On the Algebraic K-theory of The Massive D8 and M9 Branes

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ABSTRACT

We study the relation between the D8-branes wrapped on an orientable compact manifold W in a massive Type IIA supergravity background and the M9-branes wrapped on a compact manifold Z in a massive d=11 supergravity background from the K-theoretic point of view. By interpreting the D8-brane charges as elements of $K_0(C(W))$ and the (inequivalent classes of) spaces of gauge fields on the M9-branes as the elements of $K_0(C(Z) \times_{\bar{k}^*} G)$ a connection between charges and gauge fields is argued to exists. This connection is realized as a map between the corresponding algebraic K-theory groups.

1. Introduction

As the result of the analysis of the non-BPS brane states [?] the picture of the charges of the Dp-branes wrapped on an (orientable) compact manifold W as elements of the topological K-theory of W emerged [?, ?]. The charges of all possible Dp-brane configurations actually take value in the abelian groups $K^0(Y)$, $K^{-1}(Y)$ and $KO^0(Y)$ for Type IIB, Type IIA and Type I branes, respectively, where Y is the d=10 spacetime and $W \subset Y$. By imposing the tadpole anomaly cancellation in Type IIB and Type I theories, the groups reduce to $\widetilde{K^0}(Y)$ and $\widetilde{KO^0}(Y)$, respectively [?]. In Type IIA theory the non-existence of any RR boundary state guarantees that there is no RR spacetime tadpole anomaly [?].

Several important questions have already been addressed in literature in the frame of this theory. The list includes the possibility of using the Grothendieck groups and the derived categories in the study of brane charges [?], the computation of brane charges in various backgrounds [?, ?], the analysis of T-duality of non-BPS states [?, ?, ?] and the classification of descent and duality relations among branes [?].

Another important problem pointed out in [?] is the understanding of the K-theory classification of brane charges from eleven dimensional point of view. The motivation for suspecting a connection between Type IIA brane charges and some d=11 objects comes from the remark that, on one hand, the M-theory compactified on S^1 is the Type IIA theory and, on the other hand, the K-theory group of Type IIA brane charges is $\widetilde{K^{-1}}(Y) = \widetilde{K^0}(Y \times S^1)$ [1]. The presence of S^1 in both theories suggests that the circle should be actually the same. However, the major obstruction in realizing this idea in a concrete manner is the fact that there are no 10-branes in M-theory. This prevents us from giving a sensible physical interpretation of K-theory of Type IIA branes in eleven dimensions [?, ?]. One way to circumvent this difficulty is to use a K-theory that satisfies the following conditions: i) it allows a physical interpretation of its elements in d=11 and ii) it represents the Type IIA D-brane charges in d=10.

A theory that satisfies the two conditions above is the algebraic K-theory [?, ?]. The group $K_0(C(X))$ classifies the finitely generated projective $C(X)$-modules which are just the spaces of sections of vector bundles with base manifold X. Since these sections, at their turn, can be interpreted as gauge fields on X, $K_0(C(X))$ satisfies i) above. The condition ii) is automatically satisfied since by construction $K_0(C(W)) = K^0(W)$ [?].

The other crucial ingredient neccesary to describe the D-brane charges from the eleven dimensional perspective is a map between the branes and the corresponding objects in d=11. This map can alternatively be thought as a map between Type IIA theory and the

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[1]In general one can replace the $\widetilde{K^0}(Y)$ group with $K^0(Y)$ group because the corresponding K-theories are actually K$_c$-theories, i.e. the vector bundles satisfy, from physical requirements, some appropriate compact support conditions [?].

d=11 theory in which the objects are defined. We have seen above that if the algebraic K-theory is to be used the objects are the spaces of gauge fields. Then, as the previous discussion and the original formulation of the problem suggests, the sought for theory in d=11 should be M-theory or a related one. Another way to think of this is to notice that the map between d=10 and d=11 theories should appear from a natural connection between these. For M-theory and Type IIA theory there is such of connection given by the dimensional reduction.

In the following we shall illustrate these ideas on a system formed from D8 and M9 branes wrapped on compact manifolds which are embedded in massive Type IIA and massive d=11 supergravity backgrounds, respectively. The massive d=11 supergravity is the one proposed in [?]. The motivation for choosing this system relies on the following known facts. The D8-brane is the highest stable Type IIA D-brane and its charges take value in $K^0(W)$. Even if 8-brane-antibrane configurations do not contain all lower dimensional brane configurations, as was pointed out in [?], they contain lower dimensional brane charges. This makes $K^0(W)$ a nontrivial interesting object. It is also known that a D8-brane can be obtained from a M9-brane by double dimensional reduction [2]. A M9-brane moves freely in a massive d=11 supergravity background with a Killing isometry [?, ?, ?]. The massive d=11 supergravity is connected to the massive Type IIA supergravity by dimensional reduction. Moreover, its solitonic solutions include all M-branes from which all Type IIA branes can be obtained by direct or double dimensional reductions. [3] In order to construct a map between the spaces of gauge fields on the M9-brane and the D8-brane charges we have to associate to the dimensional reduction a geometrical map. In the most favorable case the same map acts simultaneously between the backgrounds and the spaces on which the branes are wrapped. As a general case we will assume that the dimensional reduction map acts only between the backgrounds. We note that because of the presence of the Killing isometry, the most natural sections of vector bundles on d=11 background and its compact submanifold are the covariant ones. These are (free generated) projective modules over the appropriate crossed algebras [?].

The organization of the paper is as follows. In Section 2 we resume some of the properties of massive Type IIA supergravity and massive d=11 supergravity. The map between the manifolds and the algebras and the apropriate algebraic K-theory groups are given in Section 3. In Section 4 we determine the correspondence between the K-theory groups of gauge fields on the M9-brane and the D8-brane charges for various types of Killing vectors. The last section is devoted to discussions. Two important mathematical results which are used throughtout the paper are collected in an Appendix.

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[2]The M9 branes have been initially discussed in [?, ?, ?, ?, ?, ?, ?].

[3]For earlier discussions on massive Type IIA supergravity see [?, ?, ?].

2. D8 and M8 branes in massive supergravity backgrounds

The massive Type IIA supergravity has the following bosonic field content: the metric g, the dilaton field ϕ, the RR one-form field $C^{(1)}$ which plays the role of a Stueckelberg field, a massive two-form field B and a three-form potential $C^{(3)}$. The massless Type IIA supergravity is obtained from the above theory in two steps. Firstly, one has to redefine the fields in order to remove the m^{-1} terms present in the supersymmetry transformation. Secondly, one has to take the limit $m = 0$. The theory displays, in string metric, a cosmological constant that does not depend on the dilaton. This suggests that the mass parameter can be viewed as the expectation value of the dual of a RR ten-form field strength.

There is a natural RR nine-form field A_9 in the spectrum of the Type IIA string theory. In order to formulate the massive Type IIA supergravity with this form field the mass parameter m should be replaced with a mass scalar field $M(x)$ that obeys the constraint $dM(x) = 0$. The field A_9 does not introduce new degrees of freedom and enters a Lagrangian term of the form $\sim dA_9 M$. Thus, the field strength of A_9 is a Lagrange multiplier for the constraints of $M(x)$. The solitonic solutions of the massive Type IIA supergravity with the nine-form field include the ones of massless as well as of massive Type IIA supergravity. In particular, the solution that carries the A_9 charge is the D8-brane. Because the solutions of this theory includes the Type IIA branes in a concise and elegant manner, it was proposed that it should be considered as the effective field theory of the Type IIA superstring [?]. For latter reference we give here the bosonic part of the massive Type IIA supergravity, in string frame, which has the following form [?, ?]

$$
\begin{aligned}
S[g, \phi, C^{(1)}, B, C^{(3)}] \;=\;\; & \frac{1}{16\pi G_N^{10}} \int d^{10}x \sqrt{|g|} \{ e^{-2\phi}[R(\omega) - 4(\partial\phi)^2 + \frac{1}{2 3!}H^2] \\
& - [\frac{1}{4}(G^{(2)})^2 + \frac{1}{2 4!}(G^{(4)})^2 + \frac{1}{8}] \\
& + \frac{1}{144}\frac{\epsilon}{\sqrt{|g|}}[\partial C^{(3)}\partial C^{(3)}B + \frac{1}{4}m\partial C^{(3)}b^3 + \frac{9}{320}m^2 B^5]\}, \quad (2.1)
\end{aligned}
$$

where the notations are the ones used in [?].

The massive d=11 supergravity proposed in [?] has the following bosonic field content: the metric $\hat{g}$ and the three-form field $\hat{C}$. The theory is characterized by an apriori given Killing isometry $\hat{k}$ of the background fields, i.e. $\hat{\pounds}_{\hat{k}}\hat{g} = \hat{\pounds}_{\hat{k}}\hat{C} = 0$. It is important to note that, as a consequence, there is a system of coordinates in which $\hat{k}^{\hat{\mu}} = \delta^{\hat{\mu}y}$ and the fields do not depend on the coordinate y on the integral curve of the Killing vector. In order to obtain the massless d=11 supergravity one has to set the mass parameter to zero and to restore the y dependence of fields. We recall that the massless d=11 supergravity represents the effective field theory of M-theory and the decompactifying limit of massless Type IIA supergravity. At its turn, the latter is the $g_s \to \infty$ limit of Type IIA superstring.

The bosonic part of the massive d=11 supergravity action [**?**] is given by

$$
\hat{S}[\hat{g}, \hat{C}] = \frac{1}{16\pi G_N^{11}} \int d^{11}x \sqrt{|\hat{g}|} \{\hat{R}(\hat{\Omega}) - \frac{1}{24!}\hat{G}^2 - \frac{1}{8}m^2|\hat{k}^2|^2
$$
$$
+ \frac{1}{(144)^2}\frac{\hat{\epsilon}}{\sqrt{|\hat{g}|}}[16\partial\hat{C}\partial\hat{C}\hat{C} + 9m\partial\hat{C}\hat{C}(i_{\hat{k}}\hat{C})^2 + \frac{9}{20}m^2\hat{C}(i_{\hat{k}}\hat{C})^4]\}, \quad (2.2)
$$

where $\hat{\Omega}$ represents the connection for the massive gauge transformation.

The M9-brane is a solitonic solution of the above massive d=11 supergravity conjectured to exist for several reasons. One of them is the non-vanishing time component of the two-form central charge $Z^{\hat{\mu}\hat{\nu}}$ of the M-theory superalgebra which also points out that the worldvolume field theory of the M9-brane should be N=1 (chiral) supersymmetric [**?**, **?**]. Another argument comes from the analysis of the eleven dimensional $E_8 \times E_8$ heterotic string [**?**]. Finally, the M9-brane fills in the place of the missing d=11 object that should fit into the pattern of generating D-branes from M-branes [**?**, **?**] (see also [**?**, **?**, **?**, **?**, **?**].) The massive d=11 supergravity is a theory in which the free M9-brane moves naturally, i.e. a world-volume field theory which contains a vector multiplet with a single scalar necessary to position the brane, can be constructed. This is possible by assuming that the Killing isometry of the background is an isometry of the world-volume of the M9-brane, too. Consequently, the world-volume field theory of the M9 should be a gauged sigma model. Actually, such of action was recently proposed by analogy with the MKK-monopole [**?**]. Like the other M-branes, the M9-brane gives rise to several Type IIA branes by different dimensional reductions: a D8-brane by reduction along Killing isometry and two gauge sigma models called KK-7A and KK-8A, respectively, by other reductions [**?**].

The Type IIA D8-brane can be identified with the solitonic solution of massive Type IIA supergravity with nine-form which has 9-dimensional Poincaré invariance. This was shown in [**?**] by proving that the massive 8-brane is T-dual with the Type IIB D7-brane as well as with the D9-brane. Another argument is that, in general, the background between two D8-branes is the massive Type IIA supergravity [**?**].

3. Algebraic K-theory of massive branes

In this section we construct a geometrical dimensional reduction map between massive Type IIA and d=11 supergravity backgrounds and give the algebraic K-theory for D8-branes and M9-branes and for the backgrounds in which they are wrapped.

3.1. The Dimensional Reduction Map

Let us consider a D8 brane-antibrane system wrapped on a compact submanifold W. The massive Type IIA supergravity background in which W is embedded is denoted by

$(Y; g, \phi, C^{(1)}; B, C^{(3)})$, where Y is the spacetime manifold on which the equations of motion for the fields of the massive Type IIA supergravity hold. There is a natural inclusion map $p : W \rightarrow Y$. In a similar way, we take the M9-branes to be wrapped on a compact submanifold Z in the massive d=11 supergravity background $(X; \hat{g}, \hat{C}; \hat{k})$ and the natural inclusion $i : Z \rightarrow X$.

In order to connect the theory in d=11 with the one in d=10, we need a map between them. Let us consider the possibility of using the dimensional reduction of the backgrounds. if we denote by $\hat{\Phi}$ the fields $\{\hat{g}, \hat{C}\}$ and by Φ the fields $\{g, \phi, C^{(1)}, B, C^{(3)}\}$, the dimensional reduction can be thought as the a map

$$\lambda : \hat{\Phi} \longrightarrow \Phi \tag{3.1}$$

which explicitely maps $\lambda(\hat{g}) = (g, \phi, C^{(1)})$ and $\lambda(\hat{C}) = (B, C^{(3)})$. The precise correspondence between the components by dimensional reduction is given by [?]

$$\begin{aligned}
\hat{g}_{yy}(\hat{x}) &= -e^{\frac{4}{3}\phi(x)} \\
\hat{g}_{\mu y}(\hat{x}) &= -e^{\frac{4}{3}\phi(x)} C_{\mu}^{(1)}(x) \\
\hat{g}_{\mu\nu}(\hat{x}) &= e^{\frac{2}{3}\phi(x)} g_{\mu\nu}(x) - e^{\frac{4}{3}\phi(x)} C_{\mu}^{(1)}(x) C_{\nu}^{(1)}(x)
\end{aligned} \tag{3.2}$$

for the d=11 metric, and

$$\begin{aligned}
\hat{C}_{\mu\nu\rho}(\hat{x}) &= C_{\mu\nu\rho}^{(3)}(x) \\
\hat{C}_{\mu\nu y}(\hat{x}) &= B_{\mu\nu}(x)
\end{aligned} \tag{3.3}$$

for the d=11 three-form field. If we introduce (??) and (??) in the eleven dimensional action (??) then we obtain the Type IIA action (2.1) by using a Palatini type identity. The two gravitational constants are related by $G_M^{10} = (2\pi l_s) G_N^{11}$ since it is assumed that the integral curve of the Killing vector is a circle of radius equal to the string length l_s [?]. Consequently, we can obtain solutions of the equations of motion of (2.1) from solutions of motion of (??) by dimensional reduction [?, ?]. This shows that if we assume the dimensional reduction map (??) between the two background fields in different dimensions we should extend it to the spacetime manifolds

$$\bar{\lambda} : Y \longrightarrow X. \tag{3.4}$$

Another way to see that is to notice that λ is equivalent to the explicit relations (??) and (??). At their turn, these represent relationships among the component fields, which are functions on X and Y, respectively. In order to have the equality between the functions in the both hand sides of (??) and (??), a relationship between the domains of the arguments $Dom(\hat{x}) = X$ and $Dom(x) = Y$ must exist. This relationship is the explicit form of the map $\bar{\lambda}$.

In the adapted coordinate system in which (**??**) and (**??**) are written, the functions on X do not depend on the coordinate y. However, since $\hat{k}$ is a Killing vector, in principle, one can extend $\bar{\lambda}$ over the entire manifold X [**?**]. The relation (**??**) defines a geometrical map between the two spacetime manifolds. Depending on the direction on which the dimensional reduction is performed we can arrive at different manifolds Y from the same X. For example, Y can inherit a Killing isometry if the dimensional reduction is performed along a direction transversal to $\hat{k}$ or does not inherit it if the direction is along $\hat{k}$.

In a standard fashion, λ induces the map between continuous (smooth) functions

$$\bar{\lambda}^* : C(X) \longrightarrow C(Y) \tag{3.5}$$

which is a homomorphism between the C*-algebras of all continuous functions on X and Y, respectively. Actually, like in the case of $\bar{\lambda}$, the map is not defined over the full $C(Y)$ since the functions depending on y are left aside. Therefore, (**??**) denotes, by an abuse of notations, the induced map by the extension over Y.

The concrete realisation of the maps (**??**) and (**??**) depends on the choice of the specific solutions of the equations of motion of the two massive supergravity theories. This choice also determines their properties. The general analysis that follows relies mainly on the homomorphism (**??**).

3.2. d=10 Algebraic K-Theory

The charges of the D8-branes belong to the topological K-theory group $K^0(W)$. Its elements are equivalence class of the complex vector bundles $E \to W$. Alternatively, they can be interpreted as the elements of the group $K_0(C(W))$ that classifies the spaces of gauge fields over W.

In order to see that, let us denote by $\Gamma(E)$ the sections of E. $\Gamma(E)$ can be thought as a finitely generated projective module over the ring (algebra) $C(W)$ of all complex-valued continuous functions over W. [4]. By a theorem by Serre and Swan [**?**], there is a complete equivalence between the category of vector bundles over a compact space and the bundle maps and the category of projective modules of finite type over commutative algebras and module morphisms. In particular, there is an isomorphism between the monoid of isomorphism classes of complex vector bundles over W (with Whitney sum) and the monoid of isomorphism classes of finitely generated projective modules over $C(W)$ (with ordinary direct sum) [**?**, **?**]. This isomorphism suggests that a group similar to $K^0(W)$ could be constructed for the projective modules of sections of bundles. The construction is standard and goes as follows [**?**, **?**, **?**, **?**].

Let us take the algebra $M_n(C(W))$ of $n \times n$ matrices with entries in $C(W)$. This can be identified with the algebra $C(W, M_n)$ of all continuous functions from W to the

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[4]Since $K^0(W)$ contains information about the topology of W, we work with $C(W)$. If the bundles are real, the module of sections is defined over $C_{\mathbb{R}}(W)$, the ring of *real*-valued functions over W.

algebra of $n \times n$ complex matrices. Every idempotent of $C(W, M_n)$ is a finitely generated projective module over $C(W)$ and it can be obtained from a bundle over W. Define next the inductive limit of finite matrices $M_n(C(W))$ by

$$M_\infty(C(W)) = \bigcup_{n=1}^{\infty} M_n(C(W)) \qquad (3.6)$$

with the natural embedding

$$\varphi : M_n(C(W)) \longrightarrow M_{n+1}(C(W)) \ , \ a \longmapsto \varphi(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} . \qquad (3.7)$$

Since $C(W)$ is an unital C*-algebra [5] every idempotent $p \in C(W)$ belongs to the set $Q(C(W)) = \{p \in C(W) : \exp(2\pi i p) = 1\}$. Two matrices $a \in Q_n(C(W))$ and $b \in Q_m(C(W))$ are equivalent if a and b have similar trivial extensions in some $Q_k(C(W))$

$$a \sim b \iff \exists \ a' = a \oplus 0_{k-n} \in Q_k(C(W)) \ , \ \exists b' = b \oplus 0_{k-m} \in Q_k(C(W))$$
$$\exists \ u \in GL_k(C(W)) : a' = ub'u^{-1}, \qquad (3.8)$$

where $GL_n(C(W))$ is the group of all invertible elements of $M_n(C(W))$. The set of eqivalence classes under $\sim$ in $\cup_n Q_n(C(W))$ is an abelian semigroup denoted by $J(C(W))$. Denote by $UJ(C(W))$ the universal group of $J(C(W))$ and by $E(C(W))$ its subgroup generated by elements of the form $\{a + b\} - \{a\} - \{b\}$, where $\{a\}$ is the equivalence class of a in $UJ(C(W))$. Then the first algebraic K-theory group is defined as

$$K_0(C(W)) = UJ(C(W))/E(C(W)). \qquad (3.9)$$

The group $K_0(C(W))$ is the Grothendieck group of the monoid of isomorphism classes of finitely generated projective $C(W))$ modules. By construction it satisfies the following identities

$$\begin{aligned} K^0(W) &= K_0(C(W)) \\ K^0_{\mathbb{R}} &= K_0(C_{\mathbb{R}}(W)) \end{aligned} \qquad (3.10)$$

for complex and real vector bundles, respectively.

Eq. (??) allows us to interpret the D8-brane charges as elements of either the topological K-theory group $K^0(W)$ or the algebraic group $K_0(C(W))$ and to shift from the complex vector bundles to sections which have the nice physical interpretation as gauge fields. The amount of information in both descriptions is the same due to the Serre-Swann theorem.

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[5] The structure of C*-algebra is given in the supremum norm $||f(x)||_\infty = sup_{x \in W} |f(x)|$. The involution $* : C(W) \to C(W)$ is the usual complex conjugation. Unital means that there is an element $\in C(W)$ such that $f = f = f \forall f \in C(W)$. All the algebras are commutative.

To $C(W)$ we can associate another algebraic abelian group $K_1(C(W))$ as follows. We construct firstly the group

$$L_n(C(W)) = GL_n(C(W))/GL_n^0(C(W)), \qquad (3.11)$$

where $GL_n^0(C(W)) = \exp(M_n(C(W))$ is the component of identity of $M_n(C(W))$. The canonical homomorphism $GL_n(C(W)) \rightarrow GL_{n+1}(C(W))$ induced by (??) yields a homomorphism $L_n(C(W)) \rightarrow L_{n+1}(C(W))$ which makes the sequence $\{L_n(C(W))\}$ into a direct limit system of groups. By definition, $K_1(C(W))$ is this limit and it is an abelian group

$$K_1(C(W)) = \varinjlim L_n(C(W)). \qquad (3.12)$$

We note that, by definiton, $K_1(C(W))$ takes into account the topology of $C(W)$[6].

Since the definitions (??) and (??) do not depend on a specific manifold W, they can be used for the compact manifold Z and for compact spacetimes X and Y. If X and Y are not compact, the above definitions no longer apply.

Assume, for definiteness, that X is just a locally compact manifold. The corresponding algebra $C_0(X)$ of complex-valued continous functions vanishing at infinity is non-unital. In this case we take the one point compactification X^+ of X and adjoin the unity to $C_0(X)$ to form the algebra $C_0(X)^+$. The map $C_0(X) \rightarrow C_0(X)^+/C_0(X)$ determines a complex homomorphism φ_0 of $C_0(X)^+$ which yields the homomorphism

$$\varphi_0^* : K_0(C(X^+)) \longrightarrow K_0(\mathbb{C}) = 0 \qquad (3.13)$$

The natural algebraic $\widetilde{K}$-theory groups are then defined in the usual manner by [?, ?, ?]

$$\begin{aligned}
\widetilde{K}_0(C_0(X)) &= ker\varphi_0^* \\
\widetilde{K}_1(C_0(X)) &= K_1(C_0(X^+)).
\end{aligned} \qquad (3.14)$$

We see that the mathematical construction of the algebraic K-theory groups is sensible to the topology of the manifolds that characterize our physical objects. In general, the properties of K- and $\widetilde{K}$-theories are not the same, so we should work with well defined topological spaces. For simplicity, we will assume in what follows that both spacetimes X and Y are compact. Moreover, when necessary, base points $x_0 \in Z \subset X$ and $y_0 \in W \subset Y$ are understood to be singled out. If the spacetime manifolds are only locally compact, one should work with their one point compactifications and with (??) but some extra care should be taken since, in general, different results are obtained.

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[6]For any ring R one can construct *two* groups K_1 by factorizing $GL_n(R)$ with $GL_n^0(R)$ like in (??) or with $E_n(R)$ which is the group of elementary matrices in $GL_n(R)$. In general, $E_n(R) \in GL_n^0(R)$ and thus the two groups differ. By construction, only (??) contains topological information about the ring R.

3.3. d=11 Algebraic K-Theory

Even for both X and Z compact, the groups $K_*(C(X))$ and $K_*(C(Z))$ are not appropriate to describe the d=11 spacetime and the M9-branes, respectively. The reason is that there is a Killing isometry $\hat{k}$ in the massive d=11 supergravity background. Also, if we want to obtain the 8-brane from the 9-brane by dimensional reduction, we have to perform it along the direction of the Killing isometry, which implies that this direction belongs to the world-volume of the M9-brane, too.

The Killing isometry, being a homeomorphism, induces the homeomorphisms $\hat{k}^*$ and $\bar{k}^*$ of $C(X)$ and $C(Z)$, respectively. Regarding $\hat{k}$ as the action of a uni-parametric group G on X and Z, $\hat{k}^*$ and $\bar{k}^*$ represent the induced actions on $C(X)$ and $C(Z)$, respectively. The most appropriate algebras for describing the sets $(C(X), \hat{k}^*, G)$ and $(C(Z), \bar{k}^*, G)$ are the crossed product algebras $C(X) \times_{\hat{k}^*} G$ and $C(Z) \times_{\bar{k}^*} G$, respectively. The crossed product algebra $C(X) \times_{\hat{k}^*} G$ is defined as the twisted convolution algebra $C_c(G, C(X))$ of continuous functions from G to $C(X)$ with a natural C*-norm [?] [7]. It is important to note that the unitary representations of the C*-algebra $C(X) \times_{\hat{k}^*} G$ correspond exactly to the covariant representations of $(C(X), \hat{k}^*)$. The same remarks are true for Z.

In order to make an explicite connection between D_8-brane system and the M_9-branes by using the K-theory, it is necessary to establish a relationship between $C(X)$ and $C(X) \times_{\hat{k}^*} G$. This mathematical problem has not been solved completely yet for an arbitrary group G. However, important results have been already derived for simply connected and solvable Lie groups, compact groups and some other groups with a simple structure [?]. In what follows we will primarily consider the cases when $G = \mathbb{R}, \mathbb{Z}$ and $\mathbb{T}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one dimensional toric group. In the last section we will discuss the more interesting case when G is an arbitrary compact group.

In the case when $\hat{k}$ represents the action of the real additive group $G = \mathbb{R}$ on X, we use a theorem due to Connes [?] that establishes the following isomorphism

$$K_p(C(X)) \times_{\hat{k}^*} \mathbb{R} \cong K_{p-1}(C(X)), \qquad (3.17)$$

where $p = 0, 1$. Similar isomorphisms exist if $\mathbb{R}$ is replaced by an arbitrary simply connected and solvable Lie group.

If the group is $G = \mathbb{Z}$, the dual group of the one-dimensional torus, a theorem by Pimsner and Voiculescu states that the following cyclic six-term exact sequence connects

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[7]The involutive algebra is given by

$$(\varphi_1 \circ \varphi_2)(g) = \int \varphi_1(h) \hat{k}_h^*(\varphi_2(h^{-1}g)) dh \qquad (3.15)$$

for any $\varphi_i \in C_c(G, C(X))$, $g, h \in G$, and

$$\varphi^*(g) = \delta(g)^{-1} \hat{k}_g^*(\varphi(g^{-1})^*), \qquad (3.16)$$

where $\delta : G \rightarrow \mathbb{R}_+^*$ is a modular function on G.

the algebraic K-theories of $C(X)$ and $C(X) \times_{\hat{k}^*} \mathbb{Z}$ [?]

$$
\begin{array}{ccccc}
K_0(C(X)) & \xrightarrow{1-\hat{k}_*^*} & K_0(C(X)) & \xrightarrow{\iota_*} & K_0(C(X) \times_{\hat{k}^*} \mathbb{Z}) \\
\\
\sigma_* \uparrow & & & & \downarrow \sigma_* \\
\\
K_1(C(X) \times_{\hat{k}^*} \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(C(X)) & \xleftarrow{1-\hat{k}_*^*,} & K_1(C(X))
\end{array}
\qquad (3.18)
$$

where $\hat{k}_*^*$ is the induced map by $\hat{k}^*$ in the K-theory and ι_* is the induced map by $C(X) \to C(X) \times_{\hat{k}^*} \mathbb{Z}$.

A more interesting case is when $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the torus group. Then the following cyclic six-term sequence is exact [?]

$$
\begin{array}{ccccc}
K_0(C(X) \times_{\hat{k}^*} \mathbb{T}) & \xrightarrow{1-\hat{k}_*^*} & K_0(C(X) \times_{\hat{k}^*} \mathbb{T}) & \xrightarrow{t_*} & K_0(C(X)) \\
\\
\rho_* \uparrow & & & & \downarrow \rho_* \\
\\
K_1(C(X)) & \xleftarrow{t_*} & K_1(C(X) \times_{\hat{k}^*} \mathbb{T}) & \xleftarrow{1-\hat{k}_*^*,} & K_1(C(X) \times_{\hat{k}^*} \mathbb{T})
\end{array}
\qquad (3.19)
$$

where ρ_* is induced by the quotient map

$$
\rho : C(X) \times_{\hat{k}^*} \mathbb{R} \longrightarrow C(X) \times_{\hat{k}^*} \mathbb{T}. \qquad (3.20)
$$

Here, we wrote explicitly all the maps in the exact sequences for later reference. Similar sequences exist for Z and $\bar{k}^*$.

Despite the fact that the six-term exact sequences have a limited general applicability, they provide useful information in many particular cases. We discuss this problem in the last section.

4. The Relationship between Type IIA and d=11 algebraic K-theories

In this section we are going to derive a map between the groups $K_0(C(W))$ and $K_0(C(Z) \times_{\bar{k}^*} G)$ which are associated with the D8-brane charges and the spaces of co-variant gauge fields on the M9-branes, respectively.

It is instructive to consider first the general situation in which the dimensional reduction map acts only between X and Y. There are natural inclusion maps for the pair (X, Z)

$$
\begin{array}{rcl}
i & : & Z \longrightarrow X \\
j & : & (X, \emptyset) \longrightarrow (X, Z)
\end{array}
\qquad (4.1)
$$

and similar ones for the pair (Y, W)

$$p \quad : \quad W \longrightarrow Y$$
$$q \quad : \quad (Y, \emptyset) \longrightarrow (Y, W). \tag{4.2}$$

A consequence of the Bott periodicity theorem (see Theorem 1 from Appendix) states that there are the following homomorphisms within the topological K-theory cohomologies of the pair (X, Z)

$$r^* \quad : \quad K^0(Z) \longrightarrow K^{-1}(X)$$
$$l^* \quad : \quad K^{-1}(Z) \longrightarrow K^0(X), \tag{4.3}$$

given by

$$r^* = j^* \circ (\delta \circ \beta) \quad , \quad l^* = j^* \circ \delta \tag{4.4}$$

Here, δ is the coboundary map and β is the Bott isomorphism. Similar homomorphisms can be constructed for the pair (Y, W), namely

$$f^* \quad : \quad K^0(W) \longrightarrow K^{-1}(Y)$$
$$g^* \quad : \quad K^{-1}(W) \longrightarrow K^0(Y) \tag{4.5}$$

given by the compositions

$$f^* = q^* \circ (\omega \circ \tau) \quad , \quad g^* = q^* \circ \omega, \tag{4.6}$$

where the coboundary map is ω and the isomorphism is τ.

Since the Connes' Thom isomorphism (**??**) and the exact sequences (**??**) and (**??**) involve higher groups K_1, we need to connect it with K^{-1}. In general, there is no isomorphism between the two groups [**?, ?**]. However, since in our setup the algebras of continuous functions on X, Z, Y and W are unital C*-algebras and since K_1 was defined using (**??**) according to Novodvorskii's theorem, the Gel'fand transform induces an isomorphism $K_p \to K^p$, $p = 0, 1$ for any of these algebras [**?**] (see also Theorem 2 from Appendix.) Let us denote these isomorphisms by γ for X, ψ for Z, φ for Y and ϵ for W. Then it is easy to verify that we have the following maps between the algebraic K-theory groups

$$m_* \quad : \quad K_0(C(Z)) \longrightarrow K_1(C(X)) \quad , \quad n_* : K_1(C(Z)) \longrightarrow K_0(C(X))$$
$$u_* \quad : \quad K_0(C(W)) \longrightarrow K_1(C(Y)) \quad , \quad v_* : K_1(C(W)) \longrightarrow K_0(C(Y)), \tag{4.7}$$

where m_*, n_*, u_* and v_* denote the following composition maps

$$m_* = \gamma \circ r^* \circ \psi^{-1} \quad , \quad n^* = \gamma \circ l^* \circ \psi^{-1}$$
$$u_* = \epsilon \circ f^* \circ \varphi^{-1} \quad , \quad v^* = \epsilon \circ g^* \circ \varphi^{-1}. \tag{4.8}$$

The properties of these maps are esentially the same as of (**??**) and (**??**) which are given by the Theorem 1 from Appendix.

While for the Type IIA algebraic groups u_* and v_* represent the end of the story, in d=11 we have to pass to the algebraic K-theory of the crossed product algebras as discussed in the precedent section.

Consider firstly the case $G = \mathbb{R}$ and denote by χ_*^X and χ_*^Z the Connes' Thom isomorphisms for X and Z, respectively. By using (**??**) to connect the K-theories of $C(X)$ and $C(Z)$ we obtain the following composition maps

$$
\begin{aligned}
\alpha_{*\mathbb{R}} &: & K_0(C(Z) \times_{\bar{k}*} \mathbb{R}) &\longrightarrow K_1(C(X) \times_{\hat{k}*} \mathbb{R}) \\
\beta_{*\mathbb{R}} &: & K_1(C(Z) \times_{\bar{k}*} \mathbb{R}) &\longrightarrow K_0(C(X) \times_{\hat{k}*} \mathbb{R}),
\end{aligned}
\tag{4.9}
$$

where $\alpha_{*\mathbb{R}}$ and $\beta_{*\mathbb{R}}$ are given by

$$
\begin{aligned}
\alpha_{*\mathbb{R}} &= \chi_*^X \circ n_* \circ \chi_*^Z \\
\beta_{*\mathbb{R}} &= \chi_*^X \circ m_* \circ \chi_*^Z.
\end{aligned}
\tag{4.10}
$$

We see that the map between the two sets of abelian groups $(\alpha_{*\mathbb{R}}, \beta_{*\mathbb{R}})$ is "twisted" in the sense that it changes the indices 0 and 1.

In the case $G = \mathbb{Z}$, the dual group of $\mathbb{T}$, we use the Pimsner-Voiculescu exact sequence (**??**) for both X and Z. Next, as above, we use m_* and n_* to relate the two sequences. From the sequences obtained we can read off the following maps

$$
\begin{aligned}
\alpha_{*\mathbb{Z}} &: & K_0(C(Z) \times_{\bar{k}*} \mathbb{Z}) &\longrightarrow K_0(C(X) \times_{\hat{k}*} \mathbb{Z}) \\
\beta_{*\mathbb{Z}} &: & K_1(C(Z) \times_{\bar{k}*} \mathbb{Z}) &\longrightarrow K_1(C(X) \times_{\hat{k}*} \mathbb{Z}),
\end{aligned}
\tag{4.11}
$$

which are given by the following compositions

$$
\begin{aligned}
\alpha_{*\mathbb{Z}} &= \iota_* \circ (1 - \hat{k}_*^*) \circ n_* \circ \sigma_* \\
\beta_{*\mathbb{Z}} &= \iota_* \circ (1 - \hat{k}_*^*) \circ m_* \circ \sigma_*.
\end{aligned}
\tag{4.12}
$$

Here, $\iota_* : K_1(C(X)) \to K_1(C(X) \times_{\hat{k}*} \mathbb{Z})$ is the map induced by the natural inclusion $\iota : C(X) \to C(X) \times_{\hat{k}*} \mathbb{Z}$ and $\sigma_* : K_1(C(Z) \times_{\bar{k}*} \mathbb{Z}) \to K_0(C(Z))$ is the vertical map of the Pimsner-Voiculescu exact sequence. $\hat{k}_*^*$ denotes the induced map in the algebraic K-theory by $\hat{k}^*$. Unlike the pair (**??**) discussed above, the pair $(\alpha_{*\mathbb{Z}}, \beta_{*\mathbb{Z}})$ is not twisted.

We proceed in the similar manner in the case $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. We construct the cyclic six-term exact sequences for X and Z using (**??**) and we connect them with m_* and n_*. As a result we obtain the pair

$$
\begin{aligned}
\alpha_{*\mathbb{T}} &: & K_0(C(Z) \times_{\bar{k}*} \mathbb{T}) &\longrightarrow K_0(C(X) \times_{\hat{k}*} \mathbb{T}) \\
\beta_{*\mathbb{T}} &: & K_1(C(Z) \times_{\bar{k}*} \mathbb{T}) &\longrightarrow K_1(C(X) \times_{\hat{k}*} \mathbb{T}),
\end{aligned}
\tag{4.13}
$$

where $\alpha_{*\mathbb{T}}$ and $\beta_{*\mathbb{T}}$ are composed by

$$
\begin{aligned}
\alpha_{*\mathbb{T}} &= \rho_* \circ m_* \circ t_* \\
\beta_{*\mathbb{T}} &= \rho_* \circ n_* \circ t_*
\end{aligned}
\tag{4.14}
$$

where $t_* : K_1(C(Z)) \times_{\bar{k}*} \mathbb{T}) \to K_1(C(Z))$ and ρ_* is the vertical map in the exact sequence (**??**) of Z. The pair (**??**) is not twisted. We also note that $\alpha_{*\mathbb{T}}$ and $\beta_{*\mathbb{T}}$ are defined in a minimal way, i.e. without any reference to the action of $\hat{k}$ on either X or Z.

By construction, the properties of α_* and β_* are given by the properties of the component maps for all the groups discussed above. It is important to notice that, although some of the components of these maps enter exact sequences for $G = \mathbb{Z}, \mathbb{T}$ and are Conne's Thom isomorphisms for $G = \mathbb{R}$, the composition maps do not belong, in general, to exact sequences due to the presence of m_* and n_*. Exactness is a supplementary condition which imposes the usual ker/im constraints on the components.

Let us return to the main problem of determining the map between the relevant d=10 and d=11 algebraic K-theory groups. We remark that for $G = \mathbb{Z}$ there is no obvious physical interpretation of the connection between the massive Type IIA and the massive d=11 supergravity branes and backgrounds in terms of dimensional reduction map. However, it is instructive to study this case because the exact sequence (**??**) in the more interesting case $G = \mathbb{T}$ is a Pimsner-Voiculescu like sequence. The case $G = \mathbb{R}$ can be thought of as the decompactifying limit of the Killing direction $R \to \infty$.

The starting point is the dimensional reduction map $\bar{\lambda}$ which induces the homomorphism $\bar{\lambda}^*$ between the unital C*-algebras $C(X)$ and $C(Y)$. This is assumed to be an unital homomorphism. Due to the categorical properties of the algebraic K-theory, $\bar{\lambda}^*$ yields the homomorphisms between the algebraic K-theory groups $K_p(C(X))$ and $K_p(C(Y))$, $p = 0, 1$, denoted by $\bar{\lambda}^*_*$. Consequently, it also connects the exact sequences in d=10 and d=11.

In the case $G = \mathbb{R}$, by connecting the sequences in different dimensions we obtain the following sequences

$$
\begin{aligned}
\ldots &\longrightarrow K_1(C(Z) \times_{\bar{k}*} \mathbb{R}) \longrightarrow K_0(C(Z)) \longrightarrow \ldots \longrightarrow K_0(C(X) \times_{\hat{k}*} \mathbb{R}) \\
&\longrightarrow K_1(C(X)) \xrightarrow{\bar{\lambda}^*_*} K_1(C(Y)) \longrightarrow K_0(C(W))
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
\ldots &\longrightarrow K_0(C(Z) \times_{\bar{k}*} \mathbb{R}) \longrightarrow K_1(C(Z)) \longrightarrow \ldots \longrightarrow K_1(C(X) \times_{\hat{k}*} \mathbb{R}) \\
&\longrightarrow K_0(C(X)) \xrightarrow{\bar{\lambda}^*_*} K_0(C(Y)) \longrightarrow K_1(C(W))
\end{aligned}
\tag{4.16}
$$

In general, (**??**) and (**??**) are not exact. However, we can define some composition maps $a_{*\mathbb{R}}$ and $b_{*\mathbb{R}}$ between the algebraic K-theory groups associated to the wrapped 8-brane

and 9-brane systems

$$
\begin{aligned}
a_{*\mathbb{R}} &: \quad K_1(C(Z) \times_{\bar{k}*} \mathbb{R}) \longrightarrow K_0(C(W)) \\
b_{*\mathbb{R}} &: \quad K_0(C(Z) \times_{\bar{k}*} \mathbb{R}) \longrightarrow K_1(C(W))
\end{aligned}
\tag{4.17}
$$

The expresion of these maps can be easily deduced from the sequences (**??**) and (**??**) and using (**??**) and (**??**) one can see that they have the following form

$$
\begin{aligned}
a_{*\mathbb{R}} &= u_*^{-1} \circ \bar{\lambda}_*^* \circ \chi_*^X \circ \beta_{*\mathbb{R}} \\
b_{*\mathbb{R}} &= v_*^{-1} \circ \bar{\lambda}_*^* \circ \chi_*^X \circ \alpha_{*\mathbb{R}}.
\end{aligned}
\tag{4.18}
$$

The properties of $a_{*\mathbb{R}}$ and $b_{*\mathbb{R}}$ are determined by those of the components and of $\bar{\lambda}_*^*$. In order to the sequences (**??**) and (**??**) be exact, $\bar{\lambda}_*^*$ as well as u_*^{-1}, v_*^{-1}, m_* and n_* should satisfy the ker/im exacteness conditions. It is an easy exercise to write them down from (**??**), (**??**) and (**??**).

In the case $G = \mathbb{Z}$ we can do a similar analysis since $\bar{\lambda}$ has a geometrical character. However, the discussion in this case is purely formal and there is no physical interpretation in terms of the dimensional reduction of the d=11 spacetime X to the d=10 spacetime Y. The sequences built out of $\bar{\lambda}_*^*$ from exact sequences in different dimensions are given by

$$
\begin{aligned}
\ldots \quad \longrightarrow \quad & K_0(C(Z)) \quad \longrightarrow \quad K_0(C(Z) \times_{\bar{k}*} \mathbb{Z}) \quad \longrightarrow \quad K_1(C(Z)) \quad \longrightarrow \quad \ldots \\
\longrightarrow \quad & K_0(C(X) \times_{\hat{k}*} \mathbb{Z}) \quad \longrightarrow \quad K_1(C(X)) \quad \xrightarrow{\bar{\lambda}_*^*} \quad K_1(C(Y)) \quad \longrightarrow \quad K_0(C(W))
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
\ldots \quad \longrightarrow \quad & K_1(C(Z)) \quad \longrightarrow \quad K_0(C(Z) \times_{\bar{k}*} \mathbb{Z}) \quad \longrightarrow \quad K_0(C(Z)) \quad \longrightarrow \quad \ldots \\
\longrightarrow \quad & K_1(C(X) \times_{\hat{k}*} \mathbb{Z}) \quad \longrightarrow \quad K_0(C(X)) \quad \xrightarrow{\bar{\lambda}_*^*} \quad K_0(C(Y)) \quad \longrightarrow \quad K_1(C(W))
\end{aligned}
\tag{4.20}
$$

The composition maps that can be constructed from these sequences act only geometrically between the K-theories in different dimensions as follows

$$
\begin{aligned}
a_{*\mathbb{Z}} &: \quad K_0(C(Z) \times_{\bar{k}*} \mathbb{Z}) \longrightarrow K_0(C(W)) \\
b_{*\mathbb{Z}} &: \quad K_1(C(Z) \times_{\bar{k}*} \mathbb{Z}) \longrightarrow K_1(C(W))
\end{aligned}
\tag{4.21}
$$

and they are composed from the following homomorphisms

$$
\begin{aligned}
a_{*\mathbb{Z}} &= u_*^{-1} \circ \bar{\lambda}_*^* \circ \sigma_* \circ \alpha_{*\mathbb{Z}} \\
b_{*\mathbb{Z}} &= v_*^{-1} \circ \bar{\lambda}_*^* \circ \sigma_* \circ \beta_{*\mathbb{Z}}.
\end{aligned}
\tag{4.22}
$$

We emphase once again that presently we cannot claim that (**??**) have any physical signifiance. However, they are helpful in understanding the next case.

When $\hat{k}$ represents the action of the group $G = \mathbb{T}$ on the manifold X, the homomorphism $\bar{\lambda}_*^*$ enters the following sequences

$$
\begin{aligned}
\ldots \quad &\longrightarrow \quad K_0(C(Z) \times_{\bar{k}^*} \mathbb{T}) \quad \longrightarrow \quad K_0(C(Z)) \quad \longrightarrow \quad \ldots \\
&\longrightarrow \quad K_1(C(X) \times_{\hat{k}^*} \mathbb{T}) \quad \longrightarrow \quad K_1(C(X)) \quad \xrightarrow{\bar{\lambda}_*^*} \quad K_1(C(Y)) \quad \longrightarrow \quad K_0(C(W))
\end{aligned}
$$

(4.23)

and

$$
\begin{aligned}
\ldots \quad &\longrightarrow \quad K_1(C(Z) \times_{\bar{k}^*} \mathbb{T}) \quad \longrightarrow \quad K_1(C(Z)) \quad \longrightarrow \quad \ldots \\
&\longrightarrow \quad K_0(C(X) \times_{\hat{k}^*} \mathbb{T}) \quad \longrightarrow \quad K_0(C(X)) \quad \xrightarrow{\bar{\lambda}_*^*} \quad K_0(C(Y)) \quad \longrightarrow \quad K_1(C(W))
\end{aligned}
$$

(4.24)

These sequences define the maps $a_{*\mathbb{T}}$ and $b_{*\mathbb{T}}$ between the corresponding K-groups of the dimensional reduction of the background X. They act as follows

$$
\begin{aligned}
a_{*\mathbb{T}} \quad &: \quad K_0(C(Z) \times_{\bar{k}^*} \mathbb{T}) \longrightarrow K_0(C(W)) \\
b_{*\mathbb{T}} \quad &: \quad K_1(C(Z) \times_{\bar{k}^*} \mathbb{T}) \longrightarrow K_1(C(W))
\end{aligned}
$$

(4.25)

From the relations (**??**) and (**??**) we obtain the following expressions

$$
\begin{aligned}
a_{*\mathbb{T}} \quad &= \quad u_*^{-1} \circ \bar{\lambda}_*^* \circ t_* \circ \alpha_{*\mathbb{T}} \\
b_{*\mathbb{Z}} \quad &= \quad v_*^{-1} \circ \bar{\lambda}_*^* \circ t_* \circ \beta_{*\mathbb{T}}.
\end{aligned}
$$

(4.26)

Like in the previous cases, the properties of $a_{*\mathbb{T}}$ and $b_{*\mathbb{T}}$ are given by the properties of the components. Another common feature with the other cases is that the sequences (**??**) and (**??**) are not, in general, exact.

Note that in the decompactifying limit of the Killing direction, the 8-brane charges are obtained rather from the K_1 associated to the 9-branes than from K_0 whose elements are equivalence classes of isomorphic gauge field spaces. This shows that the group $K_1(C(W))$ becomes important in those situations in which the massive d=11 background with M9-branes wrapped inside is subject to the dimensional reduction and ends into a massive Type IIA background with D8-branes wrapped inside, in the limit where the direction of the Killing isometry decompactifies.

If during the process of dimensional reduction the M9-branes reduce to D8-branes, a map $\bar{\mu}$ between Z and W exists. If the two reductions, of the background and of the 9-branes, respectively, hold simultaneously and along $\hat{k}$, then we can take $\bar{\mu} = \bar{\lambda}|$, where $\bar{\lambda}|$ is the reduction of $\bar{\lambda}$ to Z. However, in our analysis we can take a more general situation in which $\bar{\mu}$ is independent of $\bar{\lambda}|$. The particular case emerges from this one at equality.

Let us denote by $\bar{\mu}_*^*$ the induced map between the algebraic K-theories and consider the case $G = \mathbb{T}$. Since the wrapped 8-branes are obtained by dimensionally reducing the 9-branes, the two different gemetrical map compositions from $K_p(C(Z) \times_{\bar{k}*} \mathbb{T})$ to $K_p(C(W))$ should form commutative diagrams for $p = 0, 1$. Using (??) and (??) these diagrams can be written as

$$
\begin{array}{ccc}
K_0(C(W)) & & K_1(C(W)) \\
\bar{\mu}_*^* \uparrow \quad \searrow{\scriptstyle a}_{*\mathbb{T}} & & \bar{\mu}_*^* \uparrow \quad \searrow{\scriptstyle b}_{*\mathbb{T}} \\
K_1(C(Z)) \xrightarrow{\rho_*} K_0(C(Z) \times_{\bar{k}*} \mathbb{T}) & & K_0(C(Z)) \xrightarrow{\rho_*} K_1(C(X) \times_{\hat{k}}^* \mathbb{T})
\end{array} \tag{4.27}
$$

The resulting relations between maps represent constraints on $a_{*\mathbb{T}}$ and $b_{*\mathbb{T}}$ they are consequence of the condition that the D8-brane system is obtained from a M9-brane system. Note that these constraints do not imply $a_{*\mathbb{T}} = b_{*\mathbb{T}}$ since in the two diagrams in (??) the maps $\bar{\mu}_*^*$ and ρ_* are actually splitted, acting on different groups.

In the case $G = \mathbb{Z}$ we can also require the existence of the map $\bar{\mu}$ between Z and W as above. The difference occurs in that there are no physical reasons for taking $\bar{\mu} = \bar{\lambda}|$ and, more important, for assuming that the diagrams corresponding to (??) are commutative. If $G = \mathbb{R}$, the above construction can be repeated but in order to impose naturally the commutativity of the diagrams we have to take firstly a finite radius of the Killing direction and afterwards to let it go to infinity.

We note in the end of this section that if X and Y are only locally compact, one can repeat the above analysis if we take X^+ and Y^+ instead of the original spacetime manifolds. However, if the backgrounds are general, we must check out if the corresponding algebras of continuous functions are commutative C*-algebras with unity.

5. Discussions

It is of interest to see if the above analysis provides any information about all Type IIA D-brane charges which take values in the topological K-theory group $K^{-1}(Y)$ [?, ?]. By Novodvorskii's theorem (see Theorem 2 from Appendix) this group is isomorphic with $K_1(C(Y))$ in which $K_1(C(X))$ is mapped by the corresponding $\bar{\lambda}_*^*$ map. Consequentely, if $G = \mathbb{T}$ we can extract from the sequence (??) the following map

$$
c_{*\mathbb{T}} : K_0(C(X) \times_{\hat{k}*} \mathbb{T}) \longrightarrow K_1(C(Y)), \tag{5.1}
$$

where

$$
c_{*\mathbb{T}} = \bar{\lambda}_*^* \circ t_*. \tag{5.2}
$$

This function maps the covariant equivalent spaces of gauge fields from $K_0(C(X) \times_{\hat{k}*} \mathbb{T})$, i.e. defined on the massive d=11 supergravity background spacetime X with the toric Killing vector action, on the Type IIA D-brane charges in the massive Type IIA

background Y. Similar maps can be constructed for $G = \mathbb{R}$ and $G = \mathbb{Z}$. We note now that no reference is made to any 9-brane. As a matter of fact, the subsequence of (??) in which $c_{*\mathbb{T}}$ is defined as a composition do not include any group of Z as can be easily verified. The is also true for the other two groups.

Another important problem concerns the amount of information that can be extracted from the maps constructed in the previous section. The point here is that in constructing (??), (??) and (??) we used the Connes' Thom isomorphism, the Pimsner-Voiculescu six-term exact sequence and its consequence (??). While the isomorphism preserves the maximum of information while going from one group to another, the same is no longer true for the cyclic six-term exact sequences since the same group appears in two places. However, despite this limitation, six-terms sequences are a powerfull tool in many particular cases.

To illustrate this fact, let us consider that the 9-branes are wrapped on S^{10}. In this case the algebraic K-theory groups are given by

$$
\begin{aligned}
K_0(C(S^{10})) &= \mathbb{Z}/2\mathbb{Z} \\
K_1(C(S^{10})) &= 0.
\end{aligned}
\tag{5.3}
$$

By introducing (??) in the six-exact term sequence for, say, $G = \mathbb{Z}$ we obtain the following relations

$$
\begin{aligned}
K_0(C(S^{10}) \times_{\bar{k}*} \mathbb{Z}) &\cong (\mathbb{Z}/2\mathbb{Z})/im(1 - \bar{k}_*^*) \\
K_1(C(S^{10}) \times_{\bar{k}*} \mathbb{Z}) &\cong im(1 - \bar{k}_*^*)
\end{aligned}
\tag{5.4}
$$

which contain information about the K-groups as well as the action of the group of integers on the compact space. We can obtain similar relations for $G = \mathbb{T}$ or $G = \mathbb{R}$. These relations can be used to improve our knowledge of the maps a_* and b_*, respectively.

Another simplification of the sequences used to connect the relevant groups appears when the manifolds on which the 8- and 9-branes are wrapped are deformation retracts of the corresponding spacetime manifolds. Without presenting any details, we just note that if we take for example (X, Z) a compact pair then (X^+, Z^+) is pointed since the two base points are identified with the point at infinity and if we further assume that Z^+ is also a retract of X^+ the topological K-groups satisfy [?, ?]

$$
\widetilde{K}^{-p}(Z^+) \cong \widetilde{K}^{-p}(X^+).
\tag{5.5}
$$

Since we have the following relations

$$
\begin{aligned}
\widetilde{K}^{-p}(X^+) &= K_p(C(X)) \\
\widetilde{K}^{-p}(Z^+) &= K_p(C(Z))
\end{aligned}
\tag{5.6}
$$

supplementary relations among the maps m_* and n_* are given by the following sequence

$$
\begin{array}{ccc}
K_0(C(Z)) & \overset{m_*}{\longrightarrow} & K_1(C(X)) \\
\| & & \| \\
K_1(C(Z)) & \overset{n_*}{\longleftarrow} & K_0(C(X))
\end{array}
\tag{5.7}
$$

which connects the sequences that define α_* and β_* as composition maps.

There can be many topological configurations in which the six-term exact sequences provide more information than in their general formulation. Thus we conclude that the maps between the relevant groups which were constructed in the previous section carry signifiant information about the system.

So far we have considered mainly noncompact Killing vectors $\hat{k}$. Let us briefly discuss some of the particularities of the more realistic theories with a compact Killing isometry viewed as the action of a compact group. Consider the general case of a larger compact group of isometries G of both X and Z which are also compact manifolds [8].

Let us denote by K and action of G on X and by $\bar{K}$ an action on Z. To all actions $\{K\}$ correspond actions of G on $C(X)$ and on the vector bundles $E \to X$. A G-vector bundle is a vector bundle E for which the G-action is induced in a way that the projection $E \to X$ is equivariant. Like for usual vector bundles one can establish an exact correspondence between G-vector bundles and finitely generated projective modules of $C(X) \times_{K^*} G$. In particular, there are G K-theory groups the following isomorphisms hold [?]

$$
\begin{aligned}
\gamma_*^G &: & K_0^G(C(X)) &\longrightarrow K_0(C(X) \times_{K^*} G) \\
d_*^G &: & K_0^G(C(X)) &\longrightarrow K_0(C(X)),
\end{aligned}
\tag{5.8}
$$

where $K_0^G(X)$ is the abelian group of equivalence classes of G-vector bundles and $K_0^G(C(X))$ is the abelian (Grothendieck) group of equivalence classes of finitely generated G-projective modules over $C(X)$. The same construction can be done for Z and we denote the isomorphisms (??) in this case with χ_*^G and e_*^G, respectively.

In analogy with the case studied in the previous section, we would like to find a map between the G-covariant gauge fields on M9-branes and the D8-brane charges. Proceeding along the same line, we denote by i_*^G the map induced by the natural inclusion of Z in X in K-theory. It is not difficult to see that by using $\bar{\lambda}_*^*$ discused in the previous section we obtain the following sequence

$$
K_0(C(Z) \times_{\bar{K}^*} G) \longrightarrow K_0^G(C(Z)) \longrightarrow \cdots \longrightarrow K_0(C(Y)) \longrightarrow K_1(C(W)).
\tag{5.9}
$$

Unfortunately, this sequence defines a map between the equivalence classes of spaces of G-covariant gauge fields on M9-branes from $K_0(C(Z) \times_{\bar{K}^*} G)$ and the group $K_1(C(W))$. The result is unwanted since we do not know how to interpret the later group in terms of 8-branes.

---

[8]In the case when $K = \hat{k}$ the group is $G = U(1)$.

In order to solve this problem we pick up a subalgebra A of $C(W)$ which makes the following split sequence exact

$$0 \ \longrightarrow \ A \ \xrightarrow{\iota^A} \ \xrightarrow{\pi^A} \ C(X)/A \ \longrightarrow \ 0 \tag{5.10}$$

but otherwise arbitrary. The standard exact sequence theorem [**?**, **?**] states that the associated six-term cyclic sequence is exact. From this we can extract the K-theory map

$$\delta_*^A = \iota_*^A \circ \bar{\partial} \circ \pi_*^A, \tag{5.11}$$

where $\bar{\partial}$ is the composition of the suspended index map $\partial : K_2(C(X)/A) \to K_1(A)$ with the Bott map. Note that δ_*^A is independent of G but depends on the choice of A. Using δ_*^A we can map the equivalence classes of spaces of G-vector fields on the M9-branes on the D8-brane charges. Explicitely, we have the following map

$$\Delta_*^{GA} : K_0(C(Z) \times_{\bar{K}^*} G) \longrightarrow K_0(C(W)) \tag{5.12}$$

where the expresion of Δ_*^{GA} in terms of components is given by

$$\Delta_*^{GA} = (\delta_*^A)^{-1} \circ v_*^{-1} \circ \bar{\lambda}_*^* \circ d_*^G \circ i_*^G \circ (e_*^G)^{-1} \circ (\chi_*^G)^{-1}. \tag{5.13}$$

We note that, in general, the map (**??**) cannot be completely satisfactory because it depends on the arbitrary algebra A. From the physical point of view we can speculate that A should be identified with the algebra of continuous functions on some compact submanifold $U \subset W$ on which some lower Dp-branes are wrapped on. In any case, the choice of a submanifold U reduces further the symmetries of the system.

Appendix

In this Appendix we collect two classical mathematical results. The first theorem is a consequence of Bott's Theorem for topological K-theory of pointed compact pairs.

Theorem 1 *Let (M, N) be a pointed compact pair and let $i : N \to M$, $j : (M, \emptyset) \to (M, N)$ denote the natural inclusions. In the following diagram*

$$
\cdots \to \quad K^{-2}(M) \quad \overset{i^*}{\to} \quad K^{-2}(N) \quad \overset{\delta}{\longrightarrow} \quad K^{-1}(M, N) \quad \overset{j^*}{\to} \quad K^{-1}(M)
$$

$$
\beta \searrow \qquad\qquad \delta \circ \beta \nearrow \qquad\qquad\qquad\qquad i^* \searrow
$$

$$
K^0(N) \qquad\qquad\qquad\qquad\qquad\qquad K^1(N) \ ,
$$

$$
i^* \searrow \qquad\qquad\qquad\qquad \delta \swarrow
$$

$$
K^0(M) \qquad \overset{j^*}{\leftarrow} \quad K^0(M, N)
$$

$$(.14)$$

where the top row is the exact sequence of the pair (M, N). Then the hexagonal part of the diagram is exact. Here, δ is the coboundary map

$$
\delta : K^{-p}(N) \longrightarrow K^{-p+1}(M, N) \tag{.15}
$$

and β is the Bott isomorphism

$$
\beta : K^{-p}(M, N) \longrightarrow K^{-(p+2)}(M, N). \tag{.16}
$$

The second theorem is a particular case of a theorem by Novodvorskii [**?**, **?**] which establishes under what circumstances the topological and algebraical K-theory groups of a commutative Banach algebra are equivalent. In our case, the algebras of continuous functions in the theorem below are C*-algebras which implies that they are also Banach algebras.

Theorem 2 *The Gel'fand transform $C_0(M) \to C((\widehat{C_0(M)}))$, where M is a locally compact space, induces an isomorphism*

$$
\widetilde{K}_p(C_0(M)) \cong \widetilde{K}^{-p}(M^+) \ , \quad p = 0, 1. \tag{.17}
$$

Similarly, the Gel'fand transform $C(M) \to C(\widehat{C(M)})$, where M is a compact space, induces an isomorphism

$$
K_p(C_0(M)) \cong K^{-p}(M) \ , \quad p = 0, 1. \tag{.18}
$$

Here, $\widehat{C(M)}$ is the Gel'fand space of the algebra $C(M)$, i. e. the space of equivalence classes of irreducible representations of $C(M)$ (or the space of its maximal ideals). This space can be identified setwise as well as topologically with M. For any element $f \in C(M)$ its Gel'fand transform is the complex valued function $\hat{f} : \widehat{C(M)} \to \mathbb{C}$ given by $\hat{f}(\varphi) = \varphi(f)$ for any $\varphi \in \widehat{C(M)}$.

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