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Some Axisymmetric Stationary Solutions Generated by the Euclidon Method

J. Gariel[1*], *G. Marcilhacy*[1] *and N.O. Santos*[1,2,3 †]

[1] *LRM-CNRS/UMR 8540, Université Pierre et Marie Curie, ERGA,*
Boîte 142, 4 place Jussieu, 75005 Paris Cedex 05, France.
[2] *Laboratório Nacional de Computação Científica,*
25651-070 Petrópolis RJ, Brazil.
[3] *Centro Brasileiro de Pesquisas Físicas,*
22290-180 Rio de Janeiro RJ, Brazil.

Abstract

We apply the Euclidon method [1], for generating axisymmetric stationary solutions of Einstein's equations, to four static solutions with Newtonian potential describing semi-infinite line mass with linear mass density 1/2. The new solutions thus obtained are either the extreme Kerr black hole or the Kerr black hole.

*e-mail: gariel@ccr.jussieu.fr
†e-mail: santos@ccr.jussieu.fr and nos@cbpf.br

I The Euclidon method

GCE [1] built a method, called the Euclidon method, allowing, in principle, to obtain a new vacuum axisymmetric stationary solution from Einstein field equations, which we call *daughter-solution*, from any given vacuum solution, which we call *seed-solution*. The process of starting with a seed-solution to arrive to the daughter-solution needs an intermediary vacuum solution that we call *matrix-solution*. The matrix-solution chosen has null curvature, where the name Euclidon arises from. GCE make use of a method of variation of constants.

In the case where the seed-solution is static, the method is much simplified and the daughter-solution becomes stationary.

The Euclidon method, differently from the Herlt method [2], says nothing about the asymptotic behaviour of the new solutions thus obtained. However, in certain cases, using the corresponding Ernst potential and applying an Ehlers transformation [3], we can build a solution with a suitable asymptotic flatness.

We have presented the Euclidon method in its general form and for the case when its seed-solution is static in [4]. The stationary axisymmetric spacetime we described in its Papapetrou-Lewis metric form in Weyl coordinates, r and z, given by

$$ds^2 = f(dt - \omega d\phi)^2 - \frac{1}{f}[e^{2\gamma}(dr^2 + dz^2) + r^2 d\phi^2], \tag{1}$$

where f, ω and γ are functions of r and z. Here we apply this method for four different simple static seed-solutions, presented in the following four sections. For these static seed-solutions we choose, their Newtonian potentials, describing a semi-infinite line mass along the axis of symmetry. Its linear mass density, σ, is assumed to be 1/2. The reason for choosing this value is linked to its peculiar properties, which we briefly recall in the final section of the paper. We find that, using the same matrix-solution, the new solutions thus generated are either the extreme Kerr black hole or the Kerr black hole.

The formulae refered in [4] are followed by an asterix.

II Extreme Kerr black hole

We choose for the vacuum seed-solution (17*),

$$f_0 = q_1(\lambda - 1)(1 + \mu), \tag{2}$$

where q_1 is a real constant, and expressed in Weyl coordinates r and z (26*)-(27*), it becomes,

$$f_0 = q_1(z - z_1 + R). \tag{3}$$

We can see that (3), is a special Euclidon solution (8*), with constant $U_0 \to \infty$.

The system of partial differential equations (20*)-(21*) for U, when expressed in terms of spheroidal coordinates (26*)-(27*), becomes,

$$U_{,\lambda} = \frac{1}{\lambda-\mu}[(\lambda\mu-1)\chi_{,\lambda} + (1-\mu^2)\chi_{,\mu}], \tag{4}$$

$$U_{,\mu} = \frac{1}{\lambda-\mu}[-(\lambda^2-1)\chi_{,\lambda} + (\lambda\mu-1)\chi_{,\mu}], \tag{5}$$

where χ satisfies (19*). After integrating (4)-(5) we obtain

$$U = \ln\left[\frac{q_1(\lambda-\mu)^2}{(\lambda-\mu)(1+\mu)}\right], \tag{6}$$

where q_1 has been chosen as integration constant. Consequently the solution (14*)-(16*) becomes

$$\tilde{f} = \frac{(\lambda^2-1)(1-\mu^2) + q_1^2(\lambda-\mu)^4}{(\lambda-1)^2(1+\mu)^2 + q_1^2(\lambda-\mu)^4}, \tag{7}$$

$$\tilde{\Phi} = \frac{2q_1(\lambda-\mu)^3}{(\lambda-1)^2(1+\mu)^2 + q_1^2(\lambda-\mu)^4}, \tag{8}$$

$$\tilde{\omega} = \frac{2k}{q_1}\left\{\lambda + \frac{(1-\mu^2)[(\lambda^2-1)(1+\mu) - q_1^2(\lambda-\mu)^3]}{q_1^2(\lambda-\mu)^4 - (\lambda^2-1)(1-\mu^2)}\right\}. \tag{9}$$

The solution (7)-(9) is not asymptotically flat, like in the Demianski-Newman solution [6]. When $\lambda \to \infty$, from (9), $\tilde{\omega} \approx 2k\lambda/q_1 \to \infty$. Then we can make transformations on this solution to shape it with asymptotical flatness. Since these transformations on the solutions, producing new ones, are well known, we simply state them without making any remarks. In order to do that we first determine the Ernst potential $\xi(\lambda,\mu)$ from (7)-(9).

The Ernst equation is given by,

$$(\xi\xi^* - 1)\Delta\xi = 2\xi^*\vec{\nabla}\xi\cdot\vec{\nabla}\xi, \tag{10}$$

where

$$\xi = P(\lambda,\mu) + iQ(\lambda,\mu), \tag{11}$$

and Δ and $\vec{\nabla}$ are, respectively, the usual Laplacian and gradient operators in prolate spheroidal coordinates (26*)-(27*). The potential ξ is linked to $\tilde{f}$ and $\tilde{\Phi}$ by ζ, an intermediate potential, given by,

$$\zeta = \tilde{f} + i\tilde{\Phi}, \tag{12}$$

with,

$$\zeta = \frac{\xi-1}{\xi+1}. \tag{13}$$

From (11)-(13) we obtain,

$$P = \frac{1-\tilde{f}^2-\tilde{\Phi}^2}{(1-\tilde{f})^2+\tilde{\Phi}^2}, \quad Q = \frac{2\tilde{\Phi}}{(1-\tilde{f})^2+\tilde{\Phi}^2}, \tag{14}$$

and with (7)-(8) and (14), from (11) we have,

$$\xi = \frac{\lambda\mu - 1}{\lambda - \mu} + iq_1(\lambda - \mu). \tag{15}$$

Another solution, ξ_1, of Ernst equation can easily be obtained through the transformation

$$\xi_1 = i\xi = -q_1(\lambda + \mu) + i\frac{\lambda\mu + 1}{\lambda + \mu}, \tag{16}$$

which produces, with (11), the following solution,

$$\tilde{f} = \frac{(\lambda\mu + 1)^2 + (\lambda + \mu)^2[q_1^2(\lambda + \mu) - 1]}{(\lambda\mu + 1)^2 + (\lambda + \mu)^2[q_1(\lambda + \mu) - 1]^2}, \tag{17}$$

$$\tilde{\Phi} = \frac{2(\lambda\mu + 1)(\lambda + \mu)}{(\lambda\mu + 1)^2 + (\lambda + \mu)^2[q_1(\lambda + \mu) - 1]^2}, \tag{18}$$

$$\tilde{\omega} = -\frac{2k(\lambda^2 - 1)(1 - \mu^2)[q_1(\lambda + \mu) - 1]}{q_1[q_1^2(\lambda + \mu)^4 - (\lambda^2 - 1)(1 - \mu^2)]}. \tag{19}$$

This solution, (17)-(19), is discussed in [7], and has been found too by Das [8] and Bonanos and Kyriakopoulos [9] by using a method proposed by Herlt [2]. They have demonstrated that this solution corresponds to the extreme Kerr black hole solution, as it can be easily seen by determining the asymtotic behaviour of solution (17)-(19) by using the Boyer-Lindquist coordinates, $\mathcal{R}$ and θ,

$$\lambda = \frac{\mathcal{R} - M}{k}, \quad \mu = \cos\theta, \tag{20}$$

leading to $M = a$, where M is the mass and a is the angular momentum of the source.

III Schwarzschild and Kerr black holes

Another possible seed-solution is

$$f_0 = q_1(\lambda + 1)(1 + \mu), \tag{21}$$

which we considered in a previous article [4]. It is a static solution of the form (17*). The corresponding potential U is

$$U = \ln\left[\frac{\lambda + 1}{a_0(1 + \mu)}\right], \tag{22}$$

where a_0 is a constant of integration. The daughter-solution becomes,

$$\tilde{f} = \frac{\lambda^2 - 1 - a^2(1 - \mu^2)}{(\lambda + 1)^2 + a_0(1 + \mu)^2}, \tag{23}$$

$$\tilde{\Phi} = \frac{2a_0(\lambda - \mu)}{(\lambda + 1)^2 + a_0^2(\lambda + 1)^2}. \tag{24}$$

It follows the corresponding solution of the Ernst equation,

$$\xi = \frac{\lambda + a_0^2\mu}{1 + a_0^2} + i\frac{a_0(\lambda - \mu)}{1 + a_0^2}. \tag{25}$$

This solution can easily be transformed into the Kerr solution,

$$\xi_K = e^{i\alpha}\xi = p\lambda - iq\mu, \tag{26}$$

with

$$a_0 = -\tan\alpha, \quad p = (1 + a_0^2)^{-1/2}, \quad q = a_0(1 + a_0^2)^{-1/2}. \tag{27}$$

Furthermore, if we choose the asymptotic expression, $\lambda \to \infty$, of U in (22), we reobtain the Schwarzschild metric, as it can be directly seen from (14*)-(16*).

IV Kerr black hole

Considering,

$$f_0 = q_1(\lambda - 1)(1 - \mu), \tag{28}$$

the potential $U(\lambda, \mu)$ becomes,

$$U = \ln\left[\frac{q_1(\lambda - \mu)}{\lambda - 1}\right], \tag{29}$$

where q_1 is an integration constant. The daughter-solution then becomes,

$$\tilde{f} = -\frac{\lambda^2 - 1 - q_1(1 - \mu^2)}{(\lambda - 1)^2 + q_1^2(1 - \mu)^2}, \tag{30}$$

$$\tilde{\Phi} = \frac{2q_1(\lambda - \mu)}{(\lambda - 1)^2 + q_1^2(1 - \mu)^2}. \tag{31}$$

The solution of the Ernst equation that follows is

$$\xi = P + iQ, \tag{32}$$

where

$$P = -\frac{\lambda + q_1^2\mu}{\lambda^2 + q_1^2\mu^2}, \quad Q = \frac{q_1(\lambda - \mu)}{\lambda^2 + q_1^2\mu^2}. \tag{33}$$

We can make a unitary transformation on (32), of the form

$$\xi_1 = e^{i\theta_0} = (m + in)\xi, \tag{34}$$

where θ_0, m and n are real constants, we obtain from (32)

$$\xi = (mP - nQ) + i(nP + mQ), \tag{35}$$

and ξ_1 is still solution of the Ernst equation. Considering

$$q_1 = \frac{n}{m}, \quad m^2 + n^2 = 1, \tag{36}$$

we obtain for the daughter-solution

$$\tilde{f}_1 = \frac{1 - m^2\lambda^2 - n^2\mu^2}{(1 - m\lambda)^2 + n^2\mu^2}, \tag{37}$$

$$\tilde{\Phi}_1 = \frac{2n\mu}{(1 - m\lambda)^2 + n^2\mu^2}. \tag{38}$$

Finally making the transformation $\tilde{f}_1 \to -\tilde{f}_1$ and $m \to -m$, the solution thus obtained corresponds to the Kerr solution.

However, contrary to solution (22), the asymptotic behaviour, $\lambda \to \infty$, of (29) does no more allow us to obtain the Schwarzschild solution.

V Extreme Kerr black hole

Finally considering,

$$f_0 = q_1(\lambda + 1)(1 - \mu), \tag{39}$$

we calculate the potential $U(\lambda, \mu)$ giving

$$U = \ln\left[\frac{(\lambda + 1)(1 - \mu)}{q_1(\lambda - \mu)^2}\right], \tag{40}$$

where q_1 is an integration constant. The corresponding daughter-solution is

$$\tilde{f} = \frac{(\lambda^2 - 1)(1 - \mu^2)}{1 + \lambda)^2(1 - \mu)^2 + q_1^2(\lambda - \mu)^4}, \tag{41}$$

$$\tilde{\Phi} = \frac{2q_1(\lambda - \mu)^3}{(1 + \lambda)^2(1 - \mu)^2 + q_1^2(\lambda - \mu)^4}. \tag{42}$$

The solution of the Ernst equation, becomes

$$\xi = P + iQ, \tag{43}$$

where,

$$P = -\frac{(\lambda - \mu)(\lambda\mu - 1)}{(\lambda\mu - 1)^2 + q_1^2(\lambda - \mu)^4}, \tag{44}$$

$$Q = \frac{q_1(\lambda - \mu)^3}{(\lambda\mu - 1)^2 + q_1^2(\lambda - \mu)^4}. \tag{45}$$

We can rewrite (43) like

$$\xi = \left[\frac{(\lambda\mu - 1)^2}{(\lambda - \mu)^2} + q_1^2(\lambda - \mu)^2\right]^{-1} \xi_1, \tag{46}$$

where,

$$\xi_1 = -\frac{\lambda\mu - 1}{\lambda - \mu} + iq_1(\lambda - \mu). \quad (47)$$

We see that ξ_1 is a solution of Ernst equation, since from (46)-(47) we can further rewrite (46), becoming

$$\xi = \frac{1}{\xi_1^*}. \quad (48)$$

Through a simple transformation (46) reduces to (15), hence it represents also an extreme black hole.

VI Conclusion

For the seed-solutions of the Euclidon method, we have chosen the static solutions (2), (21), (28) and (39). They represent Newtonian potentials of semi-infinite lines of mass with different orientations in the prolate spherical coordinates (26*)-(27*). Semi-infinite lines of mass have been throughly studied by Bonnor and Martins [10]. We chose them with $\sigma = 1/2$, being σ their mass per unit lentgh. The reason for choosing this particular value lies in the peculiar properties of σ in Weyl solutions [11]. One of its family of solutions is the γ metric [12], which has limit, when its Newtonian source length tends to infinity [13], the Levi-Civita spacetime [14]. For $\sigma = 1/2$ the Levi-Civita spacetime becomes flat and the γ spacetime becomes the Schwarzschild spacetime [15].

With the semi-infinite line mass as seed-solutions, with $\sigma = 1/2$, we generated, using the Euclidon method, the extreme Kerr black hole and Kerr black hole spacetimes. This might suggest that when the Kerr metric [16] has its source length tend to infinity, we may obtain the Lewis spacetime [17] with $\sigma = 1/2$.

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