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A limit of the confluent Heun equation and the Schrödinger equation for an inverted potential and for an electric dipole

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Abstract. We reexamine and extend a group of solutions in series of Bessel functions for a limiting case of the confluent Heun equation and, then, apply such solutions to the one-dimensional Schrödinger equation with an inverted quasi-exactly solvable potential as well as to the angular equation for an electron in the field of a point electric dipole. For the first problem we find finite- and infinite-series solutions which are convergent and bounded for any value of the independent variable. For the angular equation, we also find expansions in series of Jacobi polynomials.

1. Introduction

Firstly we revise and extend some solutions in series of Bessel functions for a limiting case of the confluent Heun equation (CHE) – the CHE is also called generalized spheroidal wave equation [1, 2, 3]. Then we apply these solutions to the one-dimensional Schrödinger equation with an inverted quasi-exactly solvable potential [4], as well as to the angular equation for an electron in the field of a point electric dipole [5].

For the inverted potential we find even and odd eigenstates given by finite series of Bessel functions corresponding to the same energy eigenvalues, for any value of the parameter which characterizes the quasi-exact solvability. We obtain as well infinite-series solutions which are convergent and bounded over the entire range of the independent variable.

For the angular equation, in addition to expansions in series of Bessel functions, we get solutions in series of Jacobi polynomials, equivalent to the ones proposed by Alhaidari [6]. The latter type of solutions is inapplicable to the inverted potential and this is the reason for regarding expansions in series of Bessel functions.

The stationary one-dimensional Schrödinger equation for a particle with mass M and energy

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E is written as

$$\frac{d^2\psi}{du^2}+\left[\mathcal{E}-V(u)\right]\psi=0,\quad u=ax,\quad \mathcal{E}=\frac{2ME}{\hbar^2a^2},\qquad V(u)=\frac{2MV(x)}{\hbar^2a^2},\tag{1}$$

where a is a constant with inverse-length dimension, $\hbar$ is the Plank constant divided by 2π, x is the spatial coordinate and $V(x)$ is the potential. For the inverted potential, considered by Cho and Ho [4],

$$V(u)=-\frac{\mathrm{b}^2}{4}\sinh^2u-\frac{l^2-(1/4)}{\cosh^2u},\quad u\in(-\infty,\infty),\quad (l=1,2,3,\cdots)\tag{2}$$

where b is a positive real constant. This is a particular case of a class of potentials which gives degenerate states with opposite parity [7, 8]. It is a bottomless potential in the sense that $V(u)\to-\infty$ when $u\to\pm\infty$. If $\mathrm{b}^2<4(l+1)^2-1$, it is an inverted double well; if $\mathrm{b}^2\geq 4(l+1)^2-1$, it resembles the potential of an inverted oscillator. Then, we have the equation

$$\frac{d^2\psi}{du^2}+\left[\mathcal{E}+\frac{\mathrm{b}^2}{4}\sinh^2u+\left(l^2-\frac{1}{4}\right)\frac{1}{\cosh^2u}\right]\psi=0,\tag{3}$$

which, for the previous values assigned to l, is a quasi-exactly solvable equation.

A quantum mechanical problem is quasi-exactly solvable (QES) if one part of its energy spectrum and the respective eigenfunctions can be computed explicitly by algebraic methods [9, 10, 11]. Alternatively, a problem is QES if admits solutions given by finite series whose coefficients necessarily satisfy three-term or higher order recurrence relations [12], and it is exactly solvable if its solutions are given by hypergeometric functions (two-term recurrence relations). This viewpoint is suitable for handling problems involving Heun equations because in general these present such finite-series solutions [13]; they admit as well infinite-series solutions which could afford the rest of the spectrum.

With respect to the second problem, the Schrödinger equation for an electron in the field of a point electric dipole is used for modeling the scattering of negative ions by polar molecules [5]. When it is separated in spherical coordinates (r,θ,φ), the θ-dependence satisfies the equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+C-\beta\cos\theta-\frac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0,\tag{4}$$

where C is a separation constant to be determined, β is the dipole moment parameter and m is the angular momentum conjugate to φ.

In sections 3 and 4, respectively, equations (3) and (4) are transformed into a confluent Heun equation (CHE) having the form

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3+q(z-z_0)\right]U=0,\quad (q\neq 0)\tag{5}$$

where z_0, B_i and q, are constants (if $q = 0$ the equation degenerates into a hypergeometric equation). This equation is obtained by applying the so-called Whittaker-Ince limit [14, 15]

$$\omega \rightarrow 0, \quad \eta \rightarrow \infty, \quad \text{such that} \quad 2\eta\omega = -q, \tag{6}$$

to the following form of the confluent Heun equation [2]

$$z(z-z_0)\frac{d^2U}{dz^2} + (B_1 + B_2 z)\frac{dU}{dz} + \left[B_3 - 2\eta\omega(z-z_0) + \omega^2 z(z-z_0)\right] U = 0, \tag{7}$$

where z_0, B_i, η and ω are constants. In both cases $z = 0$ and $z = z_0$ are regular singular points having indicial exponents $(0, 1 + B_1/z_0)$ and $(0, 1 - B_2 - B_1/z_0)$, respectively. However, equations (5) and (7) differ by the behavior of their solutions at the irregular singularity $z = \infty$, namely [15],

$$U(z) \sim e^{\pm 2i\sqrt{qz}} z^{(1/4)-(B_2/2)} \text{ for Eq. (5)}, \; U(z) \sim e^{\pm i\omega z} z^{\mp i\eta - (B_2/2)} \text{ for Eq. (7)}. \tag{8}$$

We shall refer to equation (5) as Whittaker-Ince limit of the CHE (7). This equation also appears in the separation of the variables for the Laplace-Beltrami operator in an Eguchi-Hanson space [16, 17] and for the Schrödinger equation for an ion in the field of a electric quadrupole in two dimensions [18]. In addition, it includes the Mathieu equation as a particular case and, when $z_0 \rightarrow 0$, it leads to a double-confluent Heun equation which arises, for example, in the scattering of ions by polarizable targets [15].

In [15] the limits (6) were used to find solutions for equation (5) from solutions of the CHE (7). However, those solutions are not sufficient to handle the previous problems. For this reason, from known solutions in series of Hankel functions for equation (5), in section 2 we construct solutions in series of Bessel functions of the first and second kind. We find that the expansions in terms of the functions of the first kind, under certain conditions, are convergent and bounded for any value of the independent variable. The other expansions in general converge for $|z| > |z_0|$ or $|z - z_0| > |z_0|$, but under special conditions converge also at $|z| = |z_0|$ or $|z - z_0| = |z_0|$.

In section 3, the expansions in series of Bessel functions are used to obtain eigenfunctions for the equation (3) concerning the inverted potential. In section 4, we discuss the solutions for the angular equation of the point dipole. Conclusions and open issues are reported in section 5, while Appendix A contains some formulas used throughout the paper and Appendix B exhibits solutions which have been omitted in section 2 but are used in sections 3 and 4.

2. Solutions of the Whittaker-Ince limit of the CHE

All the solutions we will use for the Whittaker-Ince limit of the CHE may be generated by applying the limits (6) on solutions of the CHE (7), but may also be derived directly from (5). In this section we deal with a group of solutions given by expansions in series of four Bessel functions which are denoted by $Z_{\alpha}^{(j)}(x)$ $(j = 1, 2, 3, 4)$, according as [19]

$$Z_{\alpha}^{(1)}(x) = J_{\alpha}(x), \quad Z_{\alpha}^{(2)}(x) = Y_{\alpha}(x), \quad Z_{\alpha}^{(3)}(x) = H_{\alpha}^{(1)}(x), \quad Z_{\alpha}^{(4)}(x) = H_{\alpha}^{(2)}(x), \tag{9}$$

where J_{α} and Y_{α} are the Bessel functions of the first and second kind, respectively, whereas $H_{\alpha}^{(1)}$ and $H_{\alpha}^{(2)}$ are the first and the second Hankel functions (α is called the order of the functions). Thus, a first set of solutions has the form

$$U_1^{(j)}(z) = z^{(1-B_2)/2} \sum_{n=0}^{\infty} (-1)^n b_n^{(1)} Z_{2n+B_2-1}^{(j)}\left(2\sqrt{qz}\right), \quad [B_2 \neq 0, -1, -2, \cdots]. \tag{10}$$

where the presence of the same coefficients $b_n^{(1)}$ is due to the fact that the four Bessel functions satisfy the same difference and differential equations [20]. These are called one-sided series because the summation runs from zero to positive infinity.

In a region where the four expansions are valid, only two of them are independent. For instance, from equations (A.2) and (A.3) we get

$$U_1^{(1)}(z) = \frac{1}{2}\left[U_1^{(3)}(z) + U_1^{(4)}(z)\right], \qquad U_1^{(2)}(z) = \frac{1}{2i}\left[U_1^{(3)}(z) - U_1^{(4)}(z)\right] \tag{11}$$

up to a multiplicative constant. However, each expansion presents different behaviors at $z = \infty$ as inferred from equation (A.4).

In section 2.1 we explain how the above set of solutions has been obtained, write the general forms of the three-term recurrence relations for the series coefficients and present the transformations used to generate additional sets of solutions. In section 2.2 we analyse some properties of the solutions and discuss the case of the Mathieu equation, while in section 2.3 we study the convergence of the solutions.

2.1. Recurrence relations and transformations of the equation

The expansions in series of Hankel functions which appear in (10) are taken from Ref. [15] where they have been expressed by series of the modified Bessel functions $K_{2n+B_2-1}(\pm 2i\sqrt{qz})$ as

$$U_1^{\infty}(z) = z^{(1-B_2)/2} \sum_{n=0}^{\infty} b_n^{(1)} K_{2n+B_2-1}\left(\pm 2i\sqrt{qz}\right), \quad [B_2 \neq 0, -1, -2, \cdots].$$

In terms of Hankel functions [21],

$$K_{2n+B_2-1}(-2i\sqrt{qz}) \propto (-1)^n H^{(1)}_{2n+B_2-1}(2\sqrt{qz}),\ K_{2n+B_2-1}(2i\sqrt{qz}) \propto (-1)^n H^{(2)}_{2n+B_2-1}(2\sqrt{qz}),$$

where the proportionality constants do not depend on the summation index n. Thus, the preceding solution acquires the form of the solutions $U_1^{(3)}$ and $U_1^{(4)}$. The new solutions $U_1^{(1)}$ and $U_1^{(2)}$ result from the properties of the Bessel functions, as aforementioned, and are used in sections 3 and 4.

The recurrence relations for series coefficients b_n and the respective characteristic equations (in terms of continued fractions) assume one of the following forms [15]

$$\left.\begin{array}{l} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1) \end{array}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\gamma_1}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\ \frac{\alpha_2\gamma_3}{\beta_3 -} \cdots, \tag{12}$$

$$\left.\begin{array}{l} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_1 b_2 + \beta_1 b_1 + [\alpha_{-1} + \gamma_1]\, b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 2) \end{array}\right\} \Rightarrow \beta_0 = \frac{\alpha_0\,[\alpha_{-1} + \gamma_1]}{\beta_1 -}\, \frac{\alpha_1\gamma_2}{\beta_2 -}\, \frac{\alpha_2\gamma_3}{\beta_3 -} \cdots \tag{13}$$

or

$$\left.\begin{array}{l} \alpha_0 b_1 + [\beta_0 + \alpha_{-1}]\, b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1) \end{array}\right\} \Rightarrow \beta_0 + \alpha_{-1} = \frac{\alpha_0\gamma_1}{\beta_1 -}\ \frac{\alpha_1\gamma_2}{\beta_2 -}\, \frac{\alpha_2\gamma_3}{\beta_3 -} \cdots, \tag{14}$$

where the coefficients α_n, β_n and γ_n depend on the parameters of the equation and on the summation index n of the series. The second and the third types of recurrence relations occur only when the parameters of the equation imply that $\alpha_{-1} \neq 0$, as in the case of the Mathieu equation.

If $\gamma_n = 0$ for some $n = N + 1$, where N is a natural number or zero, the one-sided series terminate at $n = N$ giving finite-series solutions with $0 \leq n \leq N$ [19]. In particular, for $N = 0$ the series presents just the first term except when the recurrence relations are given by (13): in this case the series presents only the first term if $\alpha_{-1} + \gamma_1 = 0$. Solutions given by finite series are also called quasi-polynomial solutions or Heun polynomials. On the other hand, the previous characteristic equations were established by assuming that the summation begins at $n = 0$. However, the series begins at $n = N + 1$ if $\alpha_n = 0$ for some $n = N$. In this case we must set $n = m + N + 1$ and relabel the series coefficients in order to get series beginning at $m = 0$.

The fact that in each series all the Bessel functions must be independent imposes certain restrictions on the parameters of the differential equation as in the previous set of solutions where $B_2 \neq 0, -1, -2, \cdots$ ($Z_m^{(j)}$ and $Z_{-m}^{(j)}$ in general are proportional to each other if m is integer). Such restrictions also assure that the coefficients of the recurrence relations are well defined in the sense that there are no vanishing denominators. By transformations of variables which preserve the form of equation (5) we find new sets of solutions demanding different restrictions on the parameters. In effect, if $U(z) = U(B_1, B_2, B_3; z_0, q; z)$ denotes one solution (or set of solutions) for equation (5), then the transformations $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$, given by

$$\begin{aligned} \mathscr{T}_1 U(z) &= z^{1+(B_1/z_0)}\, U(C_1, C_2, C_3; z_0, q; z), \\ \mathscr{T}_2 U(z) &= (z - z_0)^{1-B_2-(B_1/z_0)}\, U(B_1, D_2, D_3; z_0, q; z), \\ \mathscr{T}_3 U(z) &= U(-B_1 - B_2 z_0, B_2, B_3 - q z_0; z_0, -q; z_0 - z), \end{aligned} \tag{15}$$

generate a group having eight (sets of) solutions. The constants C_i and D_i are defined by

$$C_1 = -B_1 - 2z_0, \quad C_2 = 2 + B_2 + \frac{2B_1}{z_0}, \quad C_3 = B_3 + \left(1 + \frac{B_1}{z_0}\right)\left(B_2 + \frac{B_1}{z_0}\right),$$

$$D_2 = 2 - B_2 - \frac{2B_1}{z_0}, \quad D_3 = B_3 + \frac{B_1}{z_0}\left(\frac{B_1}{z_0} + B_2 - 1\right). \tag{16}$$

The application of these rules is straightforward but leads to solutions with different properties.

2.2. Two subgroups of solutions in series of Bessel functions

Given the initial set of solutions, $U_1^{(j)}$, the rules $\mathscr{T}_1$ and $\mathscr{T}_2$ are used as

$$U_1^{(j)}(z) \overset{\mathscr{T}_1}{\longleftrightarrow} U_2^{(j)}(z) \overset{\mathscr{T}_2}{\longleftrightarrow} U_3^{(j)}(z) \overset{\mathscr{T}_1}{\longleftrightarrow} U_4^{(j)}(z) \overset{\mathscr{T}_2}{\longleftrightarrow} U_1^{(j)}(z) \tag{17}$$

in order to generate a subgroup constituted by four sets of solutions. To get a second subgroup of solutions, first we take $U_5^{(j)}(z) = \mathscr{T}_3 U_1^{(j)}(z)$ and, after this, we use the rules $\mathscr{T}_1$ and $\mathscr{T}_2$ in the same order as in (17), that is,

$$U_5^{(j)}(z) \overset{\mathscr{T}_1}{\longleftrightarrow} U_6^{(j)}(z) \overset{\mathscr{T}_2}{\longleftrightarrow} U_7^{(j)}(z) \overset{\mathscr{T}_1}{\longleftrightarrow} U_8^{(j)}(z) \overset{\mathscr{T}_2}{\longleftrightarrow} U_5^{(j)}(z). \tag{18}$$

Then, we can check the following correspondence between the solutions of the two subgroups

$$U_i^{(j)}(z) \longleftrightarrow U_{i+4}^{(j)}(z), \quad (i = 1, 2, 3, 4) \tag{19a}$$

in the sense that formally we have

$$\beta_n^{(i)} = \beta_n^{(i+4)}, \qquad \alpha_n^{(i)}\ \gamma_{n+1}^{(i)} = \alpha_n^{(i+4)}\ \gamma_{n+1}^{(i+4)}. \tag{19b}$$

Besides this, if the summation index n takes the same values in both expansions, the restrictions on the parameters, the form of the recurrence relations and the order of the Bessel functions are the same for both sets of solutions related by (19a). In this event both solutions have same characteristic equation and their series coefficients are proportional to each other. However, in some cases one series breaks off on the left while the other terminates on the right and so these statements are not true, as in the example considered in section 3.

Next we write the first and the fifth set of solutions, specify the restrictions on the parameters and the conditions to use each of the preceding recurrence relations (the other sets are in Appendix B). Thus,

$$U_1^{(j)}(z) = z^{(1-B_2)/2}\sum_{n=0}^{\infty}(-1)^n b_n^{(1)} Z_{2n+B_2-1}^{(j)}\left(2\sqrt{qz}\right),$$

$$U_5^{(j)}(z) = (z-z_0)^{(1-B_2)/2}\sum_{n=0}^{\infty}(-1)^n b_n^{(5)} Z_{2n+B_2-1}^{(j)}\left(2\sqrt{q(z-z_0)}\right), \tag{20a}$$

$$[B_2 \neq 0, -1, -2, \cdots .]$$

For $U_1^{(j)}$ we have [15]

$$\alpha_n^{(1)} = \frac{qz_0\,(n+1)\left(n-\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}\right)\left(n+\frac{B_2}{2}+\frac{1}{2}\right)},$$

$$\beta_n^{(1)} = 4B_3 - 2qz_0 + 4n\,(n+B_2-1) - \frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}-1\right)\left(n+\frac{B_2}{2}\right)},$$

$$\gamma_n^{(1)} = \frac{qz_0\,(n+B_2-2)\left(n+B_2+\frac{B_1}{z_0}-1\right)}{\left(n+\frac{B_2}{2}-\frac{3}{2}\right)\left(n+\frac{B_2}{2}-1\right)}, \tag{20b}$$

in the recurrence relations for $b_n^{(1)}$ which are given by

$$\text{Eq. (12) if } B_2 \neq 1, 2; \quad \text{Eq. (13) if } B_2 = 1; \quad \text{Eq. (14) if } B_2 = 2. \tag{20c}$$

For $U_5^{(j)}$ we get the following coefficients ($\beta_n^{(5)} = \beta_n^{(1)}$)

$$\alpha_n^{(5)} = -\frac{qz_0\,(n+1)\left(n+B_2+\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}\right)\left(n+\frac{B_2}{2}+\frac{1}{2}\right)}, \quad \gamma_n^{(5)} = -\frac{qz_0\,(n+B_2-2)\left(n-1-\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}-\frac{3}{2}\right)\left(n+\frac{B_2}{2}-1\right)},$$

in the recurrence relations (20c) for the $b_n^{(5)}$. We find the formal relation

$$b_n^{(5)} = \frac{(-1)^n\Gamma[n-(B_1/z_0)]}{\Gamma[n+B_2+(B_1/z_0)]}\, b_n^{(1)}. \tag{21}$$

Actually, this relation holds only if $(-B_1/z_0)$ and (B_2+B_1/z_0) are not zero or negative integers; on the contrary, one solution truncates on the left ($\alpha_n = 0$ for some n) and the other on the right ($\gamma_n = 0$). There are similar relations for the other solutions.

¿From relations (A.4), we get

$$\lim_{z\to\infty} U_1^{(3)}(z) \sim e^{2i\sqrt{qz}}\; z^{(1/4)-(B_2/2)}, \quad \lim_{z\to\infty} U_1^{(4)}(z) \sim e^{-2i\sqrt{qz}}\; z^{(1/4)-(B_2/2)}, \tag{22}$$

that is, each solution presents one of the possible behaviors given in equation (8). On the other side, equation (11) gives the solution $U_1^{(1)}$ as a linear combination of $U_1^{(3)}$ and $U_1^{(4)}$ and, consequently, its behavior at $z=\infty$ must be a linear combination of the previous expressions. This implies that the solution $U_1^{(1)}$ is bounded for $z=\infty$ only if both $U_1^{(3)}$ and $U_1^{(4)}$ are bounded too. The same holds for $U_1^{(2)}$.

The previous solutions lead to the usual solutions in series of Bessel for Mathieu equation. This equation has the form

$$\frac{d^2w}{du^2}+\sigma^2\left[\mathrm{a}-2k^2\cos(2\sigma u)\right]w=0, \qquad q=k^2, \tag{23}$$

where $\sigma=1$ or $\sigma=i$ for the Mathieu or modified Mathieu equation, respectively. Then, by introducing

$$w(u)=U(z), \quad z=\cos^2(\sigma u); z_0=1, \quad B_1=-\frac{1}{2}, \quad B_2=1, \quad B_3=\frac{k^2}{2}-\frac{\mathrm{a}}{4}, \quad q=k^2, \tag{24}$$

into equation (5) we obtain the above Mathieu equation. Using these relations we can express the solutions of the Mathieu equation in terms of trigonometric (hyperbolic) functions. These solutions are given by infinite series and the solutions arising from the second subgroup have exactly the same characteristic equations as the ones resulting from the first subgroup. For example,

$$w_1^{(j)}(u)=\sum_{n=0}^{\infty}(-1)^n b_n^{(1)}Z_{2n}^{(j)}[2k\cos(\sigma u)], \quad w_5^{(j)}(u)=\sum_{n=0}^{\infty}b_n^{(1)}Z_{2n}^{(j)}[2ki\sin(\sigma u)], \tag{25a}$$

with the recurrence relations

$$qb_1^{(1)}-\mathrm{a}b_0^{(1)}=0, \quad qb_2^{(1)}+[4-\mathrm{a}]b_1^{(1)}+2qb_0^{(1)}=0, \quad qb_{n+1}^{(1)}+\left[4n^2-\mathrm{a}\right]b_n^{(1)}+qb_{n-1}^{(1)}=0, \ (n\geq 2), \tag{25b}$$

which follow from the second relation given in (20c) since $B_2=1$. Both sets have the same coefficients because equation (21) gives $b_n^{(5)}=(-1)^n b_n^{(1)}$. According to the next section, the solutions $w_1^{(j)}$ converge for $|\cos(\sigma u)|\geq 1$ whereas $w_5^{(j)}$ converge only for $|\sin(\sigma u)|\geq 1$ if $j=2,3,4$. Despite this, in a common domain of convergence $w_1^{(j)}$ is a constant multiple of $w_5^{(j)}$, except if $k=0$ [19, 22].

For the general case, we have found no proof for the equivalence between the sets of solutions of the first and second subgroups. In fact, equivalence could take place only between two sets of infinite-series solutions where the summations begin at $n=0$. For instance, for the first and fifth sets, this requires that neither $(-B_1/z_0)$ nor $B_2+(B_1/z_0)$ are zero or negative integer since this implies infinite series in both sets.

On the other side, some sets of solutions admit the limit $z_0\to 0$, in which case equation (5) becomes a double-confluent Heun equation with two (irregular) singularities located at $z=0$ and $z=\infty$. In this case the second subgroup is irrelevant and only the sets $U_1^{(j)}(z)$ and $U_3^{(j)}(z)$ admit such limit, as we can see by examining the recurrence relations for the series coefficients.

2.3. Convergence of the solutions

Now the ratio test is used to get the convergence of the first set of solutions. For the other sets the convergence follows from the transformation rules. The following convergence regions refer to one-sided series; these series hold only if the differential equation has an arbitrary parameter that allows to satisfy the characteristic equations. If there is no free parameter, it is possible to obtain convergent two-sided series solutions (summation extending from negative to positive infinity) by introducing a characteristic parameter ν into the solutions but, then, the domains of convergence are changed.

For $n\to\infty$ the recurrence relations for $b_n^{(1)}$ become (we write b_n instead of $b_n^{(1)}$)

$$qz_0\left[1-\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-\frac{1}{2}\right)+O\left(\frac{1}{n^2}\right)\right]\frac{b_{n+1}}{b_n}+\left[4n(n+B_2-1)+O\left(1\right)\right]$$
$$+qz_0\left[1+\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-\frac{1}{2}\right)+O\left(\frac{1}{n^2}\right)\right]\frac{b_{n-1}}{b_n}=0,$$

whose minimal solution for $n\to\infty$ is

$$\frac{b_{n+1}}{b_n}\sim-\frac{qz_0}{4n^2}\left[1+\frac{1}{n}\left(\frac{B_1}{z_0}-\frac{3}{2}\right)+O\left(\frac{1}{n^2}\right)\right]\Rightarrow$$
$$\frac{b_{n-1}}{b_n}\sim-\frac{4n^2}{qz_0}\left[1-\frac{1}{n}\left(\frac{B_1}{z_0}+\frac{1}{2}\right)+O\left(\frac{1}{n^2}\right)\right],\quad \text{if}\quad z_0\neq 0, \tag{26a}$$

where the notation having the form $f_n\sim g_n$ means that g_n is an asymptotic approximation to f_n as $n\to\infty$, that is, $f_n/g_n=1$ when $n\to\infty$ [23]. Similarly,

$$\frac{b_{n+1}}{b_n}\sim-\frac{qB_1}{4n^3}\left[1+O\left(\frac{1}{n}\right)\right]\Rightarrow\frac{b_{n-1}}{b_n}\sim-\frac{4n^3}{qB_1}\left[1+O\left(\frac{1}{n}\right)\right],\quad \text{if}\quad z_0=0, \tag{26b}$$

Firstly we regard the expansions in series of Bessel functions of the first kind. Using the first relation given in equation (A.7), we find

$$\frac{J_{2n+B_2+1}\left(2\sqrt{qz}\right)}{J_{2n+B_2-1}\left(2\sqrt{qz}\right)}\sim\frac{qz}{4n^2}\left[1+O\left(\frac{1}{n}\right)\right].$$

Then, keeping only the leading terms, we obtain

$$\frac{b_{n+1}\ J_{2n+B_2+1}\left(2\sqrt{qz}\right)}{b_n\ J_{2n+B_2-1}\left(2\sqrt{qz}\right)}\sim\frac{z_0q^2z}{16n^4},\ \text{if}\ z_0\neq 0;\qquad \frac{b_{n+1}\ J_{2n+B_2+1}\left(2\sqrt{qz}\right)}{b_n\ J_{2n+B_2-1}\left(2\sqrt{qz}\right)}\sim\frac{B_1q^2z}{16n^5},\ \text{if}\ z_0=0.$$

Therefore, the series converges for any finite z, but the ratio test is inconclusive when $z\to\infty$. For this case we proceed as in the case of the Mathieu equation [19]. Thus, by using equation (A.4) we write ($z\to\infty$)

$$J_{2n+B_1-1}(2\sqrt{qz})\sim(-1)^n\left(\frac{1}{\sqrt{qz}}\right)^{\frac{1}{2}}\cos\left[2\sqrt{qz}-\frac{1}{2}(B_2-1)\pi-\frac{1}{4}\pi\right],\quad |\arg\sqrt{qz}|<\pi.$$

By inserting this into $U_1^{(1)}$, we find

$$\lim_{z\to\infty} U_1^{(1)}(z) = z^{(1/4)-(B_2/2)} \cos\left[2\sqrt{qz} - \frac{1}{2}(B_2-1)\pi - \frac{1}{4}\pi\right] \sum_{n=0}^{\infty} b_n^{(1)}. \tag{27}$$

Since the series $\sum_{n=0}^{\infty} b_n^{(1)}$ converges, the solution $U_1^{(1)}(z)$ also converges at $z=\infty$. However, if the cosine is expressed in terms of exponential functions, we obtain a linear combination of asymptotic behaviors of the expansion in Hankel functions, as observed after equation (22). Thus, depending on the values of B_2 and $|\arg(\sqrt{qz})|$, the solution $U_1^{(1)}$ may be bounded or unbounded at $z=\infty$.

Now we consider the expansions in series of Hankel functions and Bessel functions of the second kind. For $\alpha = 2n+B_2-1$ and $x = 2\sqrt{qz}$, relations (A.7) give

$$\frac{Z^{(j)}_{2n+B_2+1}\left(2\sqrt{qz}\right)}{Z^{(j)}_{2n+B_2-1}\left(2\sqrt{qz}\right)} \sim \frac{4n^2}{qz}\left[1+\frac{2B_2-1}{2n}+\frac{B_2(B_2-1)}{n^2}\right], \qquad j=2,3,4.$$

Combining this with the results given in equations (26a) and (26b), we find

$$\frac{b_{n+1}Z^{(j)}_{2n+B_2+1}\left(2\sqrt{qz}\right)}{b_n Z^{(j)}_{2n+B_2-1}\left(2\sqrt{qz}\right)} \sim -\frac{z_0}{z}\left[1+\frac{1}{n}\left(B_2-2+\frac{B_1}{z_0}\right)+O\left(\frac{1}{n^2}\right)\right], \ z_0\neq 0,$$

$$\frac{b_{n+1}Z^{(j)}_{2n+B_2+1}\left(2\sqrt{qz}\right)}{b_n Z^{(j)}_{2n+B_2-1}\left(2\sqrt{qz}\right)} \sim -\frac{B_1}{nz}\left[1+O\left(\frac{1}{n}\right)\right], \ z_0=0.$$

Then, the ratio test implies that the series in the solutions $U_1^{(j)}$ $(j=2,3,4)$ converge for $|z|>0$ if $z_0=0$. On the other hand, for $z_0\neq 0$ we have $(n\to\infty)$

$$\left|\frac{b_{n+1}Z^{(j)}_{2n+B_2+1}\left(2\sqrt{qz}\right)}{b_n Z^{(j)}_{2n+B_2-1}\left(2\sqrt{qz}\right)}\right| = \frac{|z_0|}{|z|}\left[1+\frac{1}{n}\mathrm{Re}\left(B_2-2+\frac{B_1}{z_0}\right)+O\left(\frac{1}{n^2}\right)\right]. \tag{28}$$

Thus, if $z_0\neq 0$ in general the series converge for $|z|>|z_0|$ since the right side of this equation is <1 but, by the Raabe test [24, 25], they converge absolutely also for $|z|=|z_0|$ if the numerator of n is <-1, that is, if $\mathrm{Re}(B_2+B_1/z_0)<1$. This possibility was not noticed before [15] because the term of order $1/n$ was not considered.

The convergence for the other sets of solutions are obtained by applying the transformations rules to $U_1^{(j)}$. For $j=2,3,4$, we find that these solutions converge for $|z|>|z_0|$ in the first subgroup, and for $|z-z_0|>|z_0|$ in the second subgroup. The special cases are summarized by:

$$|z|\geq|z_0| \text{ if } \begin{cases} \mathrm{Re}\left(B_2+\frac{B_1}{z_0}\right)<1 \text{ in } U_1^{(j)} \text{ and } U_2^{(j)}, \\ \mathrm{Re}\left(B_2+\frac{B_1}{z_0}\right)>1 \text{ in } U_3^{(j)} \text{ and } U_4^{(j)}; \end{cases} \tag{29a}$$

$$|z-z_0|\geq|z_0| \text{ if } \begin{cases} \mathrm{Re}\left(\frac{B_1}{z_0}\right)>1 \text{ in } U_5^{(j)} \text{ and } U_6^{(j)}, \\ \mathrm{Re}\left(\frac{B_1}{z_0}\right)<1 \text{ in } U_7^{(j)} \text{ and } U_8^{(j)}. \end{cases} \tag{29b}$$

3. The inverted potential

In this section we get solutions for equation (3) of the inverted potential by using the preceding solutions in series of Bessel functions of the first kind for the Whittaker-Ince limit of the CHE. We will find one pair of finite-series solutions and two pairs of infinite-series solutions, all of them convergent and bounded for any value of the variable u. The two pairs of infinite series have the same characteristic equation, but only the solutions of one pair can be expressed as a linear combination of two series of Hankel functions converging for all values of the independent variable.

The substitutions

$$\psi(u) = \big[\cosh u\big]^{-l+\frac{1}{2}}U(z), \qquad z = -\sinh^2 u, \qquad (-\infty < z \le 0) \tag{30}$$

bring equation (3) to the form

$$z(z-1)\frac{d^2U}{dz^2} + \left[-\frac{1}{2} + \left(\frac{3}{2} - l\right)z\right]\frac{dU}{dz} + \left[\frac{\mathcal{E}}{4} - \frac{\mathrm{b}^2}{16} + \left(\frac{l}{2} - \frac{1}{4}\right)^2 - \frac{\mathrm{b}^2}{16}(z-1)\right]U = 0, \tag{31}$$

which is Whittaker-Ince limit (5) of the CHE with

$$z_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{3}{2} - l, \quad B_3 = \frac{\mathcal{E}}{4} - \frac{\mathrm{b}^2}{16} + \left(\frac{l}{2} - \frac{1}{4}\right)^2, \quad q = -\frac{\mathrm{b}^2}{16}. \tag{32}$$

Thus, the Schrödinger equation (3) will be solved by replacing $U(z)$ in equation (30) by the the expansions in series of Bessel functions of the first kind with the parameters (32). Notice that for $\mathrm{b} = 0$, the potential $V(u)$ reduces to a hyperbolic Pöschl-Teller potential whose solutions are given by hypergeometric functions [26].

In section 3.1 we write down the pair of finite-series solutions which allow to get the 'quasi-solvable' part of the energy spectrum. We prove that these are degenerate for any finite value of the parameter l and adapt a method devised by Bender and Dunne [28] (for a potential which leads to a biconfluent Heun equation) to find the energy levels and the series coefficients. In section 3.2 we discuss the infinite-series solutions. In all cases, using the parameters (32) we find that the order of the Bessel functions is half-integer and so these functions are represented by finite series rather than by infinite ones [27]. Besides this, the recurrence relations are always given by equation (12) since $\alpha_{-1}^{(i)} = 0$.

3.1. Finite-series solutions

According to section 2.1, if $\gamma_{n=N+1} = 0$ in the recurrence relations (12), the series terminates at $n = N$ and we obtain a finite-series solution. In this case the recurrence relations can be written in the form

$$\begin{pmatrix} \beta_0 & \alpha_0 & 0 & \cdots & & & 0 \\ \gamma_1 & \beta_1 & \alpha_1 & & & & \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & & & \\ \vdots & & & & & & \\ & & & & \gamma_{N-1} & \beta_{N-1} & \alpha_{N-1} \\ 0 & \cdots & & & 0 & \gamma_N & \beta_N \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \\ b_N \end{pmatrix} = 0. \tag{33}$$

This system has nontrivial solutions only if the determinant of the above tridiagonal matrix vanishes. Besides this, if (as in the present problem) the elements of this matrix are real and if

$$\alpha_i \gamma_{i+1} > 0, \qquad 0 \le i \le N-1, \tag{34}$$

then the $N+1$ roots of the determinant are real and distinct [19].

Only $U_1^{(j)}$ and $U_2^{(j)}$ give finite-series wavefunctions. In order to prove the degeneracy and apply the procedure of Bender and Dunne, we redefine the series coefficients as

$$b_n^{(1)} = \frac{\Gamma\left[2n+B_2\right]\ P_n}{(-qz_0)^n\ n!\ \Gamma\left[n-(B_1/z_0)\right]}, \quad b_n^{(2)} = \frac{\Gamma\left[2n+2+B_2+(2B_1/z_0)\right]\ Q_n}{(-qz_0)^n\ n!\ \Gamma\left[n+2+(B_1/z_0)\right]}. \tag{35}$$

Then, inserting the solution $U_1^{(1)}$ and $U_2^{(1)}$ into equation (30) and using the parameters given in (32), we get the following pair $(\psi_1^{\mathrm{e}}, \psi_1^{\mathrm{o}})$ of even and odd solutions

$$\psi_1^{\mathrm{e}}(u) = \left[\tanh u\right]^{l-\frac{1}{2}} \sum_{n=0}^{l-1} (-1)^n \left(\frac{4}{\mathrm{b}}\right)^{2n} \frac{\Gamma\left[2n-l+\frac{3}{2}\right] P_n}{n!\ \Gamma\left[n+(1/2)\right]} J_{2n-l+\frac{1}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right),$$

$$\psi_1^{\mathrm{o}}(u) = \left[\tanh u\right]^{l-\frac{1}{2}} \sum_{n=0}^{l-1} (-1)^n \left(\frac{4}{\mathrm{b}}\right)^{2n} \frac{\Gamma\left[2n-l+(5/2)\right] Q_n}{n!\ \Gamma\left[n+(3/2)\right]} J_{2n-l+(3/2)}\left(\frac{\mathrm{b}}{2}\sinh u\right), \tag{36}$$

where the recurrence relations for the coefficients P_n are

$$P_{n+1} + \beta_n P_n + \gamma_n P_{n-1} = 0, \qquad (P_{-1} = 0) \tag{37a}$$

with

$$\beta_n = -E_l - n\left(n - l + \frac{1}{2}\right) - \frac{\mathrm{b}^2(2l+1)(2l-1)}{32(4n-1-2l)(4n+3-2l)},$$

$$\gamma_n = \left(\frac{\mathrm{b}^2}{8}\right)^2 \frac{n(2n-1)(2n-2l-1)(n-l)}{(4n+1-2l)(4n-3-2l)(4n-1-2l)^2}, \qquad \left[E_l := \frac{\mathcal{E}}{4} - \frac{\mathrm{b}^2}{32} + \left(\frac{l}{2} - \frac{1}{4}\right)^2\right].$$

The recurrence relations for Q_n are

$$Q_{n+1} + \tilde{\beta}_n Q_n + \tilde{\gamma}_n Q_{n-1} = 0, \quad (Q_{-1} = 0) \tag{38a}$$

with

$$\tilde{\beta}_n = -E_l - (n-l+1)\left(n+\frac{1}{2}\right) - \frac{\mathrm{b}^2(2l+1)(2l-1)}{32(4n+1-2l)(4n+5-2l)},$$

$$\tilde{\gamma}_n = \left(\frac{\mathrm{b}^2}{8}\right)^2 \frac{n(2n+1)(2n-2l+1)(n-l)}{(4n-1-2l)(4n+3-2l)(4n+1-2l)^2}. \tag{38b}$$

From the relation $J_\lambda(-x) = (-1)^\lambda J_\lambda(x)$, it turns out that these solutions are even and odd, that is, $\psi_1^{\mathrm{e}}(-u) = \psi_1^{\mathrm{e}}(u)$ and $\psi_1^{\mathrm{o}}(-u) = -\psi_1^{\mathrm{o}}(u)$.

The series are finite because the coefficients γ_n and $\tilde{\gamma}_n$ of P_{n-1} and Q_{n-1} vanish for $n = l$ and, consequently, the series terminate at $n = l-1$ as stated above. In virtue of equation (34), each eigenfunction corresponds to l distinct and real eigenvalues. In addition, from $J_\lambda(x) = (x/2)^\lambda/\Gamma(\lambda+1)$ when $x \to 0$, we find

$$\lim_{u\to 0} \psi_1^{\mathrm{e}}(u) \sim \left[\cosh u\right]^{-l+\frac{1}{2}} \to \text{finite},$$

and from the first of equations (A.4) we get

$$\lim_{u\to\pm\infty} \psi_1^{\mathrm{e}}(u) \sim \frac{[\tanh u]^{l-\frac{1}{2}}}{\sqrt{\sinh u}} \cos\left[\frac{\mathrm{b}}{2}\sinh u + \frac{1}{2}(l-1)\pi\right] \sum_{n=0}^{l-1} \left(\frac{4}{\mathrm{b}}\right)^{2n} \frac{\Gamma\left[2n-l+3/2\right] P_n}{n!\,\Gamma\left[n+1/2\right]} \to 0.$$

Thence, $\psi_1^{\mathrm{e}}(u)$ is bounded also at the singular points of the equation ($z = 0$ and $z = -\infty$). The same holds for the solutions $\psi_1^{\mathrm{o}}(u)$. In fact, expansions in series of the other Bessel functions are also bounded for all values of u but do not present definite parity.

The degeneracy of the previous solutions is established by arranging the recurrence relations in the matrix form (33). For P_n we write $\mathbb{A}\vec{P} = 0$, where

$$\vec{P} = (P_0, P_1, \cdots, P_{l-1})^t$$

(t means 'transpose') and

$$\mathbb{A} = \begin{pmatrix} \beta_0 & 1 & & & & 0 \\ \gamma_1 & \beta_1 & 1 & & & \\ & \gamma_2 & \beta_2 & \cdot & & \\ & & \cdot & & & \\ & & & \gamma_{l-2} & \beta_{l-2} & 1 \\ 0 & & & 0 & \gamma_{l-1} & \beta_{l-1} \end{pmatrix}. \tag{39}$$

For Q_n we write $\mathbb{B}\vec{Q} = 0$, where

$$\vec{Q} = (Q_0, Q_1, \cdots, Q_{l-1})^t$$

and

$$\mathbb{B}=\begin{pmatrix}\tilde\beta_0 & 1 & & & & 0\\ \tilde\gamma_1 & \tilde\beta_1 & 1 & & & \\ & \tilde\gamma_2 & \tilde\beta_2 & \cdot & & \\ & & \cdot & & & \\ & & & \tilde\gamma_{l-2} & \tilde\beta_{l-2} & 1\\ 0 & & & & \tilde\gamma_{l-1} & \tilde\beta_{l-1}\end{pmatrix}=\begin{pmatrix}\beta_{l-1} & 1 & & & & 0\\ \gamma_{l-1} & \beta_{l-2} & 1 & & & \\ & \gamma_{l-2} & \beta_{l-3} & \cdot & & \\ & & \cdot & & & \\ & & & \gamma_2 & \beta_1 & 1\\ 0 & & & 0 & \gamma_1 & \beta_0\end{pmatrix}, \tag{40}$$

where the last equality results from the identities

$$\tilde\beta_0=\beta_{l-1},\quad \tilde\beta_1=\beta_{l-2},\quad \tilde\beta_2=\beta_{l-3},\ \cdots,\quad \tilde\beta_{l-1}=\beta_0,$$

$$\tilde\gamma_1=\gamma_{l-1},\quad \tilde\gamma_2=\gamma_{l-2},\quad \tilde\gamma_3=\gamma_{l-3},\quad \cdots,\quad \tilde\gamma_{l-1}=\gamma_1.$$

Therefore, the two matrices are constituted by the same elements. To prove the degeneracy, it is sufficient to show that these matrices possess the same roots, that is, $\det\mathbb{A}=\det\mathbb{B}$. This is obvious for $l=1$ and $l=2$. For $l\geq 3$, we use the l-by-l antidiagonal matrix $\mathbb{S}$ having 1's on the antidiagonal as the only nonzero elements, that is,

$$\mathbb{S}=\mathbb{S}^{-1}=\begin{pmatrix} & & 1\\ & \cdot & \\ 1 & & \end{pmatrix},\qquad \det\mathbb{S}=-1. \tag{41}$$

Then we find the similarity relation

$$\mathbb{A}=\mathbb{S}^{-1}\mathbb{B}^t\mathbb{S}=\mathbb{S}\mathbb{B}^t\mathbb{S}, \tag{42}$$

where $\mathbb{B}^t$ is the transpose of $\mathbb{B}$. Thus, from the properties of the determinants it follows that $\det\mathbb{A}=\det\mathbb{B}$ and, therefore, the finite-series solutions are degenerate for any l.

On the other hand, the eigenvalues may be computed by equating to zero the determinants of the preceding matrices. However, the procedure of Bender and Dunne gives as well the coefficients P_n and Q_n as polynomials of degree n in the the parameter E_l. The procedure is implemented by taking $P_0=Q_0=1$ as initial conditions and by using the recurrence relations to generate the other coefficients. For a fixed l, the eigenvalues are obtained by requiring that $P_l=0$ or $Q_l=0$, since the series terminate at $n=l-1$. Thus, equations (37a) and (37b) yield

$$P_{n+1}=[E_l+k_{nl}]P_n-\gamma_nP_{n-1},\quad (P_{-1}=0,\quad P_0=1) \tag{43a}$$

with

$$k_{nl}\;=\;n\left(n-l+\frac{1}{2}\right)+\frac{\mathrm{b}^2(2l+1)(2l-1)}{32(4n-1-2l)(4n+3-2l)}, \tag{43b}$$

wherefrom

$$P_0 = 1, \quad P_1 = E_l + k_{0l}, \quad P_2 = E_l^2 + [k_{0l} + k_{1l}]E_l + k_{0l}k_{1l} - \gamma_1,$$

$$P_3 = E_l^3 + \big[k_{0l} + k_{1l} + k_{2l}\big]E_l^2 + \big[k_{0l}k_{1l} + k_{0l}k_{2l} + k_{1l}k_{2l} - \gamma_1 - \gamma_2\big]E_l \\ + k_{2l}\big[k_{0l}k_{1l} - \gamma_1\big] - k_{0l}\gamma_2, \tag{44}$$

and so on. Similarly, we write the relations (38a) and (38b) for Q_n as

$$Q_{n+1} = [E_l + \tilde{k}_{nl}]Q_n - \tilde{\gamma}_n Q_{n-1}, \quad (Q_{-1} = 0, \quad Q_0 = 1) \tag{45a}$$

where

$$\tilde{k}_{nl} = (n - l + 1)\left(n + \frac{1}{2}\right) + \frac{\mathrm{b}^2(2l + 1)(2l - 1)}{32(4n + 1 - 2l)(4n + 5 - 2l)}. \tag{45b}$$

Then, the expressions for Q_n are obtained by replacing k_{nl} and γ_n by $\tilde{k}_{nl}$ and $\tilde{\gamma}_n$ in the expressions for P_n. As an example we find the energies and the respective eigenfunctions for $l = 1$ and $l = 2$. For $l = 1$ the energy that follows from the condition $P_1 = Q_1 = 0$ is

$$E_{l=1} - \frac{\mathrm{b}^2}{32} = 0 \quad \Rightarrow \quad \mathcal{E} = \frac{1}{4}\left(\mathrm{b}^2 - 1\right), \tag{46a}$$

corresponding to the degenerate pair of eigenfunctions

$$\psi_1^{\mathrm{e}}(u) = \sqrt{\tanh u}\; J_{-\frac{1}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right), \quad \psi_1^{\mathrm{o}}(u) = \sqrt{\tanh u}\; J_{\frac{1}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right). \tag{46b}$$

For $l = 2$ the condition $P_2 = Q_2 = 0$ leads to

$$E^{\pm}_{l=2} = \frac{1}{4}\left[\frac{\mathrm{b}^2}{8} + 1 \pm \sqrt{1 + \mathrm{b}^2}\right] \quad \Rightarrow \quad \mathcal{E}^{\pm} = \frac{1}{4}\left[\mathrm{b}^2 - 5\right] \pm \sqrt{1 + \mathrm{b}^2}. \tag{47a}$$

Since, $P_1 = E_2 + (3\mathrm{b}^2/32)$ and $Q_1 = E_2 - (1/2) - (5\mathrm{b}^2/32)$ these energies are associated with the eigenstates

$$\psi_1^{\mathrm{e}\pm}(u) = (\tanh u)^{\frac{3}{2}}\left[J_{-\frac{3}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right) + \frac{2}{\mathrm{b}^2}\left(\frac{\mathrm{b}^2}{2} + 1 \pm \sqrt{\mathrm{b}^2 + 1}\right)J_{\frac{1}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right)\right],$$

$$\psi_1^{\mathrm{o}\pm}(u) = (\tanh u)^{\frac{3}{2}}\left[J_{-\frac{1}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right) + \frac{2}{\mathrm{b}^2}\left(\frac{\mathrm{b}^2}{2} + 1 \mp \sqrt{\mathrm{b}^2 + 1}\right)J_{\frac{3}{2}}\left(\frac{\mathrm{b}}{2}\sinh u\right)\right], \tag{47b}$$

up to normalization factors.

These energies for $l = 1, 2$ are the same found by Cho and Ho [4]. However, the eigenfunctions differ from theirs, as we can see by writing the Bessel functions in terms of elementary functions via the formulas written in Appendix A. Indeed, their solutions may be obtained from another group of expansions in series of Bessel functions given in [14].

Notice that the coefficients of the preceding finite series factorize in the same manner as the coefficients of the problem considered by Bender and Dunne [28], that is, for a fixed l one has

$$P_{l+i}(E_l) = p_i(E_l)P_l(E_l), \quad Q_{l+i}(E_l) = q_i(E_l)Q_l(E_l), \quad (i \geq 0) \tag{48}$$

where p_i and q_i are polynomials of degree i in E_l. For example, by taking $n = l$ ($\gamma_l = 0$) and $n = l + 1$ in equation (43a), we find

$$P_{l+1} = (E_l + k_{l,l})P_l = p_1 P_l, \qquad P_{l+2} = [(E_l + k_{l+1,l})(E_l + k_{l,l}) - \gamma_{l+1}]P_l = p_2 P_l,$$

and, by induction, we obtain the previous expression for P_{l+i}.

3.2. Infinite-series solutions

The solutions $U_3^{(1)}$ and $U_4^{(1)}$ lead respectively to odd and even infinite-series wavefunctions which are bounded for any value of u, namely,

$$\psi_2^{o}(u) = \left[\tanh u\right]^{-l-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(3)} J_{2n+l+(3/2)}\left(\frac{\mathrm{b}}{2}\sinh u\right),$$

$$\psi_2^{e}(u) = \left[\tanh u\right]^{-l-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(4)} J_{2n+l+(1/2)}\left(\frac{\mathrm{b}}{2}\sinh u\right). \tag{49a}$$

In the recurrence relations (12) for $b_n^{(3)}$ the coefficients are

$$\alpha_n^{(3)} = -\frac{\mathrm{b}^2(n+1)(2n+3)}{2\,(4n+2l+5)\,(4n+2l+7)},$$

$$\beta_n^{(3)} = \mathcal{E} - \frac{\mathrm{b}^2}{8} + \left(l - \frac{1}{2}\right)^2 + 4(n+1)\left(n + l + \frac{1}{2}\right) + \frac{\mathrm{b}^2\,(2l+1)\,(2l-1)}{8\,(4n+2l+1)\,(4n+2l+5)},$$

$$\gamma_n^{(3)} = -\frac{\mathrm{b}^2(2n+2l+1)\ (n+l)}{2\,(4n+2l-1)\,(4n+2l+1)}. \tag{49b}$$

and for $b_n^{(4)}$ the coefficients are

$$\alpha_n^{(4)} = -\frac{\mathrm{b}^2(n+1)(2n+1)}{2\,(4n+2l+3)\,(4n+2l+5)},$$

$$\beta_n^{(4)} = \mathcal{E} - \frac{\mathrm{b}^2}{8} + \left(l - \frac{1}{2}\right)^2 + 4(n+l)\left(n + \frac{1}{2}\right) + \frac{\mathrm{b}^2\,(2l+1)\,(2l-1)}{8\,(4n+2l-1)\,(4n+2l+3)}$$

$$\gamma_n^{(4)} = -\frac{\mathrm{b}^2(2n+2l-1)\ (n+l)}{2\,(4n+2l-3)\,(4n+2l-1)}. \tag{49c}$$

According to section 2.3, the corresponding expansions given by series of the Bessel functions Y_α and $H_\alpha^{(1,2)}$ converge in the domain $|z| = \sinh^2 u > 1$ which does not include all the values of u.

On the other side, from the solutions $U_7^{(1)}$ and $U_8^{(1)}$ we find another pair of bounded eigenfunctions given by

$$\psi_3^{o}(u) = \tanh u \sum_{n=0}^{\infty} \frac{\Gamma[n+(3/2)]}{(n+l)!}\,(-1)^n b_n^{(3)} J_{2n+l+(3/2)}\left(\frac{\mathrm{b}}{2}\cosh u\right),$$

$$\psi_3^{\mathrm{e}}(u) = \sum_{n=0}^{\infty} \frac{\Gamma[n+(1/2)]}{(n+l)!}\,(-1)^n b_n^{(4)} J_{2n+l+(1/2)}\left(\frac{\mathrm{b}}{2}\cosh u\right). \tag{50}$$

As the series coefficients are proportional to the coefficients of the previous pair, the characteristic equations are the same as in that pair. In addition, since in this case $\mathrm{Re}(B_1/z_0) = -1/2 < 1$ in $U_7^{(1)}$ and $U_8^{(1)}$, equation (29b) implies that now the expansions in terms of Y_α and $H_\alpha^{(1,2)}$ converge in the domain $|1-z| = \cosh^2 u \geq 1$, that is, over the entire range of u.

By using the solutions $U_5^{(j)}$ and $U_6^{(j)}$ we would find solutions equivalent the preceding ones. Actually, we would find $\alpha_n^{(5,6)} \propto (n-l+1)$ in the recurrence relations for $b_n^{(5)}$ and $b_n^{(6)}$ what means that the series begin at $n = l$. Thence, by setting $n = m + l$ we may conclude that such solutions are proportional to the above ones.

At last, notice that the previous considerations take into account only the analytical properties of the wavefunctions. The full solution of the problem requires the computation of the characteristic equation resulting from the three-term recurrence relations, which is represented by an infinite continued fraction having the form given in equation (12) or by the determinant of an infinite tridiagonal matrix.

4. The electron in the field of a point dipole

For an electron with mass M, charge e and energy E in the field of a point electric dipole, the time-independent Schrödinger equation is [5]

$$\left(-\frac{\hbar^2}{2M}\nabla^2 + \mathrm{e}\frac{\vec{D}\cdot\vec{r}}{r^3} - E\right)\psi = 0,$$

where D is the dipole moment. This equation is separable in spherical coordinates (r, θ, φ). Choosing the z axis along the dipole moment and performing the separation

$$\Psi(r,\theta,\varphi) = \frac{1}{r}R(r)\ \Theta(\theta)\ e^{\pm im\varphi}, \quad [0 \leq \theta \leq \pi, \quad m = 0, 1, 2, \cdots]$$

one gets

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + C - \beta\cos\theta - \frac{m^2}{\sin^2\theta}\right]\Theta = 0, \qquad \left[\frac{d^2}{dr^2} - \frac{C}{r^2} + \mathcal{E}\right]R = 0, \tag{51}$$

where C is the separation constant, $\beta = 2MeD/\hbar^2$ and $\mathcal{E} = 2ME/\hbar^2$. The energies are determined from the solutions of the radial equation, but firstly it is necessary to determine the parameter C from the solutions of the angular equation. The substitutions

$$\Theta(\theta) = (\sin\theta)^m\ U(z), \quad z = \sin^2(\theta/2), \tag{52}$$

give the equation

$$z(z-1)\frac{d^2U}{dz^2} - \big[m+1-2(m+1)z\big]\frac{dU}{dz} + \big[m(m+1) - \beta - C - 2\beta(z-1)\big]U = 0, \tag{53}$$

which is the Whittaker-Ince limit (5) of the CHE, with the following set of parameters

$$z_0 = 1, \quad B_1 = -m-1, \quad B_2 = 2m+2, \quad B_3 = m(m+1) - \beta - C, \quad q = -2\beta. \tag{54}$$

Therefore, the solutions $\Theta(\theta)$ can be constructed by introducing solutions of equation (5) into equation (52). Then, the admissible values for C are determined from the characteristic equations which follow from the recurrence relations for the series coefficients.

For $\beta = 0$ the angular equation (51) has solutions regular in the interval $0 \le \theta \le \pi$ if $C = \ell(\ell+1)$, where ℓ is a non-negative integer such that $\ell \ge m$. These solutions are given by the associated Legendre polynomials $P_\ell^m(\cos\theta)$. For this trivial case, a closed form for energy spectrum can be obtained from boundary conditions on the radial part of the wavefunction. However, if $\beta \ne 0$ there is no analytic formula for $\mathcal{E}$ since the constant C must be determined from a transcendental equation (characteristic equation).

In section 4.1 we write the expansions in series of Bessel functions for the angular equation. We find two periodic expansions having period 2π and the same characteristic equation. However, in section 4.2 we find only one expansion in series of associated Legendre polynomial; this has the same characteristic equation as the solutions in terms of Bessel functions.

4.1. Expansions in series of Bessel functions

For $\beta \ne 0$ the solutions $\Theta(\theta)$ can be obtained by inserting into (52) the solutions in series of Bessel functions of the first. The other expansions are unsuitable in virtue of their domain of convergence.

Thus, using the parameters (54) we find out that only $U_1^{(1)}$, $U_2^{(1)}$, $U_5^{(1)}$ and $U_6^{(1)}$ are valid. Moreover, both $U_1^{(1)}$ and $U_2^{(1)}$ afford the same solution, denoted by Θ_1. Analogously, both $U_5^{(1)}$ and $U_6^{(1)}$ yield another solution, denoted by Θ_2. These solutions are

$$\Theta_1(\theta) = \left(\sin\frac{\theta}{2}\right)^{-1}\left(\cot\frac{\theta}{2}\right)^m \sum_{n=0}^{\infty}(-1)^n b_n^{(1)} J_{2n+2m+1}\left(i\sqrt{8\beta}\ \sin\frac{\theta}{2}\right), \tag{55}$$

$$\Theta_2(\theta)=\left(\cos\frac{\theta}{2}\right)^{-1}\left(\tan\frac{\theta}{2}\right)^m\sum_{n=0}^{\infty}b_n^{(1)}J_{2n+2m+1}\left(\sqrt{8\beta}\ \cos\frac{\theta}{2}\right), \tag{56}$$

with the recurrence relations for $b_n^{(1)}$ given by $(\alpha_{-1}=0)$

$$\begin{aligned}&\frac{\beta(n+1)}{(2n+2m+3)}b_{n+1}^{(1)}-\Big[n\big(n+2m+1\big)+m\big(m+1\big)-C\Big]b_n^{(1)}+\\&\frac{\beta(n+2m)}{(2n+2m-1)}b_{n-1}^{(1)}=0,\quad \Big(b_{-1}^{(1)}=0\Big).\end{aligned} \tag{57}$$

These solutions have the same coefficients $b_n^{(1)}$ because equation (21) implies that $b_n^{(5)}=(-1)^n b_n^{(1)}$. Thus, there is only one characteristic equation to determine the values of the parameter C.

The solutions Θ_1 and Θ_2 are connected by the substitutions $\theta\rightarrow\theta+\pi$ and $\beta\rightarrow-\beta$ which leave invariant the angular equation (notice that this is equivalent to the change of b_n by $(-1)^n b_n$ in the recurrence relations). Both are convergent in the interval $0\leq\theta\leq\pi$ and are regular at the singular points $\theta=0$ and $\theta=\pi$. We have found no criterion to discard one of these solutions, neither have found a proof that they are equivalent (the problem mentioned in section 2.2). However, in the following we will find only one solution in series of Jacobi polynomials; this has the same characteristic equation as the above solutions.

4.2. Expansions in series of Jacobi polynomials

The expansions in series of Bessel functions given in section 2 appear associated with a group of expansion in series of Gauss hypergeometric functions $F(a,b;c;x)$, which are obtained by applying the transformations rules to the solution [15]

$$\mathbb{U}_1(z)=\sum_{n=0}^{\infty}b_n^{(1)}F\left(-n,n+B_2-1;B_2+\frac{B_1}{z_0};1-\frac{z}{z_0}\right),\quad [B_2\neq0,-1,-2,\cdots] \tag{58}$$

where $b_n^{(1)}$ is the same as in (20a) and satisfies the recurrence relations (20c). This series converges for finite values of z. The restrictions on the values of B_2 assure independence of the hypergeometric functions. However, we must demand as well that $c\neq0,-1,-2,\cdots$ because in general $F(a,b;c;x)$ is not defined if c is a negative integer or zero. For this reason, this group is less general than the group formed by series of Bessel functions. Now, putting $\mathbb{U}_5(z)=\mathscr{T}_3\mathbb{U}_1(z)$, we find

$$\mathbb{U}_5(z)=\sum_{n=0}^{\infty}b_n^{(5)}F\left(-n,n+B_2-1;-\frac{B_1}{z_0};\frac{z}{z_0}\right),\quad [B_2\neq0,-1,-2,\cdots] \tag{59}$$

where the coefficients $b_n^{(5)}$ are formally connected with $b_n^{(1)}$ by equation (21). If B_2+B_1/z_0 and $(-B_1/z_0)$ are not zero or negative integers, both solutions are valid and are given by infinite series.

Then, setting $z = z_0 \cos^2(\sigma u)$, using the relation (21) and rewriting the hypergeometric functions in terms of Jacobi's polynomials $P_n^{(\alpha,\beta)}$ through equation (A.8), we find

$$\mathbb{U}_1(z) = \mathbb{W}_1(u) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{n!\; b_n^{(1)}}{\Gamma(n+\alpha+1)}\; P_n^{(\alpha,\beta)}\big[\cos(2\sigma u)\big],$$

$$\mathbb{U}_5(z) = \mathbb{W}_5(u) = \Gamma(\beta+1)\sum_{n=0}^{\infty}\frac{(-1)^n\; n!\; b_n^{(1)}}{\Gamma(n+\alpha+1)}\; P_n^{(\beta,\alpha)}\big[-\cos(2\sigma u)\big], \quad (60)$$

where $\alpha = B_2 - 1 + B_1/z_0$ and $\beta = -1 - B_1/z_0$ (this β should not be confused with the parameter of the angular equation). Then, relation (A.9) implies that $\mathbb{U}_1$ is a multiple of $\mathbb{U}_5$. This conclusion does not hold if only one solution is valid. The same can be said of the other pairs of solutions, $(\mathbb{U}_i, \mathbb{U}_{i+4})$. In this manner, the linear dependence of solutions having the same characteristic equation is almost trivial in the present case.

The solution $\mathbb{U}_1$ (equivalent to $\mathbb{U}_5$) is the only one applicable to the angular equation of the point dipole. Thus, by inserting this into (52) and using the parameters (54), we find

$$\Theta(\theta) = (\sin\theta)^m \sum_{n=0}^{\infty} b_n^{(1)} F\big[-n, n+2m+1; m+1; \cos^2(\theta/2)\big],$$

where the coefficients $b_n^{(1)}$ again satisfy the recurrence relations (57). In terms of Jacobi polynomials (A.8) or associated Legendre polynomials (A.10), we find that

$$\Theta(\theta) = (\sin\theta)^m \sum_{n=0}^{\infty}\frac{(-1)^n n!}{(n+m)!} b_n^{(1)} P_n^{(m,m)}(\cos\theta) \propto \sum_{n=0}^{\infty}\frac{(-1)^n n!}{(n+2m)!} b_n^{(1)} P_{n+m}^{m}(\cos\theta). \quad (61)$$

If $m = 0$, this reduces to the solutions in series of ordinary Legendre polynomials $P_n = P_n^0$ given by Lévy-Leblond [5]. If $m \neq 0$, these are the Alhaidari solutions in terms of Jacobi polynomials [6] up to a redefinition of the series coefficients. On the other side, by taking into account the solutions $\mathbb{U}_2$, $\mathbb{U}_3$ and $\mathbb{U}_4$, written in [15], we find that these are not valid by one of the following reasons: the last parameter of the hypergeometric functions is a negative integer or $\Theta(\theta)$ is not regular at $\theta = 0$ or $\theta = \pi$.

Finally, the expansions in hypergeometric functions are inapplicable to the Schrödinger equation with the inverted potential (2). In effect, infinite-series solutions are inappropriate because they converge only for finite values of the arguments of the hypergeometric functions. On the other side, only $\mathbb{U}_1(z)$ and $\mathbb{U}_2(z)$ could afford finite-series solutions but in these cases the hypergeometric functions are not defined since the last parameter is zero or negative integer, that is, $c = B_2 + B_1/z_0 = 1 - l$.

5. Conclusion

We have dealt with one-sided solutions for the Whittaker-Ince limit (5) of the confluent Heun equation (CHE) and possible applications of these solutions. Specifically, in section 2 we have considered expansions in series of Bessel functions of the first and second kind, in addition to solutions in series of Hankel functions given in a previous paper [15]. In sections 3 and 4 we have established solutions in series of Bessel functions for an inverted potential and for an angular equation for a point electric dipole, respectively.

We have noticed the possible coexistence of different sets of expansions in series of Bessel functions having the same characteristic equation. For the Mathieu equation it is known that these solutions are linearly dependent in a common region of convergence [19, 22]. The same holds for expansions in series of Jacobi polynomials, as we have shown in section 4.2. If this dependence is valid for the general case, we can avoid the duplicity of solutions in infinite series of Bessel functions. Nevertheless, we have found no proof for such conjecture.

In section 3 we have obtained eigenstates given by finite and by infinite series of Bessel functions for the Schrödinger equation with the inverted potential. A proof that the generacy of finite series with opposite parity takes place for any value of the parameter l was possible because we have found a general expression for the eingenstates. These quasi-polynomial solutions permit to determine only a part of the energy spectrum by using, for instance, the procedure of Bender and Dunne presented in section 3.1.

Also the infinite series are convergent and bounded for any value of the variable u. As in the case of finite series, odd and even solutions are given by expansions in series of Bessel functions of the first kind, while other kinds of Bessel functions give solutions without definite parity. In principle, the solutions of the characteristic equations may afford the remaining part of the energy spectrum.

In section 4 we have found solutions given either by series of Bessel functions or by series of Jacobi polynomials for the θ-dependence of the scattering of electrons by the field of the point electric dipole. The expansion in series of Jacobi polynomials is equivalent to the one found by Alhaidari [6] and includes, as a particular case, the solution in series of Legendre polynomials proposed by Lévi-Leblond for the case $m = 0$ [5]. We have found two expansions in series of Bessel functions corresponding to the same characteristic equation. We have also explained why expansions in series of Jacobi polynomials are inappropriate for the inverted potential.

We observe that, we can find different solutions with the same characteristic equation also for expansions in series of confluent hypergeometric functions for the CHE, given in [29] – these are the ones which lead to the expansions in series of Bessel functions discussed in section 2. Then, the issue concerning the linear dependence or independence of solutions arises in this case as well.

Finally, although we have mentioned two other problems governed by Eq. (5), it would be interesting to find new ones since this could motivate further investigation on the CHE. For instance, there are expansions in two-sided infinite series ($-\infty < n < \infty$) of hypergeometric and confluent hypergeometric functions for the CHE [2, 29] which admit of the Whittaker-Ince limit [15], but need to be extended in order to incorporate all the Meixner expansions in series of Legendre and Bessel functions [21, 30] for the ordinary spheroidal wave equation, that is, for the CHE (7) with $\eta = 0$. Such extension would lead to two-sided solutions for Eq. (5) as well.

Appendix A. Some mathematical formulas

Firstly we give some properties of the Bessel functions $J_\alpha(x)$, $Y_\alpha(x)$, $H_\alpha^{(1)}(x)$ and $H_\alpha^{(2)}(x)$. After this we write some formulas concerning the hypergeometric functions. The power-series representation for $J_\alpha(x)$ is

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(\alpha+k+1)} \left(\frac{x}{2}\right)^{2k}. \tag{A.1}$$

This function is connected with the Hankel functions by the relation

$$J_\alpha(x) = \frac{1}{2}\left[H_\alpha^{(1)}(x) + H_\alpha^{(2)}(x)\right]. \tag{A.2}$$

Similarly, the Bessel functions of the second kind can be expressed as

$$Y_\alpha(x) = \frac{1}{2i}\left[H_\alpha^{(1)}(x) - H_\alpha^{(2)}(x)\right], \tag{A.3}$$

On the other hand, for a fixed α the behaviors of the Bessel functions when $|x| \to \infty$ are given by [21]

$$\begin{aligned}
&J_\alpha(x) \sim \sqrt{2/(\pi x)}\ \cos\left(x - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right), \ |\arg\ x| < \pi;\\
&H_\alpha^{(1)}(x) \sim \sqrt{2/(\pi x)}\ e^{i(x-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)}, \quad -\pi < \arg\ x < 2\pi;\\
&H_\alpha^{(2)}(x) \sim \sqrt{2/(\pi x)}\ e^{-i(x-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)}, \ -2\pi < \arg\ x < \pi.
\end{aligned} \tag{A.4}$$

The behavior for $Y_\alpha(x)$ is obtained by changing the cosines by sines in the expression for $J_\alpha(x)$. The Bessel functions are given by finite series of elementary functions if their order α is half an odd integer; then the restrictions on $\arg x$ are unnecessary in (A.4) [20]. For instance, if $m = 0, 1, 2, \cdots$, the Bessel functions of the first kind can be expressed as

$$J_{-m-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\ x^{m+\frac{1}{2}} \left(\frac{d}{xdx}\right)^m \frac{\cos x}{x}, \quad J_{m+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\ (-1)^m x^{m+\frac{1}{2}} \left(\frac{d}{xdx}\right)^m \frac{\sin x}{x}. \tag{A.5}$$

which give, in particular,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ \cos x, \quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x,$$

$$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right), \quad J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right). \tag{A.6}$$

Since, for x fixed and $\alpha \to \infty$,

$$J_\alpha(x) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^\alpha, \quad Y_\alpha(x) \sim -iH_\alpha^{(1)}(x) \sim iH_\alpha^{(2)}(x) \sim -\frac{1}{\pi}\Gamma(\alpha)\left(\frac{2}{x}\right)^\alpha,$$

we find

$$\frac{J_{\alpha+2}(x)}{J_\alpha(x)} \sim \frac{x^2}{4(\alpha+1)(\alpha+2)}; \quad \frac{Z_{\alpha+2}^{(j)}(x)}{Z_\alpha^{(j)}(x)} \sim \frac{4\alpha(\alpha+1)}{x^2}, \quad (j=2,3,4). \tag{A.7}$$

These relations have been used in section 2.3.

In section 4, the relation between hypergeometric functions and the Jacobi polynomials $P_n^{(\alpha,\beta)}$ was obtained from [27]

$$F(-n, n+1+\alpha+\beta; 1+\alpha; y) = \frac{n!\ \Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}\ P_n^{(\alpha,\beta)}(1-2y), \tag{A.8}$$

where n is a non-negative integer. The Jacobi polynomials can be expressed as

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right],$$

whereby we find the relation

$$P_n^{(\beta,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x), \tag{A.9}$$

used in section 4.2.

On the other side, the relation between hypergeometric functions and the associated Legendre functions P_ν^k, when k is a positive integer or zero, is given by

$$F(k-\nu, \nu+k+1; k+1; y) = \frac{(-1)^k k!\Gamma(\nu+1-k)}{\Gamma(\nu+k+1)}\left(y-y^2\right)^{-\frac{k}{2}} P_\nu^k(1-2y). \tag{A.10}$$

Thence, by setting $\alpha = \beta = k$ in (A.8) and $\nu = n+k$ in (A.10) we obtain

$$P_n^{(k,k)}(\xi) = \frac{(-2)^k (n+k)!}{(n+2k)!}\left(1-\xi^2\right)^{-k/2} P_{n+k}^k(\xi). \tag{A.11}$$

where we have put $\xi = 1-2y$.

Appendix B. The other solutions for equation (5)

¿From the solutions (20a), the others are obtained by using the transformations $\mathscr{T}_1$ and $\mathscr{T}_2$ as indicated in the sequences (17) and (18). In this manner, we get

$$U_2^{(j)}(z) = z^{(1-B_2)/2} \sum_{n=0}^{\infty} (-1)^n b_n^{(2)} Z^{(j)}_{2n+1+B_2+(2B_1/z_0)} (2\sqrt{qz}),$$

$$U_6^{(j)}(z) = z^{1+\frac{B_1}{z_0}} (z-z_0)^{-\frac{1}{2}-\frac{B_1}{z_0}-\frac{B_2}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(6)} \times Z^{(j)}_{2n+1+B_2+(2B_1/z_0)} \left(2\sqrt{q(z-z_0)}\right), \qquad \text{(B.1)}$$
$$\left[\frac{B_2}{2}+\frac{B_1}{z_0} \neq -1, -\frac{3}{2}, -2, \cdots\right]$$

where

$$\alpha_n^{(2)} = \frac{qz_0(n+1)\left(n+2+\frac{B_1}{z_0}\right)}{\left(n+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right)\left(n+\frac{3}{2}+\frac{B_2}{2}+\frac{B_1}{z_0}\right)},$$

$$\beta_n^{(2)} = 4B_3 - 2qz_0 + 4\left[n+1+\frac{B_1}{z_0}\right]\left[n+B_2+\frac{B_1}{z_0}\right] - \frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+\frac{B_2}{2}+\frac{B_1}{z_0}\right)\left(n+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right)},$$

$$\gamma_n^{(2)} = \frac{qz_0\left(n+B_2+\frac{B_1}{z_0}-1\right)\left(n+B_2+\frac{2B_1}{z_0}\right)}{\left(n-\frac{1}{2}+\frac{B_2}{2}+\frac{B_1}{z_0}\right)\left(n+\frac{B_2}{2}+\frac{B_1}{z_0}\right)}, \qquad \text{(B.2)}$$

in the recurrence relations for $b_n^{(2)}$, which are given by the equations

$$(12) \text{ if } \frac{B_2}{2}+\frac{B_1}{z_0} \neq 0, -\frac{1}{2}; \; (13) \text{ if } \frac{B_2}{2}+\frac{B_1}{z_0} = -\frac{1}{2}; \; (14) \text{ if } \frac{B_2}{2}+\frac{B_1}{z_0} = 0. \qquad \text{(B.3)}$$

These relations also hold for the coefficients $b_n^{(6)}$ with $\beta_n^{(6)} = \beta_n^{(2)}$ and

$$\alpha_n^{(6)} = \frac{-qz_0(n+1)\left(n+B_2+\frac{B_1}{z_0}\right)}{\left(n+\frac{B_1}{z_0}+\frac{B_2}{2}+\frac{1}{2}\right)\left(n+\frac{B_1}{z_0}+\frac{B_2}{2}+\frac{3}{2}\right)}, \quad \gamma_n^{(6)} = -\frac{qz_0\left(n+B_2+\frac{2B_1}{z_0}\right)\left(n+1+\frac{B_1}{z_0}\right)}{\left(n-\frac{1}{2}+\frac{B_1}{z_0}+\frac{B_2}{2}\right)\left(n+\frac{B_1}{z_0}+\frac{B_2}{2}\right)}, \qquad \text{(B.4)}$$

in the recurrence relations (B.3) for the coefficients $b_n^{(6)}$.

For the third and seventh sets we find

$$U_3^{(j)}(z) = (z-z_0)^{1-B_2-\frac{B_1}{z_0}} z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(3)} Z^{(j)}_{2n+3-B_2} (2\sqrt{qz}),$$

$$U_7^{(j)}(z) = z^{1+\frac{B_1}{z_0}} (z-z_0)^{-\frac{1}{2}-\frac{B_1}{z_0}-\frac{B_2}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(7)} Z^{(j)}_{2n+3-B_2} \left(2\sqrt{q(z-z_0)}\right), \; [B_2 \neq 4, 5, 6, \cdots] \qquad \text{(B.5)}$$

with the coefficients

$$\alpha_n^{(3)} = \frac{qz_0\ (n+1)\left(n+2+\frac{B_1}{z_0}\right)}{\left(n+2-\frac{B_2}{2}\right)\left(n+\frac{5}{2}-\frac{B_2}{2}\right)},$$

$$\beta_n^{(3)} = 4B_3 - 2qz_0 + 4(n+1)(n+2-B_2) - \frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n+1-\frac{B_2}{2}\right)\left(n+2-\frac{B_2}{2}\right)},$$

$$\gamma_n^{(3)} = \frac{qz_0\ (n+2-B_2)\left(n+1-B_2-\frac{B_1}{z_0}\right)}{\left(n+\frac{1}{2}-\frac{B_2}{2}\right)\left(n+1-\frac{B_2}{2}\right)}. \tag{B.6}$$

in the recurrence relations for $b_n^{(3)}$ which are given by

$$\text{Eq. (12) if } B_2 \neq 2,3; \quad \text{Eq. (13) if } B_2 = 3; \ \text{Eq. (14) if } B_2 = 2. \tag{B.7}$$

We obtain the coefficients ($\beta_n^{(7)} = \beta_n^{(3)}$)

$$\alpha_n^{(7)} = -\frac{qz_0\ (n+1)\left(n+2-B_2-\frac{B_1}{z_0}\right)}{\left(n+2-\frac{B_2}{2}\right)\left(n+\frac{5}{2}-\frac{B_2}{2}\right)}, \ \gamma_n^{(7)} = -\frac{qz_0\ (n+2-B_2)\left(n+1+\frac{B_1}{z_0}\right)}{\left(n+\frac{1}{2}-\frac{B_2}{2}\right)\left(n+1-\frac{B_2}{2}\right)}. \tag{B.8}$$

in the recurrence relations (B.7) for $b_n^{(7)}$. Finally we obtain

$$U_4^{(j)} = (z-z_0)^{1-B_2-\frac{B_1}{z_0}} z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(4)} Z_{2n+1-B_2-(2B_1/z_0)}^{(j)} \left(2\sqrt{qz}\right),$$

$$U_8^{(j)}(z) = z^{\frac{1-B_2}{2}} \sum_{n=0}^{\infty} (-1)^n b_n^{(8)} Z_{2n+1-B_2-\frac{2B_1}{z_0}}^{(j)} \left(2\sqrt{q(z-z_0)}\right), \ \left[\frac{B_2}{2}+\frac{B_1}{z_0} \neq 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots\right] \tag{B.9}$$

with

$$\alpha_n^{(4)} = \frac{qz_0\ (n+1)\left(n-\frac{B_1}{z_0}\right)}{\left(n+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n+\frac{3}{2}-\frac{B_2}{2}-\frac{B_1}{z_0}\right)},$$

$$\beta_n^{(4)} = 4B_3 - 2qz_0 + 4\left[n-\frac{B_1}{z_0}\right]\left[n-B_2+1-\frac{B_1}{z_0}\right] - \frac{2qz_0\left(\frac{B_2}{2}-1\right)\left(\frac{B_2}{2}+\frac{B_1}{z_0}\right)}{\left(n-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)},$$

$$\gamma_n^{(4)} = \frac{qz_0\ \left(n+1-B_2-\frac{B_1}{z_0}\right)\left(n-B_2-\frac{2B_1}{z_0}\right)}{\left(n-\frac{1}{2}-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n-\frac{B_2}{2}-\frac{B_1}{z_0}\right)}, \tag{B.10}$$

in the recurrence relations given by the following equations

$$(12) \text{ if } \frac{B_2}{2}+\frac{B_1}{z_0} \neq 0, \frac{1}{2}; \quad (13) \text{ if } \frac{B_2}{2}+\frac{B_1}{z_0} = \frac{1}{2}; \quad (14) \text{ if } \frac{B_2}{2}+\frac{B_1}{z_0} = 0. \tag{B.11}$$

The coefficients $b_n^{(8)}$ also satisfy the recurrence relations (B.11) with $\beta_n^{(8)} = \beta_n^{(4)}$ and

$$\alpha_n^{(8)} = \frac{-qz_0\ (n+1)\left(n+2-B_2-\frac{B_1}{z_0}\right)}{\left(n+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n+\frac{3}{2}-\frac{B_2}{2}-\frac{B_1}{z_0}\right)},$$

$$\gamma_n^{(8)} = -\frac{qz_0\ \left(n-1-\frac{B_1}{z_0}\right)\left(n-B_2-\frac{2B_1}{z_0}\right)}{\left(n-\frac{1}{2}-\frac{B_2}{2}-\frac{B_1}{z_0}\right)\left(n-\frac{B_2}{2}-\frac{B_1}{z_0}\right)}. \tag{B.12}$$

References

[1] E. Fisher, "Some differential equations involving three-term recursion formulas", *Phil.Mag.* **24**, 245 (1937).

[2] E. W. Leaver, "Solutions to a generalized spheroidal wave equation: Teukolsky equations in general relativity, and the two-center problem in molecular quantum mechanics," *J. Math. Phys.* **27**, 1238 (1986).

[3] A. H. Wilson, "A generalised spheroidal wave equation, " *Proc. Roy. Soc. London* **A118**, 617 (1928).

[4] H. T. Cho and C. L. Ho, "Self-adjoint extensions of the Hamiltonian operator with symmetric potentials which are unbounded from below," *J. Phys. A: Math. Theor.* **41**, 255308 (2008).

[5] J. M. Lévy-Leblond, "Electron capture by polar molecules," *Phys. Rev.* **153**, 1 (1967).

[6] A. D. Alhaidari, "Analytic solution of the wave equation of an electron in the field of a molecule with an electric dipole moment," *Annals of Physics* **323**, 1709 (2008).

[7] S. Kar and R. R Parwani, "Can degenerate bound states occur in one-dimensional quantum mechanics?, " Europhys. Lett., **80**, 30004 (2007).

[8] R. Koley and S. Kar, "Exact bound states in volcano potentials," Phys. Lett. A, **363**, 369 (2007).

[9] A. V. Turbiner, "Quantum mechanics: problems intermediate between exactly solvable and completely unsolvable," *Sov. Phys. JETP* **67**, 230 (1988).

[10] A. V. Turbiner, "Quasi-exactly-solvable problems and $sl(2)$ algebra," *Commun. Math. Phys.* **118**, 467 (1988).

[11] A. G. Ushveridze, "Quasi-exactly solvable models in quantum mechanics," *Sov. J. Part. Nucl.* **20**, 504 (1989).

[12] E. G. Kalnins, W. Miller and G. S. Pogosyan, "Exact and quasiexact solvability of second-order superintegrable quantum systems: I. Euclidian space preliminaries," *J. Math. Phys.* **47**, 033502 (2006).

[13] A. Ronveaux (editor), *Heun's Differential Equations* (Oxford University Press, 1995).

[14] L. J. El-Jaick and B. D. B. Figueiredo, "Solutions for confluent and double-confluent Heun equations," *J. Math. Phys.* **49**, 083508 (2008); e-print arXiv: 0800.2219v2.

[15] B. D. B. Figueiredo, "Ince's limits for confluent and double-confluent Heun equations," *J. Math. Phys.* **46**, 113503 (2005).

[16] S. Mignemi, "Classical and quantum motion on an Eguchi-Hanson space," *J. Math. Phys.* **32**, 3047 (1991).

[17] A. Malmendier, "The eigenvalue equation on the Eguchi-Hanson space," *J. Math. Phys.* **44**, 4308 (2003).

[18] A. D. Alhaidari, "Charged particle in the field of an electric quadrupole in two dimensions," *J. Phys. A: Math. Theor.* **40**, 14843 (2007).

[19] F. M. Arscott, *Periodic Differential Equations* (Pergamon Press, 1964).

[20] Y. L. Luke, *Integrals of Bessel functions* (McGraw-Hill, 1962).

[21] A. Erdélyi, W. Magnus, F. Oberhettingerm and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, 1963).

[22] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Dover, 1964).

[23] F. W. J. Olver, *Asymptotics and Special Functions* (Academic Press, 1974).

[24] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, 1954).

[25] K. Knopp, *Infinite Sequences and Series* (Dover, 1956).

[26] S. Flügge, *Practical Quantum Mechanics* (Springer, 1994).

[27] M. Abramowitz M and I. A. Stegun I A (eds.) *Handbook of Mathematical Functions* (Dover, 1965).

[28] C. M. Bender and G. W. Dunne, "Quasi-exactly solvable systems and orthogonal polynomials," *J. Math. Phys.* **37**, 6 (1996).

[29] B. D. B. Figueiredo, "On some solutions to generalized spheroidal wave equations and applications," *J. Phys. A: Math. Gen.* **35**, 2877 (2002).

[30] J. Meixner, "Reihenentwicklungen von Produkten zweier Spharoidfunktionen nach Produkten von Zylinder- und Kugelfunktionen," *Math. Nachr.* **3**, 193 (1950).