CBPF-NF-022/00

The Angular Size Distance-Redshift Relation for a Flat Accelerating Universe

by

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Abstract

We derive the exact expression of the angular size-distance redshift relation for a flat Friedmann-Robertson-Walker universe with a cosmological constant, depending on the parameters Ω_Λ and Ω_M. Expressions in terms of $\gamma = (\Omega_\Lambda/\Omega_M)^{1/6}$ and its inverse are given, in particular reproducing Mattig's relation with a $O(\gamma^5)$ correction due to the cosmological constant.

Key-Words: Accelerating universe; Distance-redshift relation.

There is now strong observational evidence that the expansion of the universe is accelerating [Perlmutter et al. 1999; Perlmutter, Riess et al., 1998; Schmidt et al., 1998; Turner and White, 1999]. The results are fully consistent with the existence of a cosmological constant, whose contribution to the total energy density of the universe amounts to 70% of the critical density ($\Omega_\Lambda \sim 0.7$). Since matter alone is considered to contribute $\Omega_M \sim 0.4. \pm 0.1$, taken together matter and cosmological constant energy account for a critical density Universe, consistent with measurements of the anisotropy of the cosmic microwave background. In this context, we may be led to consider that the large scale geometry of the Universe is approximately described by a flat Friedmann-Robertson-Walker universe with a cosmological constant and that large scale measurements (e.g., deep surveys at $z \geq 1$) should take into account this spacetime structure.

In this vein, the aim of the present note is precisely to derive an exact analytic expression of the proper angular size distance-redshift relation for a FRW universe with dust plus a cosmological constant, having zero curvature spatial sections. The interest of such exact analytic expression lies in the possibility of expanding it to any order of perturbation in the parameters $\sigma = (\Omega_M/\Omega_\Lambda)^{1/6}$ or $\gamma = (\Omega_\Lambda/\Omega_M)^{1/6}$ obtaining analytically the distante-redshift relation constribution to any order of perturbation in these parameters and testing all possible orders of perturbations in fitting experimental data. The present observational data obviously favours the expansion in σ. There is also an interest per se in the exact relation since it is new in the literature. As we will see, the expansion of our expression in $O(\gamma^2)$ will furnish the exact well-known Mattig's relation for a matter dominated flat universe, and further terms in the expansion in γ will furnish the first corrections to Mattig's relation in the case the universe has a small component Ω_Λ. The expansion in $O(\sigma^2)$ gives the distance-redshift relation in a DeSitter universe; higher order terms in the expansion in σ will show us the corrections to the DeSitter distance-redshift relation due to the presence of a matter distribution component. Actually, in the latter case, corrections to all orders in σ may straightforwardly be obtained by a direct expansion of our exact relation.

Our derivation considers basically the FRW flat universe

$$ds^2 = dt^2 - a^2(t)\left\{dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\} \ , \tag{1}$$

with $a(t)$ the scale factor of the universe, and satisfying Einstein's equations with dust and a cosmological constant Λ. The first integral of Einstein's equations gives

$$(da/dt)^2 = 8\pi G\rho_0 a_0^3/3a - (\Lambda/3)a^2 \ , \tag{2}$$

where ρ_0 is the present matter density of the universe and a_0 is the present value of the scale factor of the universe. Defining the Hubble constant H_0 as the present value of the paramenter $H = [(da/dt)/a]$, equation (2) can be expressed as

$$H^2 = H_0^2 \left[\Omega_M (a_0/a)^3 - \Omega_\Lambda\right] \tag{3}$$

where we have defined

$$\Omega_M = 8\pi G\rho_0/3H_0^2 \quad , \quad \Omega_\Lambda = \Lambda/3H_0^2 \ . \tag{4}$$

Let us consider now that we are an observer located at the origin of the coordinate system $r = 0$ and receive light signals from a source located at (r_1, θ, ϕ). Radial null signals emitted by the source must satisfy (cf. (1))

$$dt/a(t) = \pm dr \tag{5}$$

and we obviously adopt the minus sign since the observer is located at the origin $r = 0$. If the source emitted a signal at t_1 with corresponding scale factor a_1, we receive this signal at the present time t_0 with corresponding scale factor a_0. Due to the cosmological expansion, the wavelenght spectrum of the signal emitted will reach us redshifted according to

$$1 + z = \lambda_0/\lambda_1 = a_0/a_1 \ . \tag{6}$$

From (2) equation (5) can be straightforwardly integrated

$$\int_0^{r_1} dr = \int_{a_0(1+z)}^{a_0} \frac{da}{\sqrt{[(8\pi G\rho_0/3)\, a_0^3 a + (\Lambda/3)a^4]}} \ . \tag{7}$$

The crux of our problem corresponds to solve analytically the integral equation (7). Although a nontrivial task, we were able to realize it by using standard methods in the treatment of elliptic integrals [Abramowitz and Stegun, 1989]. Equation (7) is solved by the Jacobian elliptic function $cn(u/m)$ according to

$$F(z) = \sqrt{(7+4\sqrt{3})} \left\{ cn \left(K_0 + (\sqrt{\sqrt{3}})\Omega_\Lambda^{1/6}\Omega_M^{1/3} H_0 a_0 r_1 \Big| (2+\sqrt{3})/4 \right) \right\} \tag{8}$$

where

$$F(z) = \frac{\left[1 - (\sqrt{3}+1)(\Omega_M/\Omega_\Lambda)^{1/3}(1+z)/2\right]}{\left[1 - (\sqrt{3}-1)(\Omega_M/\Omega_\Lambda)^{1/3}(1+z)/2\right]} \tag{9}$$

and K_0 is a pure number, depending on σ^2 only, defined by

$$K_0 \equiv cn^{-1}\left(\frac{F(z=0)}{(\sqrt{(7+4\sqrt{3})})}\Big|(2+\sqrt{3})/4\right) \tag{10}$$

Eqs. (8)-(10) constitute the basic results of this note, relating the redshift z to the proper angular size distance [Peebles, 1993] $a_0 r_1$ in a flat accelerating universe.

For illustration, let us now expand (8)-(9) in powers of σ^2; according to recent observational data the paramete σ^2 is evaluated to be a number of the order of 0.75. We obtain from the expansion up to $O(\sigma^8)$ that the redshift-distance law is given by

$$\begin{aligned} z - \left\{(\sqrt{3}-1)^3(1+z)^4/8\right\}\sigma^6 &= \sqrt{\Omega_\Lambda}\ H_0 a_1 r_1 + \\ + \ \frac{1}{8}\left\{\left[(11-6\sqrt{3})\left(1+\sqrt{\Omega_\Lambda}H_0 a_0 r_1\right)^4 - 1\right]\right\}&\sigma^6 + 0(\sigma^8)\ . \end{aligned} \tag{11}$$

This expression holds for all values of z and $a_0 r_1$. We note that at any order of expansion obtained we have that $z = 0$ corresponds to $a_0 r_1 = 0$. For neglegible σ^2 the expression reduces to the usual redshift-distance relation in a DeSitter Universe, namely.

$$z - \sqrt{\Omega_\Lambda}\ H_0 a_0 r_1 = 0\ .$$

In the case of a matter dominated universe, in which the value of $\gamma = (\Omega_\Lambda/\Omega_M)^{1/6}$ is small, we would obtain form the expansion of (8)-(9) that

$$1/(1+z) = \left[1 - \sqrt{\Omega_M}\ H_0 a_0 r_1/2\right]^2 + O(\gamma^5) \tag{12}$$

Neglecting the contributions of $O(\gamma^5)$, this gives the well-known Mattig's relation [Mattig, 1958] for a spatially flat universe with dust. The first correction $O(\gamma^5)$ to Mattig's relation due to the presence of a cosmological constant is only of pure academic interest, due to the present observational status, and will not be given here. It can however be derived without difficulty, in a procedure analogous in obtaining expansion (11).

Finally, for z small, the distance-redshift relation (11) simplifies to

$$z - \sqrt{\Omega_\Lambda}\ H_0 a_0 r_1 \frac{\left\{1 + 4(11-6\sqrt{3})\sigma^6\right\}}{\left\{1 - 4(\sqrt{3}-1)^3\sigma^6\right\}} = 0(\sigma^8)$$

All our calculations were checked with Maple V package programs.

APPENDIX

A useful relation in expansing the equations (8)-(9) in terms of the parameters is the exact expression of the derivative of K_0.

$$dK_0/d(\sigma^2) = \frac{(\sqrt{\sqrt{3}})}{\sqrt{(1+\sigma^6)}} \; .$$

The corresponding derivative with respect to γ can be easily obtained by noting that $\sigma = 1/\gamma$.

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