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The Thermal Gap Equation in the Massless $\lambda\varphi^4_D$ Model

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Abstract

Using the Dyson-Schwinger equation, we re-examinate the behavior at finite temperature of the massless $\lambda\varphi^4$ model in a generic D-dimensional Euclidean space. An analysis of the thermal behavior of the renormalized self-energy is done for all temperatures. It results that the thermal renormalized self-energy is positive and increases monotonically with the temperature.

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1 Introduction

Studying the Casimir effect for a massless scalar field in a D-dimensional spacetime [1], it was obtained a generalization of the Debye integrals [2]. The purpose of this letter is to show how this result can be used in the resummation program in the massless $\lambda\varphi^4$ theory. It recent papers it has been used this above metioned useful result [3][4], together with the CJT formalism [5][6] and the Dyson-Schwinger equations, in order to obtain the behavior of the thermal mass and coupling constant assuming that the quantum system is in equilibrium with a thermal reservoir.

The Dyson-Schwinger equations (DSE) provide a non-perturbative approach to solving quantum field theories. The DSE are a infinite tower of coupled equations for the n-point functions of the theory, and a tractable problem is only obtained if one truncates the system. One systematic way of truncating the system is the weak coupling expansion. In this way the DSEs contain perturbative theory, since the weak coupling expansion of the equations generates all the diagrams obtained in perturbation theory. There are many different examples of DSEs. For example, since the phenomenon of supercondutivity in the BCS theory [7] is related to electron pair condensation and display a non-perturbative character of such condensation, the gap equation that describes Copper pairing is simply a truncated DSE for the fermion field two-point function. Bethe-Salpeter equations are also DSEs for four-point functions.

In interacting relativistic quantum field theories two types of divergences appear. The first one are the ultraviolet divergences that arises because quantum fields are operator-value distributions. Consequently, the singular ultraviolet behavior of a theory is independent of the sector (vacuum, thermal, etc.). The secong group are the infrared divergences and they strongly depends on the sector in which a given theory is being examinated [8].

Using dimensional regularization [9] it is very hard to see the physical significance of the infrared divergences. It is clear that infrared divergences should be absence in the cross section of a physically observed process. In $(QED)_4$ is refered to the Bloch-Nordsieck theorem [10]. In QCD the same mechanism is expected to work. Nevertheless the situation is quite different since in the $(QED)_4$ one enconters only soft divergences and in the second one appear colinear divergences. In QCD soft cancelations was demonstrated at the one-loop level in some processes where also the colinear divergences cancel out. There is a powerful theorem: The Kinoshita-Poggio-Quinn (KPT) theorem [11], i.e. the absence of infrared divergences in off-shell Green's functions in massless renormalizable field theories. In other words: the proper (one-particle irreducible) Green's functions with Euclidean non-exceptional external momenta are free of infrared divergences in massless renormalizable theories.

Temperature effects also can solve the infrared problem in some models in QFT [12]. For a recent treatment in non-abelian gauge theories at high temperature see for example Ref.[13] Also in the massless $\lambda\varphi^4$ theory, if we assume thermal equilibrium with a reservoir the infrared problem can be solved after a ressumation procedure [14][15][16][17]. For a recent reatment of the $\lambda\varphi^4$ theory, see for example Ref.[4] [17]. The purpose of this letter is to calculate the gap equation in the massless $(\lambda\varphi^4)_D$ theory at finite temperature. In this paper we use $\hbar = c = k_b = 1$.

2 The thermal gap equation in the massless $\lambda\varphi^4_D$ model

In this section we will calculate the temperature dependent self-energy $\Pi(\beta)$ using an alternative method developed by de Calan et al [3] and Ananos et al [4]. Let us suppose that our system is in equilibrium with a thermal bath. At the one-loop approximation the thermal mass and coupling constant for the $\lambda\varphi^4$ model in a d-dimensional Euclidean space have been obtained in a previous work [18] and are given by

$$m^2(\beta) = m^2 + \frac{g}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left(\frac{m}{\beta n}\right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1}(mn\beta) \tag{1}$$

and

$$\lambda(\beta) = g - \frac{3}{2}\frac{g^2}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left(\frac{m}{\beta n}\right)^{\frac{D}{2}-2} K_{\frac{D}{2}-2}(mn\beta), \tag{2}$$

where $K_\nu(z)$ is the modified Bessel function and m^2 and g are the zero temperature renormalized squared mass and coupling constant respectively. Note that since we are using dimensional regularization there is implicitly in the definition of the coupling constant g a factor μ^{4-d}. It is possible to improve the above results using the CJT formalism or the Dyson-Schwinger equations. In imaginary time formalism, the Dyson-Schwinger equation give us a self-consistent equation. (In this case we adopt the notation $\Pi(\beta)$ for the temperature dependent self-energy and we are in D=4).

$$\Pi(\beta) = \frac{g}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{p}^{\,2} + \Pi(\beta)} \tag{3}$$

Note that we are working in the massless case and the Matsubara frequencies are $\omega_n = \frac{2n\pi}{\beta}$. Since we use a mix between dimensional and zeta function analytical regularizations to evaluate formally the integral over the continuous momenta and the summation over the Matsubara frequencies, let us write the above expression in a general $D = d + 1$ dimensional Euclidean space and introduce $g = \mu^{4-D}\lambda$. Since we are working in imaginary time formalism, the Euclidean time is restricted to the interval $0 \leq \tau \leq \beta = \frac{1}{T}$, and the functional integral is define over the field $\varphi(\tau, \vec{x})$ satisfying periodic boundary in Euclidean time. All the Feynman rules are the same as in zero temperature case, except that that the momentum space integral over the zeroth component is replaced by a sum over discrete frequencies. In a D dimensional Euclidean space the gap equation becomes

$$\Pi(\beta) = \frac{g}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\omega_n^2 + \vec{p}^{\,2} + \Pi(\beta)} \tag{4}$$

To perform the sum over n many authors use a countour integral in the complex energy plane, and after they perform the integral in the continua momenta. We use an alternative procedure. First we will use dimensional regularization an the continua momenta and after this we use the principle of the analytic extension to evaluate the divergent sums over the Matsubara frequencies.

3 The solution of the thermal gap equation

In this section we will show how the generalization of the Debye integrals can be used to solve the finite temperature gap equation. As we discussed, from the Dyson-Schwinger equation it is possible to write the self-energy gap equation:

$$\Pi(\beta) = \frac{g}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\omega_n^2 + \vec{p}^{\,2} + \Pi(\beta)} \tag{5}$$

To use dimensional regularization we have that $\Pi(\beta) \neq 0$. If we assume $\Pi(\beta) = 0$ we have an trivial identity, consequentely let us assume $\Pi(\beta) \neq 0$. After a straighforward calculation using dimensional regularization and defining $f(d) = \frac{\pi^{\frac{d}{2}-2}}{8}$ we have:

$$\Pi(\beta) = g\beta^{1-d} f(d)\Gamma(1-\frac{d}{2}) \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + \frac{\beta^2\Pi(\beta)}{4\pi^2})^{1-\frac{d}{2}}} \tag{6}$$

Let us define the Hurwitz zeta function $\zeta(z,a)$ as:

$$\zeta(z,a) = \sum_{n=-\infty}^{\infty} \frac{1}{(n^2+a^2)^z} \tag{7}$$

The Hurwitz zeta function $\zeta(z,a)$ is analytic for $Re(z) > \frac{1}{2}$. It is possible to analytic extend it for the whole complex plane and an useful representation which is valid for $Re(z) < 1$ was given by Ford and used by Ford and Svaiter [19]:

$$\sum_{n=-\infty}^{\infty} \left(n^2+a^2\right)^{-z} = a^{1-2z} \left[\sqrt{\pi}\, \frac{\Gamma(z-\frac{1}{2})}{\Gamma(z)} + 4\sin\pi z \int_1^{\infty} \frac{(t^2-1)^{-z} dt}{e^{2\pi a t}-1} \right]. \tag{8}$$

It is clear that the most interesting case is the even dimensional Euclidean space. Consequently, let us study the even dimensional case. Substituting the analytic extension of the Hurwitz zeta function in the self-energy we get a sum of a polar (temperature independent) term plus a thermal analytic correction. The pole is suppressed by the renormalization procedure. Then after some technical manipulations, if we define $h(d) = 4\Gamma(1-\frac{d}{2})\frac{\pi^{-\frac{d}{2}-1}}{2^{d+2}}$, the gap equation may be rewritten in the form,

$$\Pi(\beta) = g(\Pi(\beta))^{\frac{d-1}{2}} h(d) \int_1^{\infty} dt (t^2-1)^{\frac{d}{2}-1} \frac{1}{e^{\sqrt{\Pi(\beta)}\beta t}-1}. \tag{9}$$

From the above expression some authors claim that there is a breakdown of the perturbation theory due to infrared divergences and the consequence is the fact that the functional dependence of $\Pi(\beta)$ is not in powers of g as we would expect from perturbation theory, but appears an non-analytic behavior. The purpose of this calculations is analyse the exact form of $\Pi(\beta)$. Defining a new variable $\tau = \sqrt{\Pi(\beta)}\beta t$ it is easy to show that

$$(\Pi(\beta))^{4-d} = g\frac{h(d)}{\beta} \int_{\sqrt{\Pi(\beta)}\beta}^{\infty} d\tau \left((\frac{\tau}{\sqrt{\Pi(\beta)}\beta})^2 - 1 \right)^{\frac{d}{2}-1} \frac{1}{e^{\tau}-1}. \tag{10}$$

Since we are studying the even dimensional case the use of the Newton binomial theorem will give a very direct way for evaluating $\Pi(\beta)$. The expansion of $\left((\frac{\tau}{\sqrt{\Pi(\beta)}\beta})^2-1\right)^{\frac{d}{2}-1}$ yields a infinite power series, and the expression for the thermal squared mass becomes

$$\Pi(\beta) = g\beta^{1-d}\sum_{k=0}^{\infty} c(d,k)\left(\sqrt{\Pi(\beta)}\beta\right)^{2k}\int_{\sqrt{\Pi(\beta)}\beta}^{\infty} d\tau \frac{\tau^{d-2-2k}}{e^{\tau}-1}, \tag{11}$$

where

$$c(d,k) = h(d)C_p^k(-1)^k, \tag{12}$$

and

$$C_p^0 = 1, C_p^1 = \frac{p}{1!}, ..C_p^k = \frac{p(p-1)..(p-k+1)}{k!}, \tag{13}$$

are a generalization of the binomial coeficients. Note that for small values of k the integral that appear in eq.(11) is a Debye integral of the type

$$I_1(x,n) = \int_x^{\infty} d\tau \frac{1}{e^{\tau}-1}\tau^n = \sum_{q=1}^{\infty} e^{-qx}\left(\frac{x^n}{q} + \frac{nx^{n-1}}{q^2} + ...\frac{n!}{q^{n+1}}\right), \tag{14}$$

which is valid for $x > 0$ and $n \geq 1$ [2]. For k satisfying $k > \frac{D-4}{2}$, which corresponds to $n < 1$ in the preceeding equation, it is necessary to generalize the Debye integral (the case $n = 0$ is trivial). Let us investigate the case $n < 0$. This generalization has been done by Svaiter and Svaiter [1] and the result reads,

$$I_2(x,n) = \int_x^{\infty} d\tau \frac{1}{e^{\tau}-1}\frac{1}{\tau^n} = -\sum_{q=0,q\neq n}^{\infty} \frac{B_q}{q!}\frac{x^{q-n}}{q-n} - \frac{1}{(n!)}B_n \ln x + \gamma_{\frac{n-1}{2}}, \tag{15}$$

(for odd n), $Re(x) > 0$, $2\pi > |x| > 0$ and $\gamma_{\frac{n-1}{2}}$ being a constant. The quantites B_n are the Bernoulli numbers. Note that this generalization can be used only for high-temperatures i.e. $m(\beta)\beta < 2\pi$. Thus, in the high temperature regime, if we define

$$I(x, D-3-2k) = \begin{cases} I_1(x, D-3-2k), & for \quad x > 0, \quad k \leq \frac{D-4}{2} \\ I_2(x, D-3-2k), & for \quad 0 < x < \pi, \quad k > \frac{D-4}{2}, \end{cases} \tag{16}$$

we may write

$$\Pi(\beta) = g\beta^{2-D} \sum_{k=0}^{\infty} f(d,k) \left(\sqrt{\Pi(\beta)}\beta\right)^{2k} I\left(\sqrt{\Pi(\beta)}\beta, D-3-2k\right). \tag{17}$$

The above equation gives a non-perturbative expression for the thermal squared mass in the high temperature regime in the case of even dimensional Euclidean space. All the calculations can be done in a odd dimensional space, and in tis case the summation in k finishes at $\frac{D-3}{2}$. For any dimension, it is possible to perform a numerical analysis of the behavior of the renormalized self-energy for all temperatures using eq.(11). It was found that in both cases $D = 3$, and $D = 4$, the thermal squared mass appears as a positive monotonically increasing function of the temperature.

4 Conclusions

We have done in this paper an analysis of the $\lambda\varphi^4$ model in a flat D-dimensional Euclidean space in equilibrium with a thermal bath. The form of the thermal corrections to the self-energy have been discussed using resummation methods. We have chosen this way of working, in order to get answers as much as possible of a non-perturbative character. In what concerns the thermal behavior of the self-energy, we have shown that it is a monotonic increasing function of the temperature for any Euclidean dimension.

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