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Division Algebras and Extended SuperKdVs*.

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Abstract

The division algebras **R**, **C**, **H**, **O** are used to construct and analyze the $N = 1, 2, 4, 8$ supersymmetric extensions of the KdV hamiltonian equation. In particular a global $N = 8$ super-KdV system is introduced and shown to admit a Poisson bracket structure given by the "Non-Associative $N = 8$ Superconformal Algebra".

Key-words: Division Algebras, Extended Supersymmetries.

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1 Introduction

In the last several years integrable hierarchies of non-linear differential equations in $1+1$ dimensions have been intensely explored, mainly in connection with the discretization of the two-dimensional gravity (see [1]).

Supersymmetric extensions of such equations have also been largely investigated [2]-[7] using a variety of different methods. Unlike the bosonic theory, many questions have not yet been answered in the supersymmetric case. In this talk I report a recent result, obtained in collaboration with H.L. Carrion and M. Rojas [8], concerning the formulation of the N-extended supersymmetric versions of the bosonic integrable equations (for $N = 1, 2, 4, 8$) in terms of the division algebras $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}, \mathbf{O}$ respectively.

To be precise we focused our investigation on the supersymmetric extensions of KdV. At first the standard results of Mathieu [3] concerning $N = 2$ KdV are reviewed in the language of division algebras, while a full analysis of the global $N = 4$ extensions of KdV is performed. The Delduc and Ivanov [9] result is recovered as a special case. Later it is proven that a unique $N = 8$ hamiltonian extension of KdV can be found. It admits as a Poisson brackets structure the so-called "Non-Associative $N = 8$ Superconformal Algebra" introduced for the first time in [10]. It is quite a special superconformal algebra, since it does not satisfy the Jacobi property (this is why it was named "non-associative"). In order to have an $N = 8$ KdV, the "non-associativity" of the underlining superconformal algebra of Poisson brackets is mandatory. This is due to the fact that no central extension, which is responsible for the inhomogeneous character of KdV, is allowed by jacobian superconformal algebras.

It is worth noticing that in the following the problem of the integrability of the hamiltonian super-KdVs system is not addressed, only the issue of their global supersymmetric invariances is of concern here.

2 Division Algebras and Extended Superconformal Algebras

The N-extended superconformal algebras, for $N = 1, 2, 4, 8$, can be recovered from division algebras. Let us here present the largest of such conformal algebras, the "$N = 8$ Non-associative SCA" of reference [10]. The $N = 1, 2, 4$ SCA's are recovered as subalgebras. The "$N = 8$ Non-associative SCA" can be defined via octonionic structure constants. A generic octonion x is expressed as $x = x_a\tau_a$ (throughout the text the convention over repeated indices is understood), where x_a are real numbers while τ_a denote the basic octonions, with $a = 0, 1, 2, ..., 7$.

$\tau_0 \equiv \mathbf{1}$ is the identity, while τ_α, for $\alpha = 1, 2, ..., 7$, denote the imaginary octonions. In the following a Greek index is employed for imaginary octonions, a Latin index for the whole set of octonions (identity included).

The octonionic multiplication can be introduced through

$$\tau_\alpha \cdot \tau_\beta \quad = \quad -\delta_{\alpha\beta}\tau_0 + C_{\alpha\beta\gamma}\tau_\gamma, \tag{1}$$

with $C_{\alpha\beta\gamma}$ a set of totally antisymmetric structure constants which, without loss of generality, can be taken to be

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1. \tag{2}$$

and vanishing otherwise.

When α, β, γ are restricted to, let's say, the values $1, 2, 3$ we recover the quaternionic subalgebra, which is associative. The $N = 8$ extension of the Virasoro algebra is constructed in terms of the above structure constants. Besides the spin-2 Virasoro field, it contains eight fermionic spin-$\frac{3}{2}$ fields Q, Q_α and 7 spin-1 bosonic currents J_α. It is explicitly given by the following Poisson brackets

$$\begin{aligned}
\{T(x), T(y)\} &= -\frac{1}{2}{\partial_y}^3\delta(x-y) + 2T(y)\partial_y\delta(x-y) + T'(y)\delta(x-y),\\
\{T(x), Q(y)\} &= \frac{3}{2}Q(y)\partial_y\delta(x-y) + Q'(y)\delta(x-y) + (X1),\\
\{T(x), Q_\alpha(y)\} &= \frac{3}{2}Q_\alpha(y)\partial_y\delta(x-y) + {Q_\alpha}'(y)\delta(x-y),\\
\{T(x), J_\alpha(y)\} &= J_\alpha(y)\partial_y\delta(x-y) + {J_\alpha}'(y)\delta(x-y),\\
\{Q(x), Q(y)\} &= -\frac{1}{2}{\partial_y}^2\delta(x-y) + +\frac{1}{2}T(y)\delta(x-y),\\
\{Q(x), Q_\alpha(y)\} &= -J_\alpha(y)\partial_y\delta(x-y) - \frac{1}{2}{J_\alpha}'(y)\delta(x-y),\\
\{Q(x), J_\alpha(y)\} &= -\frac{1}{2}Q_\alpha(y)\delta(x-y),\\
\{Q_\alpha(x), Q_\beta(y)\} &= -\frac{1}{2}\delta_{\alpha\beta}{\partial_y}^2\delta(x-y) + C_{\alpha\beta\gamma}J_\gamma(y)\partial_y\delta(x-y) +\\
&\quad +\frac{1}{2}(\delta_{\alpha\beta}T(y) + C_{\alpha\beta\gamma}{J_\gamma}'(y))\delta(x-y),\\
\{Q_\alpha(x), J_\beta(y)\} &= \frac{1}{2}(\delta_{\alpha\beta}Q(y) - C_{\alpha\beta\gamma}Q_\gamma(y))\delta(x-y),\\
\{J_\alpha(x), J_\beta(y)\} &= \frac{1}{2}\delta_{\alpha\beta}\partial_y\delta(x-y) - C_{\alpha\beta\gamma}J_\gamma(y)\delta(x-y).
\end{aligned}\tag{3}$$

This superconformal algebra can be recovered via Sugawara construction of the affine octonionic algebra (see [11] for details). The failure in closing the Jacobi identity is in consequence of the non-associativity of the multiplication between octonions.

3 Extended SuperKdVs

A natural question to be raised is whether the (3) SCA (and its superconformal subalgebras) can be regarded as a Poisson brackets structure for a supersymmetric hamiltonian extension of KdV. This amounts to determine the most general globally supersymmetric hamiltonian of a given dimension 4 (i.e. the second hamiltonian). In the case of $N = 2$, this result is known since the works of Mathieu [3]. In [8] the extension to the $N = 4$ case has been completely worked out. The moduli space of inequivalent $N = 4$ KdV equations has been classified. The special integrable point of Delduc-Ivanov [9] has been recovered in this framework. The complete solution to the $N=8$ case was given as well.

To get these results an extensive use of computer algebra (with Mathematica and the Thielemans' package for classical OPEs computations) was required. The final results are presented here.

The most general $N=4$-supersymmetric hamiltonian for the $N=4$ KdV depends on 5 parameters (plus an overall normalization constant). However, if the hamiltonian is further assumed to be invariant under the involutions of the $N=4$ minimal SCA, three of the parameters have to be set equal to 0. I recall here that the involutions of the $N=4$ minimal SCA are induced by the involutions of the quaternionic algebra. Three such involutions exist (any two of them can be assumed as generators), the α-th one (for $\alpha=1,2,3$) is given by leaving τ_α (together with the identity) invariant and flipping the sign of the two remaining τ's.

What is left is the most general hamiltonian, invariant under $N=4$ and the involutions of the algebra. It is given by

$$\begin{aligned} H_2 &= T^2 + Q'_a Q_a - J''_\alpha J_\alpha + x_\alpha T {J_\alpha}^2 + \\ & \quad 2x_\alpha Q_0 Q_\alpha J_\alpha - C_{\alpha\beta\gamma} x_\alpha J_\alpha Q_\beta Q_\gamma + \\ & \quad 2x_1 J_1 J'_2 J_3 - 2x_2 J'_1 J_2 J_3. \end{aligned} \tag{4}$$

here $a=0,1,2,3$ ($Q_0 \equiv Q$) and $\alpha,\beta,\gamma \equiv 1,2,3$.

The $N=4$ global supersymmetry requires the three parameters x_a to satisfy the condition

$$x_1 + x_2 + x_3 = 0, \tag{5}$$

so that only two of them are independent. Since any two of them, at will, can be plotted in a real $x-y$ plane, it can be proven that the fundamental domain of the moduli space of inequivalent $N=4$ KdV equations can be chosen to be the region of the plane comprised between the real axis $y=0$ and the $y=x$ line (boundaries included). There are five other regions of the plane (all such regions are related by an S_3-group transformation) which could be equally well chosen as fundamental domain.

In the region of our choice the $y=x$ line corresponds to an extra global $U(1)$-invariance, while the origin, for $x_1=x_2=x_3=0$, is the most symmetric point (it corresponds to a global $SU(2)$ invariance associated to the generators $\int dx J_\alpha(x)$).

The involutions associated to each given imaginary quaternion allows to consistently reduce the $N=4$ KdV equation to an $N=2$ KdV, by setting simultaneously equal to 0 all the fields associated with the τ's which flip the sign, e.g. the fields $J_2=J_3=Q_2=Q_3=0$ for the first involution (and similarly for the other couples of values $1,3$ and $1,2$). After such a reduction we recover the $N=2$ KdV equation depending on the free parameter x_1 (or, respectively, x_2 and x_3).

The integrability is known for $N=2$ KdV to be ensured for three specific values $a=-2,1,4$, discovered by Mathieu [3], of the free parameter a. We are therefore in the position to determine for which points of the fundamental domain the $N=4$ KdV is mapped, after any reduction, to one of the three Mathieu's integrable $N=2$ KdVs. It turns out that in the fundamental domain only two such points exist. Both of them lie on the $y = x$ line. One of them produces, after inequivalent $N = 2$ reductions, the $a = -2$ and the $a = 4$ $N = 2$ KdV hierarchies. The second point, which produces the $a = 1$ and the $a = -2$ $N = 2$ KdV equations, however, does not admit at the next order an $N = 4$ hamiltonian which is in involution w.r.t. H_2. This has been explicitly proven in [8]. The treatment of [8] is more complete than the one in [9] since it is based on an exhaustive component-fields analysis, rather than on an extended superfield formalism.

The most general equations of motion of the $N = 4$ KdV are directly obtained from the hamiltonian (4) together with the (3) Poisson brackets.

4 The $N = 8$ SuperKdV

A similar analysis can be extended to the $N = 8$ case based on the full $N = 8$ non-associative SCA. At first the most general hamiltonian with the right dimension has been written down. Later, some constraints on it have been imposed. The first set of constraints requires the invariance under all the 7 involutions of the algebra. In the case of octonions the total number of involutions is 7 (with 3 generators) each one being associated to one of the seven combinations appearing in (2). In the case, e.g., of the 123 combination the corresponding τ_α's are left invariant, while the remaining four τ's, living in the complement, have the sign flipped.

The second set of constraints requires the invariance under the whole set of $N = 8$ global supersymmetries. Under this condition there exists only one hamiltonian, up to the normalization factor, which is $N = 8$ invariant. It does not contain any free parameter and is quadratic in the fields. It is explicitly given by

$$H_2 = T^2 + Q'_a Q_a - J''_\alpha J_\alpha \tag{6}$$

(here $a = 0, 1, ..., 7$ and $\alpha = 1, 2, ..., 7$).

The hamiltonian corresponds to the origin of coordinates (confront the previous case) which is also, just like the $N = 4$ case, the point of maximal symmetry. This means that the hamiltonian is invariant under the whole set of seven global charges $\int dx J_\alpha(x)$, obtained by integrating the currents J_α's.

Despite its apparent simplicity, it gives an $N = 8$ extension of KdV which does not reduce (for any $N = 2$ reduction) to the three Mathieu's values for integrability. Nevertheless it is a highly non-trivial fact that an $N = 8$ extension of the KdV equation indeed exists and that it is unique. Explicitly, the associated equations of motion are given by

$$\begin{aligned}
\dot{T} &= T''' + 12T'T + 6Q''_a Q_a - 4J''_\alpha J_\alpha, \\
\dot{Q} &= Q''' + 6T'Q + 6TQ' + 4Q''_\alpha J_\alpha - 2Q_\alpha J''_\alpha, \\
\dot{Q}_\alpha &= Q_\alpha''' + 2QJ''_\alpha + 6TQ'_\alpha + 6T'Q_\alpha - 2Q'J'_\alpha - 4Q''J_\alpha + \\
&\quad + C_{\alpha\beta\gamma}(Q_\beta J''_\gamma - Q'_\beta J'_\gamma - 2Q''_\beta J_\gamma), \\
\dot{J}_\alpha &= J_\alpha''' + 4T'J_\alpha + 4TJ'_\alpha + 2QQ'_\alpha + C_{\alpha\beta\gamma}(2J_\alpha J_\beta J''_\gamma + Q_\beta Q'_\gamma).
\end{aligned} \tag{7}$$

Besides the $N = 8$ KdV, the fields entering (3) admit globally $N = 4$ supersymmetric invariant hamiltonians, which depend on free parameters. Two such classes of hamiltonians are individuated. The first class consists of the hamiltonians invariant under supersymmetries related with the quaternionic subalgebra which, without loss of generality, can be assumed to be given by $\int dx Q_i(x)$, for $i = 0, 1, 2, 3$. The second class of invariances is associated to the $N = 4$ supersymmetries associated to the remaining generators, i.e. those living in the complement (in the following, without loss of generality, these supersymmetries are labeled by $i = 1, 2, 4, 5$). For completeness we report here the results. The first class of $N = 4$-invariant hamiltonians is given by

$$\begin{aligned} H_2 &= T^2 + Q'Q + Q'_pQ_p + xQ'_rQ_r - J''_pJ_p - xJ''_rJ_r + 2x_pQQ_pJ_p + x_pTJ_pJ_p - \\ &\quad -2x_3Q_1Q_2J_3 + 2x_2Q_1Q_3J_2 - 2x_1Q_2Q_3J_1 + 2x_1J_1J'_2J_3 - 2x_2J'_1J_2J_3. \end{aligned} \tag{8}$$

with x_1, x_2, x free parameters, while $x_3 = -x_1 - x_2$.

In the above expression $p = 1, 2, 3$ and $r = 4, 5, 6, 7$.

The second class of $N = 4$ invariant hamiltonians is explicitly given by

$$\begin{aligned} H_2 &= T^2 + x(Q'Q + Q'_3Q_3 + Q'_6Q_6 + Q'_7Q_7) + Q'_1Q_1 + Q'_2Q_2 + Q'_4Q_4 + Q'_5Q_5 - \\ &\quad -x(J''_1J_1 + J''_2J_2 + J''_4J_4 + J''_5J_5) - J''_3J_3 - J''_6J_6 - J''_7J_7 + \\ &\quad +y(TJ_7J_7 - TJ_3J_3) + 2yJ_3J'_6J_7 + 2y(Q_1Q_2J_3 - Q_1Q_4J_7 - Q_2Q_5J_7 - Q_4Q_5J_3) \end{aligned} \tag{9}$$

where in this case two free parameters, x and y, appear.

5 CONCLUSIONS

In this paper I have presented some new results concerning an explicit connection of division algebras and the extended supersymmetrizations of the KdV equation.

In particular it has been proven, following [8], the existence of a unique $N = 8$ KdV equation of hamiltonian type based on the $N = 8$ non-associative SCA as a generalized classical Poisson brackets structure.

Division algebras are a natural ingredient when dealing with extended supersymmetries. It is therefore likely that supersymmetric extensions of other classes of bosonic integrable equations could be studied with the tools furnished by division algebras. Besides KdV, the next simplest equations to be investigated are the mKdV and the NLS. In view of the results of [11], where the $N = 8$ Non-associative SCA is recovered from a singular limit of a Sugawara construction based on the superaffine octonionic algebra of superMalcev type, it is almost for granted that such extensions indeed exist.

Whether such constructions could be applied to other classes of integrable equations is still an open problem. In any case division algebras look as a promising and elegant tool to unveil some of the mysterious features still surrounding the supersymmetrization of the bosonic hierarchies.

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