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Student's t- and r-Distributions: Unified Derivation From an Entropic Variational Principle

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Abstract

The optimization of the recently generalized entropy $S_q \equiv \{1 - \int dx[p(x)]^q\}/(q-1)$ with the constraints $\int dx p(x) = 1$ and $< x^2 >_q \equiv \int dx x^2 [p(x)]^q = 1$ yields the Student's t-distribution for $q > 1$, and the r-distribution for $q < 1$.

Keywords: Generalized entropy; Variational principle, Student's t-distribution; r-distribution.

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The normal (Gaussian) distribution can be derived through a great variety of manners. One of the most elegant no doubt is through the optimization of the Boltzmann-Gibbs-Shannon entropy

$$S_1[p] \equiv -\int dx p(x) ln p(x) \tag{1}$$

with the constraints

$$\int dx p(x) = 1 \tag{2}$$

and

$$< x^2 >_1 \equiv \int dx x^2 p(x) = 1 \tag{3}$$

(the subindex 1 will become clear later on). Indeed, if we impose $\delta S_1[p] = 0$ and introduce the Lagrange parameters α and β (to take account of constraints (2) and (3) respectively) we obtain straightforwardly

$$p(x) = \sqrt{\frac{\beta}{\pi}} e^{-\beta x^2} \tag{4}$$

where we have already used Eq. (2) to eliminate α. If we introduce now the new variable

$$y \equiv \sqrt{2\beta} x \tag{5}$$

we obtain

$$\phi^{(G)}(y) = \frac{1}{\sqrt{2\beta}} p(y/\sqrt{2\beta}) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \tag{6}$$

as desired. The optimization of $S_1[p]$ is equivalent of course to the optimization of the *likelihood function*

$$L_1[p] = e^{S_1[p]}. \tag{7}$$

The question we focus in the present paper is how could we obtain the well known Student's t- and r-distributions from a similar variational principle. For physical purposes (related to multifractals and long-range interactions), one of us introduced [1] the following generalized entropy

$$S_q[p] = \frac{1 - \int dx [p(x)]^q}{q-1} \quad (q \in \Re) \tag{8}$$

which, in the $q \to 1$ limit, recovers Eq. (1) (by using $[p(x)]^{q-1} = e^{(q-1) ln p(x)} \approx 1 + (q-1) ln p(x)$). This entropy satisfies a great variety of interesting mathematical and physical properties[2-15]. Let us just recall here that it is nonnegative ($\forall q$), concave (convex) if $q > 0$ ($q < 0$), and also that

$$S_q[p_1 p_2] = S_q[p_1] + S_q[p_2] + (1-q) S_q[p_1] S_q[p_2]. \tag{9}$$

where p_1 and p_2 refer to independent systems. In other words, in contrast with $S_1[p]$ (and with the so-called Renyi entropy), $S_q[p]$ is generically *nonextensive (nonadditive)*.

In what concerns physical applications, it has been successfully used in astrophysical systems[16,17], in Lévy-flight-like anomalous diffusion[18], in learning tasks accomplished by perceptrons[19], in statistical inference[20], possibly in biological systems[21], economical ones, among others. More specifically, this generalized entropy has enabled a consistent generalization of Statistical Mechanics and of Thermodynamics[2] (while preserving the standard Legendre-transform framework). It comes out that the standard average $< O >_1 \equiv \int dx O(x) p(x)$ of an arbitrary observable O must be generalized into the *q-expectation value* $< O >_q \equiv \int dx O(x) [p(x)]^q$.

We are now prepared to come back to our initial aim related to Student's t- and r-distributions. If we optimize $S_q[p]$ with the constraints given by Eq. (2) and, following [18], by

$$< x^2 >_q \equiv \int dx x^2 [p(x)]^q = 1 \tag{10}$$

(instead of Eq. (3)), we straightforwardly obtain

$$p(x) = \frac{[1-\beta(1-q)x^2]^{1/(1-q)}}{\int dx [1-\beta(1-q)x^2]^{1/(1-q)}}. \tag{11}$$

Let us discuss now separately the $q > 1$ and $q < 1$ cases.

$\underline{q > 1}$:

Eq. (11) becomes

$$p(x) = \frac{[1+\beta(q-1)x^2]^{1/(1-q)}}{\int_{-\infty}^{\infty} dx [1+\beta(q-1)x^2]^{1/(1-q)}}$$

$$= \sqrt{\frac{\beta(q-1)}{\pi}} \frac{\Gamma(\frac{1}{q-1})}{\Gamma(\frac{1}{q-1} - \frac{1}{2})} \frac{1}{[1+\beta(q-1)x^2]^{1/(q-1)}} \quad (1 < q < 3). \tag{12}$$

($p(x)$ is not normalizable if $q \geq 3$).

If we introduce now (see also Eq. (14) of Ref. [18])

$$q \equiv \frac{3+m}{1+m} \quad (0 < m < \infty) \tag{13}$$

we obtain

$$p(x) = \sqrt{\frac{2\beta}{\pi(1+m)}} \frac{\Gamma(\frac{1+m}{2})}{\Gamma(\frac{m}{2})} \frac{1}{(1+\frac{2\beta}{1+m}x^2)^{\frac{1+m}{2}}}. \tag{14}$$

Introducing now

$$y \equiv \sqrt{\frac{2\beta m}{1+m}}\, x \quad , \tag{15}$$

we obtain

$$\phi_m^{(t)}(y) = \sqrt{\frac{1+m}{2\beta m}}\; p(y\sqrt{\tfrac{1+m}{2\beta m}})$$

$$= \frac{1}{\sqrt{\pi m}} \frac{\Gamma(\frac{1+m}{2})}{\Gamma(\frac{m}{2})} \frac{1}{(1+\frac{y^2}{m})^{\frac{1+m}{2}}}, \tag{16}$$

which precisely is (see [22]) the *Student's t-distribution with m degrees of freedom*! In the limit $q \to 1$ (i.e., $m \to \infty$), we recover the Gaussian distribution given by Eq. (6). In the limit $q \to 3$ (i.e., $m \to +0$), we obtain a completely flat nonnormalizable distribution.

$\underline{q < 1}$:

Eq. (11) becomes

$$p(x) = \frac{[1-\beta(1-q)x^2]^{1/(1-q)}}{\int_{-1/\sqrt{\beta(1-q)}}^{1/\sqrt{\beta(1-q)}} dx [1-\beta(1-q)x^2]^{1/(1-q)}}$$

$$= \sqrt{\frac{\beta(1-q)}{\pi}} \frac{\Gamma(\frac{1}{1-q}+\frac{3}{2})}{\Gamma(\frac{1}{1-q}+1)} [1-\beta(1-q)x^2]^{1/(1-q)} \quad (-\infty < q < 1). \tag{17}$$

This distribution vanishes for $|x| \geq [\beta(1-q)]^{-1/2}$.

If we introduce now

$$q \equiv \frac{n-6}{n-4} \qquad (4 < n < \infty) \tag{18}$$

we obtain

$$p(x) = \sqrt{\frac{2\beta}{\pi(n-4)}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} [1 - \frac{2\beta}{n-4}x^2]^{\frac{n-4}{2}}. \tag{19}$$

Introducing now

$$r \equiv \sqrt{\frac{2\beta}{n-4}}\, x \quad , \tag{20}$$

we obtain

$$\phi_n^{(r)}(r) = \sqrt{\frac{n-4}{2\beta}}\, p(\sqrt{\tfrac{n-4}{2\beta}} r)$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} (1-r^2)^{\frac{n-4}{2}} \qquad (|r| \leq 1), \tag{21}$$

which precisely is (see [22]) the *r-distribution with n-2 degrees of freedom*! In the limit $q \to 1$ (i.e., $n \to \infty$), we recover the Gaussian distribution given by Eq. (6). In the limit $q \to -\infty$ (i.e., $n \to 4$), we obtain a Dirac-delta distribution.

The distributions $p(x)$ are represented in the Figure for typical values of $q \epsilon [-\infty, 3]$. The optimization of S_q is equivalent to the optimization of the generalized likelihood function L_q given by [10,14]

$$L_q[p] = \{1 + (1-q) S_q[p]\}^{\frac{1}{1-q}}. \tag{22}$$

Indeed, L_q is a monotonically increasing function of S_q for *all* values of q. So, for $q > 0$ ($q < 0$), we must maximize (minimize) the entropy S_q, hence we must maximize (minimize) the likelihood function L_q.

Summarizing, we have shown that the standard variational principle applied to the generalized entropy S_q with simple auxiliary constraints, straightforwardly yields the well known Students's t- and r-distributions.

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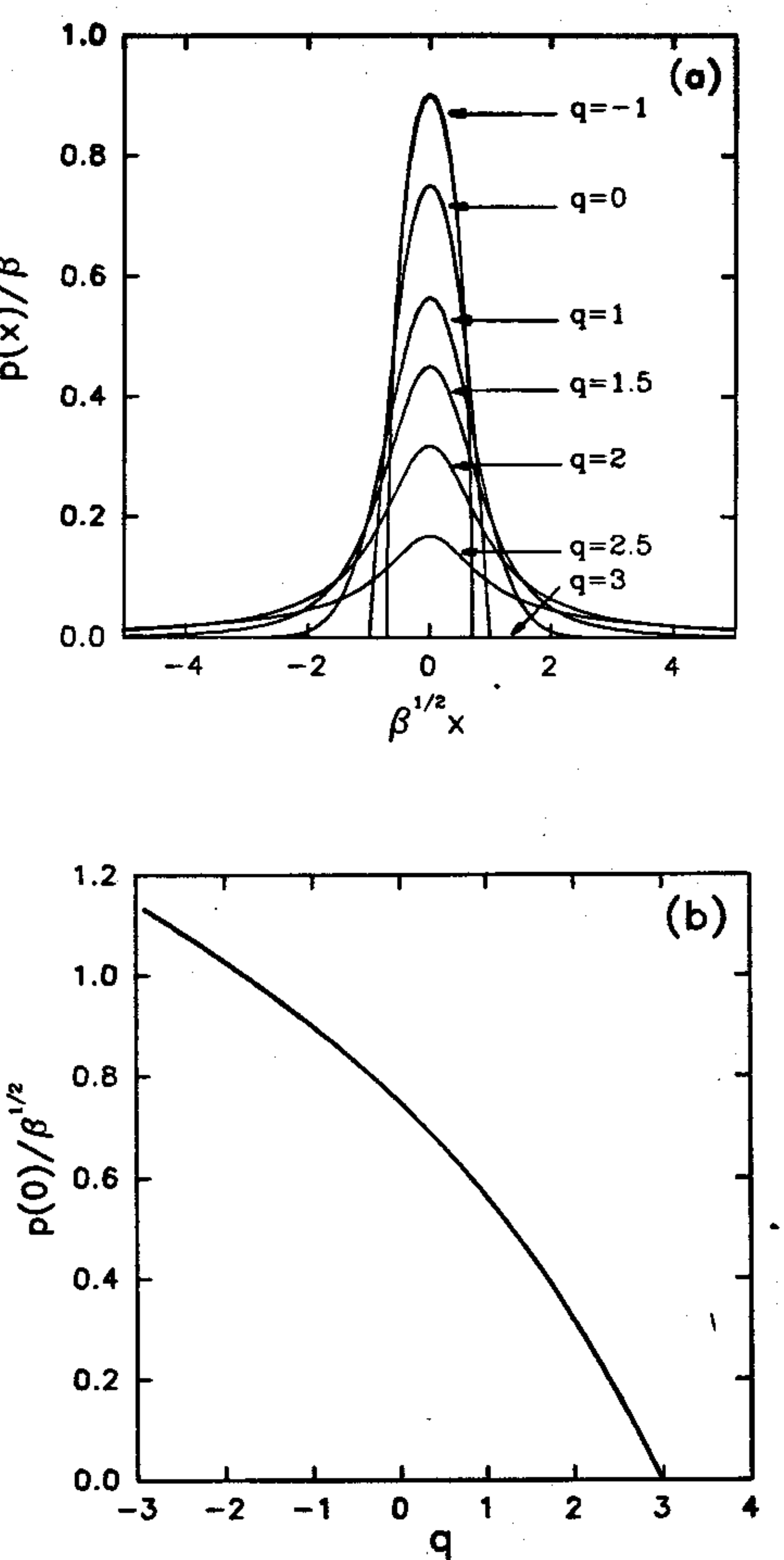
Figure Caption

(a) Distributions $p(x)/\sqrt{\beta}$ vs. $\sqrt{\beta}x$ for typical values of q; (b) Values of $p(0)/\sqrt{\beta}$ as a function of q.


(a)
q=−1
q=0
q=1
q=1.5
q=2
q=2.5
q=3
$p(x)/\beta^{1/2}$
$\beta^{1/2}x$
(b)
$p(0)/\beta^{1/2}$
q

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