is a small arbitrary parameter. Solutions (10) may, in principle, change the algebraic type of the Weyl tensor, and introduce a radiative sector in the Weyl tensor, namely, a Petrov type N region in the spacetime. If this is the case, (10) constitutes a true perturbation of the geometry, in the sense that Weyl scalars (6) which are zero for the unpertubed metric become of $O(\varepsilon)$ for the perturbed metric. These radiative solutions *restore* the algebraic structure (8) of the Weyl tensor for the Robinson-Trautman metrics.

Let us examine (10) for the case $\varepsilon = 0$. From (4) we obtain

$$b'(u) = c \ , \tag{11.a}$$

$$(g_\theta \sin\theta)_\theta = -2ck^2 \sin\theta \ , \tag{11.b}$$

where c is a separation constant. The functions $g(\theta)$ and $k(\theta)$ are related through Eq. (3)

$$1 - \frac{1}{\sin\theta}\left(\sin\theta\frac{K_\theta}{K}\right)_\theta = k^2 g \tag{12}$$

Equation (11a) can be immediately integrated,

$$b(u) = -2M + cu \ , \tag{13}$$

where M is an arbitrary constant. For $c = 0$, it is easy to see that the only non-singular solution of (11b) is g constant, which we take $g = 1$ due to the assumed compact topology of the $u, r =$ const surfaces. In this case, a general solution of (12) can be reduced to $k = 1$ by a convenient coordinate transformation, and it corresponds to the Schwarzschild spacetime, with M interpreted as the geometrical mass.

In general, the function $b(u)$ will be associated to the mass function of the configuration. Although its form is dependent on the coordinates used, in the present coordinate system it may be unambiguously interpreted as the mass function of the configuration in the sense that its u-derivative is proportional to the flux of matter (neutrinos and strings) emitted in the equatorial plane $\theta = \pi/2$, as we shall see. We also note that $b(u)$ typically plays the role of mass function as it appears as the factor in the $O\left(\frac{1}{r^3}\right)$ Petrov D components of the Weyl tensor (6).

The case $c \neq 0$ corresponds to a situation in which the spacetime is asymptotically flat but not asymptotically Minkowskian. To solve Eqs. (11b) and (12) for $c \neq 0$ turns out to be very difficult. However, if we assume that the solutions are analytic in c, it can be shown [27] that these solutions are singular at $\theta = 0$ and/or $\theta = \pi$. A concrete example of this is provided by exhibiting an approximate solution of $(11)-(12)$ for $|c| << 1$, namely,

$$\begin{aligned} g(\theta) &= 1 + c\,[-1 + 2\ln(1 \pm \cos\theta)] \\ k(\theta) &= 1 - 2c\ln(1 \pm \cos\theta)\ . \end{aligned} \tag{14}$$

The + solution is regular at $\theta = 0$, and the – solution is regular at $\theta = \pi$. In order to get rid of the undesirable singularities at $\theta = 0$ or $\theta = \pi$, we cover the whole spacetime by using the + solutions for $0 \le \theta \le \pi/2$ and the – solutions for $\pi/2 \le \theta \le \pi$. This is carried out by matching the two sets at the equatorial plane $\theta = \pi/2$, since are continuous there, yielding a metric that is of class C^0. Noticing that the first derivatives of the metric are not continuous at $\theta = \pi/2$, a timelike shell must therefore be present at the equatorial plane, in accord to Israel formalism [27]. Evidently the appeal to the introduction of the equatorial shell is only justified if we are able to give a satisfactory physical interpretation to the shell dynamics. The introduction of a shell and the physical modeling of it will be discussed in the next Section.

Solutions of type (10) must satisfy equations (3)-(4); it then follows that, to first order in ε, the functions W and Y must be separable.

$$\begin{aligned} W(u,\theta) &= w(\theta)N(u) \\ Y(u,\theta) &= w(\theta)N(u)\ , \end{aligned}$$

resulting in the temporal equations

$$3b(u)\,\frac{N'}{N} + 2b' = a_0 \tag{15.a}$$

$$\frac{Z'}{N} = b_0 \tag{15.b}$$

and in the angular equations

$$(y_\theta \sin\theta)_\theta = -\,(w + 2g(\theta)y)\,k^2\sin\theta\ , \tag{16.a}$$

$$(w_\theta \sin\theta)_\theta = -2k^2\sin\theta(b_0 + a_0 y)\ , \tag{16.b}$$

where a_0 and b_0 are arbitrary separation constants. Here the functions $k(\theta)$ and $g(\theta)$ are the solutions of the $\varepsilon = 0$ problem (11)-(13). The integration of (15) is immediate and results

$$N(u) = N_0\Big[-6M + 3cu\Big]^{\dfrac{a_0 - 2c}{3c}} \tag{17.a}$$

$$Z(u) = Z_0 + \frac{3b_0 c N_0}{(a_0 + c)}\,\Big[-6M + 3cu\Big]^{\dfrac{a_0 + c}{3c}} \tag{17.b}$$

where N_0 and Z_0 are integration constants. Integration of (16) for several situations will be discussed in the next Sections.

4 The Physics of the Non-Analytic Solutions and the Shell Structure

We first approach the case $c = 0$, with the functions $k(\theta) = 1 = g(\theta)$. This corresponds to $O(\varepsilon)$ solutions of RT equations, and yields $0(\varepsilon)$ perturbations of the Schwarzchild metric, including the analytic Foster-Newman solution [4] given here in another coordinate system. In the limit $c = 0$ solutions (17) result

$$\begin{aligned} N(u) &= N_0 \exp\left(-\frac{a_0 u}{6M}\right) \\ Z(u) &= Z(u) + \left(-\frac{6Mb_0}{a_0}\right) N_0 \exp\left(-\frac{a_0 u}{6M}\right) \end{aligned}$$

with $a_0 \neq 0$. Regularity requirements at $\theta = 0$ and $\theta = \pi$ demand that the solutions of (16) have the form

$$\begin{aligned} w_\ell &= \frac{2b_0}{a_\ell} + w_{0\ell} P_\ell(co\theta) \\ y_\ell &= -\frac{b_0}{a_\ell} + \frac{\ell(\ell+1)}{2a_\ell}\, w_{o\ell} P_\ell(\cos\theta) \\ a_0 &\equiv a_\ell = 2\left(\frac{\ell(\ell+1)}{2}\right)\left(\frac{\ell(\ell+1)}{2} - 1\right) \end{aligned}$$

where ℓ is a non-negative integer and $w_{0\ell}$, is an arbitrary constant. The solution with $w_{0\ell} = 0$ corresponds to the Schwarzschild geometry in another temporal gauge. Here $P_\ell(\cos\theta)$ is the Legendre polynomial of order ℓ. The condition $a_0 \neq 0$ implies that $\ell \geq 2$, that is, only quadrupole or higher order poles gravitational radiation fields are present, as expected, with wave zone defined by the $O(1/r)$ non-zero components of the Weyl tensor

$$C_{0202} = -C_{0303} = -\frac{\ell(\ell+1)a_0 w_{0\ell}}{24Mr}\left[-2\cos\theta \frac{dP_{d\ell}}{d\theta} + \ell(\ell+1)P_\ell\right] N_0 \exp\left(-\frac{a_\ell u}{6M}\right)$$

The non-radiative modes $l \leq 1$ correspond to the case $a_0 = 0$. Solutions of eqs. (16) for $a_0 = 0$ and $b_0 \neq 0$ are in general given singular at $\theta = 0$ and/or $\theta = \pi$, formally analogous to solutions (13), being also interpreted as having a shell structure at $\theta = \pi/2$, as we will see below. As an example, we select the particular set.

$$w = w_0 + 2b_0 \ln(1 + \cos\theta)\ , \quad y = (w_0 + b_0)/2 - b_0 \ln(1 + \cos\theta)\ ,$$

for $0 \le \theta < \pi/2$, and

$$w = w_0 + 2b_0 \ln(1 - \cos\theta) \ , \quad y = (w_0 + b_0)/2 - b_0 \ln(1 - \cos\theta) \ ,$$

for $\pi/2 < \theta \le \pi$, which are regular at $\theta = 0$ and $\theta = \pi$ respectively. These nonanalytic solutions complete the Foster-Newman class for all ℓ.

Let us now examine the general properties of the solutions of Eqs. (11b)-(12) and (16), and the nature of the resulting spacetime when non-analyticity is admitted. We initially restrict ourselves to Eqs. (11b)-(12), but our discussion will equally apply to Eqs. (16), where $\varepsilon \neq 0$. As we remarked before, if we assume that the solutions of (11b)-(12) are analytic in c, then these solutions are either singular at $\theta = 0$ and/or $\theta = \pi$. Let us consider the exact solution $\{g(\theta), k(\theta)\}$ singular at $\theta = 0$, say. Due to the symmetry of equation (11b) under the change $\theta \to \pi - \theta$, the solution $\{g(\pi - \theta), k(\pi - \theta)\}$ is singular at $\theta = \pi$. We can get rid of the undesirable singularities at $\theta = 0$ or $\theta = \pi$ if we use the set $\{g(\pi - \theta), k(\pi - \theta)\}$, defined for $0 \le \theta < \pi/2$ and $\{g(\theta), k(\theta)\}$, defined for $\pi/2 < \theta \le \pi$ to cover the whole spacetime. The sets are matched at the equatorial plane $\theta = \pi/2$, since they are continuous there, resulting in a metric that is of class C^0. However, the first derivatives of the metric are not continuous at $\theta = \pi/2$ implying that a timelike shell must therefore be present at the equatorial plane. To describe the geometry of the $1 + 2$ spacetime of the shell, we introduce the local triad basis on the 3-dim surface $\Sigma : \theta = \pi/2$

$$\begin{aligned} e^{\alpha}_{(0)} &= \left(\hat{\alpha}^{-1}(u, r), 0, 0, 0\right) \\ e^{\alpha}_{(1)} &= \left(\hat{\alpha}^{-1}(u, r), \hat{\alpha}(u, u), 0, 0\right) \\ e^{\alpha}_{(3)} &= \left(0, 0, 0, \ \frac{1}{r\hat{k}}\right) \end{aligned} \tag{18}$$

with $e^{\alpha}_{(a)} e_{\alpha(b)} = diag(1, -1, -1), a, b = 0, 1, 3,$ where $\hat{}$ denotes the restriction of the respective function to $\sum$. In accord to Israel's formalism [28], the discontinuity of the first derivatives of the geometry across the surface $\sum$ demands the presence of a shell of matter in $\sum$, with energy-momentum tensor given in the triad basis (18) by [29]

$$\hat{T}_{ab} = [K_{ab}]_{\Sigma} - g_{ab} \left[g^{cd} K_{cd}\right]_{\Sigma}$$

where K_{ab} is the extrinsic curvature of $\sum$ defined by $K_{ab} = -e^{\alpha}_{(a)} n_{\alpha;\beta} e^{\beta}_{(b)}$, with n^{α} the unit normal the $\sum$, and $[\ ]_{\Sigma}$ denoting discontinuity across Σ. Using (2) and (18) we derive that

$$\hat{T}_{ab} = {}_{N}T_{ab} + {}_{S}T_{ab} + {}_{G}T_{ab} \tag{19}$$

where

$$_{N}T_{ab} = -\left\{\frac{[\lambda_\theta]_\Sigma}{2r\hat{K}\hat{\alpha}^2}\right\}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;, {}_{G}T_{ab} = -\left\{\frac{\left[\frac{K'}{K_\theta}\right]_\Sigma}{\hat{K}\hat{\alpha}^2}\right\}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$_{S}T_{ab} = \frac{[K_\theta]_\Sigma}{r\hat{K}^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Our task now is to show that this energy-momentum tensor can be modelled by neutrinos, strings and gravitational waves propagating radially on the $1+2$ spacetime $\sum$ of the shell.

To this end let us discuss the dynamics of massless neutrinos firstly in the $1+3$ spacetime, and secondly in the $1+2$ spacetime $\sum$ of the shell, which are basically distinct. Neutrinos in interaction with the gravitational field are described by spinor fields in curved spacetime via the prescription of Brill and Wheeler [30]. In a local tetrad basis, Dirac's equation for neutrinos is expressed as

$$-i\gamma^A\left(e_{\alpha(A)}\partial_\alpha - \Gamma_A\right)\Psi = 0, \tag{20}$$

where the Γ_A are the Fock-Ivanenko coefficients associated to the tetrad field. If we restrict our considerations to radial neutrinos only, defined by $\gamma^0\Psi = \gamma^1\Psi$ such that the null four current $J^A = (\Psi^\dagger)\gamma^A\Psi$ has components $J^A = (\Psi^\dagger\Psi)(1,1,0,0)$, it is straightforward to check that Dirac's equation (20) for the four dimensional RT metric has no solution for radial neutrinos, even as test particles.

However in $\sum$ radial neutrinos are admissible and generate the first parcel of the right hand side of the energy-momentum tensor (19) of the shell. Radial neutrinos in $\sum$ are also defined by $\gamma^0\Psi = \gamma^1\Psi$ (with γ^A now given in Ref. [31]) resulting in the radial null current on the shell $J^A = \Psi^\dagger\Psi(1,1,0)$. For these neutrinos the general solution of Dirac's equation in the triad basis (18) is given by the two-spinors

$$\Psi = \frac{1}{\sqrt{r\hat{\alpha}}}\begin{pmatrix} -i\chi(u) \\ \chi(u) \end{pmatrix} \tag{21}$$

where $\chi(u)$ is an arbitrary complex function. The corresponding surface stress-energy tensor, ${}_N T_{ab} = i\left[\psi^+\gamma^0\gamma_{(a}\mathcal{D}_{b)}\psi - \mathcal{D}_{(a}\psi^+\gamma^0\gamma_{b)}\psi\right]$, associated with (21) reads

$$ {}_N T_{ab} = -\frac{4i}{r\hat{\alpha}^2}\ \left((\partial_u\chi^*)\chi - \chi^*\partial_u\chi\right)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ , \tag{22} $$

where $\mathcal{D}_a = e_\alpha(a)\partial_\alpha - \Gamma_a$ is the spinor covariant in the $1+2$ spacetime of the shell. The above tensor models the first parcel of $\hat{T}_{ab}$ provided we identify

$$ \frac{[\lambda_\theta]_\Sigma}{2\hat{K}} \equiv 4i\left((\partial_u\chi^*)\chi - \chi^*\partial_u\chi\right)\ . \tag{23} $$

By construction this surface stress-energy tensor is independently conserved, and the null radial current associated with the two-spinor solution (21), $j^a = \dfrac{2\chi^*\chi}{r\hat{\alpha}}\,(1,1,0)$ propagates outwardly, in the direction of increasing r (cf. the definition of $e_{\alpha(1)}$ in eqs. (18)). We note that the LHS of (23) is constant

$$ E(c) \equiv \frac{[g_\theta]_\Sigma}{2\hat{K}} \simeq -2c = -4i\left((\partial_u\chi^*)\chi - \chi^*\partial_u\chi\right) $$

the second equality holding for the approximate solutions (14). In view of eqs. (11a) or (13) we can see the outward radial propagation of neutrinos in the shell corresponds to a decrease of the mass function of the configuration.

The second term in (19) has the structure of a perfect fluid stress-energy tensor in which the pressure is negative. This suggests that we take it as corresponding to a gas of strings with energy density ρ_S given in the form

$$ {}_S T_{ab} = \rho_S(-\gamma)^{-1/2}\Sigma_{ac}\Sigma^c{}_b\ . \tag{24} $$

Here the skew-symmetric tensor $\sum^{ab}$ represents the kinematics of the gas of strings and must satisfy the normalization condition $\sum_{ab}\sum^{ab} = 2\gamma$. Due to the symmetry of the shell we assume that $\sum^{01}$ is the only nonvanishing component on the tensor $\sum^{ab}$. As a consequence, (24) takes the form

$$ {}_S T_{ab} = \rho_S(-\gamma)^{1/2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{25} $$

which could model the second term in (19). Let us first restrict ourselves to the case $\varepsilon = 0$. Noting that the stress-energy tensor of the neutrinos is independently conserved, we have from the conservation law $(\rho_S \sum_{ab})_{||b} = 0$ that $\rho_S(-\gamma)^{1/2} \equiv$const$/r$. Comparing (25) with the second term in (19) we are led to identify this constant with

$$G(c) \equiv \frac{[k_\theta]_\Sigma}{\hat{k}^2}$$

and the modelling is complete. We remark that the present result holds for the case of solution with $\varepsilon = 0$, where gravitational waves are absent and ${}_GT_{ab} = 0$. In the use of the approximate solutions (14)

$$G(c) \simeq -2c = r\rho_S(-\gamma)^{1/2} \ .$$

It follows then that the decrease of the mass function of the configuration corresponds to the outward radial emission of neutrinos and strings on the shell.

Our task now is to interpret the third term in (19), ${}_GT_{ab}$ occurring in the class of solutions with $\varepsilon \neq 0$, where gravitational waves are present. In the realm of these solutions, the form of the stress-energy tensors on the shell ${}_NT_{ab}$ and ${}_ST_{ab}$ will be same, with the corrections

$$\frac{[\lambda_\theta]_\Sigma}{2\hat{K}} = E(c) \quad \rightarrow \quad \frac{[\lambda_\theta]_\Sigma}{2\hat{K}} = E(c) + \varepsilon\left([w_\theta]_\Sigma - [y_\theta]_\Sigma\right)\frac{N}{\hat{\alpha}^2} + O(\varepsilon^2) \tag{26.a}$$

$$\frac{[K_\theta]_\Sigma}{\hat{K}^\alpha} = G(c) \quad \rightarrow \quad \frac{[K_\theta]_\Sigma}{\hat{K}^2} = G(c) + \varepsilon\left(k[y_\theta]_\Sigma - y[k_\theta]_\Sigma\right)N + O(\varepsilon^2) \tag{26.b}$$

${}_NT_{ab}$ is still modelled by the class of radial neutrinos, with the identifications (23) and (26a), and by construction it is independently conserved. The parcel ${}_ST_{ab}$ can still be modelled by radial strings with the identification $[K_\theta]_\Sigma/K^2_\# = \rho_S(-\gamma)^{1/2}$ but now it is not in general independently conserved and must satisfy the local conservation law $\left({}_ST^{ab} + {}_GT^{ab}\right)_{||b} = 0$. It results that

$${}_ST^{ab}{}_{||b} = J^a = -{}_GT^{ab}{}_{||b} \ , \tag{27}$$

where $J^a = (J, J, 0)$, with

$$J = \left\{\frac{\left[\frac{K'}{K_\theta}\right]_\Sigma}{r\hat{K}\hat{\alpha}}\right\} = \varepsilon[y_\theta]_\Sigma \frac{N'}{r\hat{\alpha}} \tag{28}$$

From (6)-(7) and (27) we can see that whenever gravitational waves are present the stress-energy tensor of the radial strings is not conserved, this non-conservation being at the expenses of gravitational waves (an exception would be gravitational waves for which wave fronts are not discontinuous at $\theta = \pi/2$, in $O(\varepsilon)$ of perturbation; this in principle would be possible but we were not able to exhibit a solution with the mentioned property). Furthermore

$$ {}_G T^{ab} = -\varepsilon [y_\theta]_\Sigma \frac{N'}{\hat{\alpha}^2} \left(\delta_0^a \delta_0^a + \delta_1^b \delta_1^a + \delta_0^a \delta_1^b + \delta_1^a \delta_0^b \right) \tag{29} $$

is different of zero if and only if gravitational waves are present. These facts led us to interpreted the stress-energy tensor ${}_G T^{ab}$ as describing the dynamics of the discontinuity of gravitational wave fronts in $\sum$, and which behave dynamically as a null fluid intrinsic to the $1+2$ spacetime $\sum$ of the shell; we were not able to model this tensor from the dynamics of a geometrical object defined intrinsically in $\sum$.

For the approxiamte solutions with $c = 0$ we have $[w_\theta]_\Sigma = [y_\theta]_\Sigma = 2b_0$. We may note that the $O(\varepsilon)$ corrections in this case reproduce the same structure of the shell modelled by radially propagating neutrinos, strings and gravitational waves, with the $N(u)$ function having the standard exponential from and b_0 playing a role analogous to c in measuring both the rate of decrease of the mass function and the break of analyticity.

5 The Collapse of a Pulse of Gravitational Waves

There is still a class of solutions of *RT* equations with gravitational waves ($\varepsilon \neq 0$) which may be of physical interest. This class corresponds to taking $g_(\theta) = 1 = k(\theta)$ and $b(u) = 0$. In $O(\varepsilon)$ eqs. (15) and (16) reduce to

$$ \frac{Z'}{N} = b_0 \tag{30} $$

$$ \begin{aligned} (y_\theta \sin\theta) &= -(w + 2y)\sin\theta \\ (w_\theta \sin\theta)_\theta &= -2b_0 \sin\theta \ . \end{aligned} \tag{31} $$

The associated geometry is given by the line element

$$ ds^2 = \left[1 + \varepsilon w(\theta) N(u) + \varepsilon \ \frac{Z(u)}{r} - 2\varepsilon r N' \right] du^2 - 2du dr - r^2 \left[1 + \varepsilon y(\theta) N(u) \right]^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{32} $$

Our use of $e = -1$ (cf. eq. (1)) here implies that the coordinate u has the asymptotic nature of an advanced Eddington-Finkelstein coordinate, that allows us to give a physical significance to the present solution, as we will see.

The C^0 solution constructed from the above functions exhibits a $O(\varepsilon)$ shell in the equatorial plane $\theta = \pi/2$, which can be modelled exactly analogous to previous cases examined, where the parameter b_0 is associated with quantities intrinsic to the shell through $[w_\theta]_\Sigma = -2[y_\theta]_\Sigma = -[y_\theta]_\Sigma = -4b_0$. However for the geometry (33) and its restriction to the $1+2$ spacetime of the shell, the vector $e^{(1)}_\alpha$ of the triad basis in which the stress-energy tensors (19) are expressed now has the form

$$e^\alpha_{(1)} = \left(-\hat{\alpha}^{-1}(u,r), -\hat{\alpha}(u,r)0, 0,\right) \ .$$

Therefore the flux of neutrinos, strings and gravitational waves discontinuities now propagate radially inwards, namely, in the direction of decreading r.

Now, in the order of approximation considered the function $N(u)$ is arbitrary, and can be prescribed at the past null infinity as an advanced time analytic pulse of gravitational waves with finite duration $\Delta u\, :\, \sigma_1 \leq u \leq \sigma_2$. The pulse is sent inwards from the past null infinity of the RT geometry (33). The past null asymptotic region can be here approximately defined using the retarded null coordinate of the underlying $\varepsilon = 0$ Minkowski geometry[1]. For $u \leq \sigma_1$ we have the Minkowski spacetime in the null advanced coordinate system (u, r, θ, φ), In interval $\sigma_1 \leq u \leq \sigma_2$ the spacetime is described by (33); a shell forms in $u = \sigma_1$ with neutrinos, strings and gravitational wave front discontinuties propagating inwards, and disappears in $u = \sigma_2$, when an event horizon forms. The mass function $Z(u)$ is given by

$$Z(u) = -\varepsilon b_0 \int_{\alpha_1}^{u} N(u) du + O(\varepsilon^2) \ .$$

It is zero for $u \leq \sigma_1$ and increases continuously in the interval $\sigma_1 \leq u \leq \sigma_2$ reaching the constant value $Z(\sigma_2)$. For $u \geq \sigma_2$ an event horizon forms and the solution (33) describes a Schwarzschild black-hole with infinitesimal mass $-\varepsilon Z(\sigma_2)/2$. For $\sigma_1 \leq u \leq \sigma_2$ an apparent horizon is present in (33), described by $r = -\varepsilon Z(u) + O(\varepsilon^2)$.

The above solution may be considered a simple but contrived model of the collapse of a finite pulse of gravitational waves to form a Schwarzschild black hole of inifinisimal mass. During the duration of the pulse there occurs the formation of an equatorial shell of matter, modelled by strings, neutrinos and gravitational wave front discontinuities

[1]The geometry (33) is a time perturbation of the Minkowski geometry, in the sense that its non-null curvature components are all $O(\varepsilon)$. This could be seen move immediately from the geometry itself with the use of Bondi-Sachs coordinates [26].

propagating radially inwards. The apparent horizon of the geometry as well as its mass function increases continuously, and reach a constant value when an event horizon forms.

6 Final Comments and Conclusions

In this paper we have studied a class of Robinson-Trautman metrics, solutions of Einstein's equations, for which the angular dependence of the metric functions is non-analytic. These functions are chosen to be of class C^0 in the angular coordinates having finite discontinuous first derivatives through the $\theta = \pi/2$ timelike hypersurface $\sum$. We show that in general such discontinuities in the Robinson-Trautman geometries demand the presence of a shell of matter localized in $\sum$ and that the associated stress-energy tensor of the shell can be modelled by neutrinos, strings and gravitational wave front discontinuities propagating radially on $\sum$. The bulk of the paper are Sections 3 and 4, where discuss we and exhibit solutions describing the above configurations. Due to the difficuties in the equations for the gravitational dynamics of the problem, our explicit solutions were given as perturbative configurations of the Minkowski solution or of the Schwarzschild solution, the pertubations involving expansion in two parameters, a separation constant c closely connected to the non-analyticity of the angular funtions and a perturbative parameter ε describing gravitational wave solutions of the RT equation (in some class of solutions a unique ε parameter has both roles). The separation constant c turns out to be a physically meaningful quantity related to the ratio of decreasing of the mass function of the configuration due to the emission of neutrinos and strings radially in the equatorial plane of the shell. This fact furnishes us with one possible physical characterization of the mass function $b(u)$ in the sense that the time derivative of the latter (cf. Eqs. (11a) or (13)) is proportional to the flux of radial neutrinos in $1+2$ spacetime of the shell, defined by the (01) component of the stress-energy tensor (22). The (u, r, θ, ϕ) coordinate system in which $b(u)$ has the from (13) seems to be more appropriate for the physical interpretation of the solutions. Non-analytic solutions with gravitational waves ($\varepsilon \neq 0$) present the same shell structure $\sum$, but the associated stress-enegy tensor of the shell has one additional parcel which describes the dynamics of gravitational wave front discontinuities occuring in $\sum$, and is different of zero if and only if gravitational waves are present. This of the stress-energy tensor is dynamically the one of a null intrinsic to the $1+2$ spacetime $\sum$

of the shell, but we were not able to model it from the dynamics of a geometrical object intrinsically in $\sum$. It also has the important property that it is not independetly conserved but it changes current with the strings present in the shell. In other words, the non-conservation of the flux strings is made at the expenses of gravitational waves and vice-versa. Non-analyticity in metric functions of bounded gravitational configurations emitting gravitational waves and matter appears to be intuitively a demand for the balance of energy in the spacetime, and is the source of the presence of line singularities in exact Robinson-Trautman solutions for Einstein's vacuum equations. In our present model, the shell structure actually substitutes lines of singularities in the metric functions through which a balance of energy emitted or absorbed by the source could be realized, and allows us to characterize unambiguously the mass-energy loss or mass-energy accretion due to the emission or absorption of neutrinos, strings and gravitational waves.

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$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \ \gamma^3 = \begin{pmatrix} 0 & 1 \\ -1 & -0 \end{pmatrix} \ .$$