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ν-Dimensional Ideal Quantum q-Gas: Bose-Einstein Condensation and λ-Point Transition

M.R-Monteiro, Itzhak Roditi and *Ligia M.C.S. Rodrigues*

Centro Brasileiro de Pesquisas Físicas – CPBF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro, RJ – Brazil

Abstract

We consider an ideal quantum q-gas in ν spatial dimensions and energy spectrum $\omega_i \ \alpha \ p^\alpha$. Departing from the Hamiltonian $H = \omega[N]$, we study the effect of the deformation on thermodynamic functions and equation of state of that system. The virial expansion is obtained for the high temperature (or low density) regime. The critical temperature is higher than in non-deformed ideal gases. We show that Bose-Einstein condensation always exists (unless when $\nu/\alpha = 1$) for finite q but not for $q = \infty$. Employing numerical calculations and selecting for ν/α the values $3/2$, 2 and 3, we show the critical temperature as a function of q, the specific heat C_V and the chemical potential μ as functions of T/T_c^q for $q = 1.05$ and $q = 4.5$. C_V exhibits a λ-point discontinuity in all cases, instead of the cusp singularity found in the usual ideal gas. Our results indicate that physical systems which have quantum symmetries can exhibit Bose-Einstein condensation phenomenon, the critical temperature being favoured by the deformation parameter.

Key-words: Quantum groups; q-Oscillators; Bose-Einstein condensation; Critical phenomena.

1 Introduction

The purpose of this paper is to analyse some thermodynamic properties of a quantum ideal q-gas [1–5]. This system generalizes the ideal boson gas using a set of independent q-oscillators (objects that satisfy deformed Heisenberg algebra) [6, 7], instead of the standard bosonic oscillator. Our main motivation to discuss deformed ideal quantum gases comes from the role played by the theory of ideal gases in many different physical phenomena as superfluidity, superconductivity, blackbody radiation, phonons in a cristal lattice, etc. [8].

Because of the relevance of the Heisenberg algebra in physics, its deformed versions [6, 7] have attracted attention in the last few years; they are non-trivial generalizations of the Heisenberg algebra through the introduction of one (or more [9]) parameter(s) such that the non-deformed case is recovered for a special limit of the parameter(s). Only recently has their interesting connection with quantum algebras and superalgebras been established [6, 7], as well as their derivation from the contraction of quantum algebras [7].

On the other hand there has been a great interest in quantum algebras and quantum groups [10–13] in the last years. This mathematical structure, also called Quasitriangular Hopf algebras, has emerged as an appealing non-trivial generalization of Lie algebras and groups which are recovered when the deformation parameter (or a set of parameters) goes to one. Quantum groups have left their trace in several areas of physics [14] and, interesting simple physical systems, those made with physical particles, have quantum group symmetry [15, 16].

In section 2 we summarize the basic formalism of q-oscillators and discuss some of their properties in the large q limit. Section 3 is devoted to the ν-dimensional ideal quantum q-gas: we find the virial expansion for its equation of state for large q and show the existence of Bose-Einstein condensation. In section 4 we show that the specific heat exhibits a λ-point transition, analyse our results and present our conclusions.

2 q-Oscillators

One calls bosonic q-oscillators (or deformed Heisenberg algebra) [6, 17] the associative algebra generated by the elements α, α^+ and N satisfying the relations

$$\begin{aligned} [N, \alpha^+] &= \alpha^+ \quad , \quad [N, \alpha] = -\alpha \\ [\alpha, \alpha^+]_\alpha &= f_\alpha(N) \ . \end{aligned} \tag{2.1}$$

We are going to consider here the following forms of the above algebra (2.1):

$$[a, a^+]_a \equiv aa^+ - qa^+a = q^{-N} \tag{2.2a}$$

$$[A, A^+]_A \equiv AA^+ - q^2A^+A = 1 \ , \tag{2.2b}$$

which are related to each other via

$$A = q^{N/2}a \quad , \quad A^+ = a^+q^{N/2} \ , \tag{2.3}$$

in the case of real q.

It is possible to construct the representation of relation (2.2) in the Fock space $\mathcal{F}$ generated by the normalized eigenstates $|n\rangle$ of the number operator N as

$$\begin{aligned} \alpha|0\rangle &= 0, \quad , \quad N|n\rangle = n|n\rangle \; ; \quad , \quad n = 0, 1, 2 \cdots \\ |n\rangle &= \frac{1}{\sqrt{[n]_\alpha!}} \, (\alpha^+)^n |0\rangle \; ; \end{aligned} \tag{2.4}$$

where $[n]_\alpha! \equiv [n]_\alpha \cdots [1]_\alpha$, $[n]_a = (q^n - q^{-n})/(q - q^{-1})$, $[n]_A = (q^{2n} - 1)/(q^2 - 1)$ and $[n]_\alpha \to n$ as $q \to 1$.

In $\mathcal{F}$ it is possible to express the deformed oscillators in terms of the standard bosonic ones b, b^+, which obey the usual Heisenberg algebra, as [17]

$$\alpha = \left(\frac{[N+1]_\alpha}{N+1} \right)^{1/2} b \quad , \quad \alpha^+ = b^+ \left(\frac{[N+1]_\alpha}{N+1} \right)^{1/2} \; ; \tag{2.5}$$

and it can easily be shown in $\mathcal{F}$ that

$$\alpha\alpha^+ = [N+1]_\alpha \quad , \quad \alpha^+\alpha = [N]_\alpha \; . \tag{2.6}$$

If we now consider two independent q-oscillators, for instance a_1, a_2, one can realize the $SU_q(2)$ algebra [10–12]

$$[J_0, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = [2J_0]_a \; , \tag{2.7}$$

à la Schwinger as [6]

$$\begin{aligned} J_+ &= a_1^+ a_2 \quad , \quad J_- = a_2^+ a_1 \\ J_0 &= \frac{1}{2}(N_1 - N_2) \neq \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2) \; . \end{aligned} \tag{2.8}$$

Further with

$$n_1 = j + m \quad , \quad n_2 = j - m \; , \tag{2.9}$$

one can realize the $|j, m\rangle$ basis of $SU_q(2)$ by means of the above q-oscillators as:

$$|j, m\rangle = |n_1\rangle|n_2\rangle = \frac{(a_1^+)^{j+m}}{\sqrt{[j+m]_a!}} \, \frac{(a_2^+)^{j-m}}{\sqrt{[j-m]_a!}} \, |0\rangle \; . \tag{2.10}$$

As expected, in the $q \to 1$ limit the non-deformed algebras are recovered.

Analogously to the above construction for $SU_q(2)$, all the deformed algebras of type A, B, C and D [18], the quantum superalgebras [7] and the deformed exceptional algebras [19] can be realized à la Schwinger. q-Oscillators can be obtained from the deformed algebras [7], similarly to what happens in the non-deformed case, by the Schwinger contraction method. Due to these procedures, it is expected that q-oscillators have the same status with respect to quantum algebras as classical oscillators with respect to classical Lie algebras.

We are now going to discuss real $q > 1$ deformed q-oscillators. In the $q \to \infty$ limit for $n \geq 2$, $[n]_\alpha \to \infty$ and as a result when $q = \infty$ Fock space

$$|n\rangle = \frac{1}{\sqrt{[n]_\alpha !}} (\alpha^+)^n |0\rangle \ ,$$

is reduced to a fermionic one since $|n\rangle$ vanishes for $n \geq 2$. Consequently, the statistical properties of q-oscillators (2.2) become those of fermions. This can be clearly seen by writing the canonical partition function

$$Z = \sum_{n=0}^{\infty} e^{-\beta\omega[n]_A} = 1 + e^{-\beta\omega} + e^{-\beta\omega(1+q^2)} + \cdots + e^{-\beta\omega[1+q^2+\cdots+q^{2(n-1)}]} + \cdots \ , \qquad (2.11)$$

for the Hamiltonian

$$H = \omega A^+ A = \omega [N]_A \ , \qquad (2.12)$$

with $\beta = (kT)^{-1}$, where k is the Boltzmann constant. Indeed, inspection of (2.11) reveals that as q grows the higher states are drastically less occupated. In the limit $q = \infty$

$$Z = 1 + e^{-\beta\omega} \ , \qquad (2.13)$$

which is the fermionic canonical partition function. Notice that a similar process but in the opposite sense happens in the case of parastatistics [20, 21]: as the maximal number of particle per state, p, increases, higher energy states are occupied and in the limit $p = \infty$ the bosonic regime is reproduced. Thus for q-oscillators we have a bosonic regime for all $q \geq 1$ unless when $q = \infty$ where the regime is fermionic whereas for parastatistics we have a fermionic-like regime for all p, unless when $p = \infty$ where the system is bosonic.

The Hamiltonian (2.12) is the simplest non-trivial deformation of the harmonic oscillator leading to thermodynamic functions that depend on the deformation [2, 3, 22] and, it recovers the usual bosonic behaviour for $q = 1$. In this sense, it is not a unique choice. In particular, for

$$H = \omega N \ , \qquad (2.14)$$

the energy spectrum is not changed - the partition function being the bosonic one - and the effect of the deformation only appears in correlation functions.

3 ν-Dimensional Ideal Quantum q-Gas

In this section we are going to analyse ideal quantum q-gas in ν spatial dimensions. We find the virial expansion for its equation of state for large q and investigate the Bose-Einstein condensation phenomenon [23, 5].

Let us consider an ideal deformed system described by the Hamiltonian:

$$H = \sum_i \omega_i A_i^+ A_i = \sum_i \omega_i [N_i]_A \ , \qquad (3.1)$$

where A_i, A_i^+ and N_i are interpreted respectively as annihilation, creation and occupation number operators of particles in level i, with energy ω_i, which satisfy the algebra (2.2b) and commute for different levels.

The grand canonical partition function is:

$$Z = Tr \exp[-\beta(H - \mu N)] = \exp(-\beta\Omega) \, , \tag{3.2}$$

where $N, N = \sum_i N_i$, is the total number operator; μ is the chemical potential and Ω is the grand canonical potential. For the above system Z factorizes and the grand canonical potential is given by a sum over single level partition functions [2]

$$\Omega = -\frac{1}{\beta} \sum_i \ln Z_i^0(\omega_i, \beta, \mu) \, , \tag{3.3}$$

where

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i [n]_A - \mu n)} \, . \tag{3.4}$$

We enclose the system in a large ν-dimensional volume V and replace, as in the usual procedure, the sum over levels by an integral over a $\vec{p}$-space:

$$\sum_i \to \frac{V}{h^\nu} \int d^\nu p \, . \tag{3.5}$$

The energy spectrum of each q-particle obeys the dispersion law

$$\omega_i \to \gamma p^\alpha \; ; \tag{3.6}$$

for $\alpha = 1(2)$ one recovers the ultrarelativistic (non-relativistic) case, with $\gamma = 1(1/2m)$.

The general expansion for the grand canonical potential (3.3) is then

$$\Omega = \frac{-V}{h^\nu \beta} \int d^\nu p \ln \sum_{n=0}^{\infty} e^{-\beta(\gamma p^\alpha [n]_A - \mu n)} \, . \tag{3.7}$$

Integrating over the angular variables and defining the new variable $\eta = \beta\gamma p^\alpha$, Ω can be rewritten as

$$\Omega = -\frac{-2\pi^{\nu/2} V}{\Gamma(\frac{\nu}{2}) h^\nu \beta \alpha (\beta\gamma)^{\nu/\alpha}} \int_0^\infty d\eta \; \eta^{\frac{\nu}{\alpha}-1} \ln \left\{ \sum_{n=0}^{\infty} z^n e^{-[n]_A \eta} \right\} \, , \tag{3.8}$$

where $z = e^{\beta\mu}$ is the fugacity. Integrating (3.8) by parts, we finally have

$$\Omega = -\frac{-\Lambda^{-\nu}}{\Gamma(\frac{\nu}{\alpha}+1)\beta} \int_0^\infty d\eta \; \eta^{\nu/\alpha} \; \frac{\sum_{n=0}^{\infty} [n]_A z^n e^{-[n]_A \eta}}{\sum_{n=0}^{\infty} z^n e^{-[n]_A \eta}} \, , \tag{3.9}$$

where the thermal wavelength Λ, $\Lambda^{-\nu} = \frac{\pi^{\nu/2}\Gamma(\frac{\nu}{\alpha}+1)}{\Gamma(\frac{\nu}{2}+1) h^\nu (\beta\gamma)^{\nu/\alpha}}$, is the relevant expansion parameter in the thermodynamic functions.

The pressure $P = -\frac{\Omega}{V}$ and the density $n = \frac{\partial P}{\partial \mu}|_{T,V}$ for the q-oscillator in ν-spatial dimensions and energy spectrum given by (3.6) are then:

$$P(T,z) = kT\Lambda^{-\nu} Y_q(z) \tag{3.10a}$$

$$n(T,z) = \Lambda^{-\nu} y_q(z)\ . \tag{3.10b}$$

where

$$Y_q(z) = \frac{1}{\Gamma(\frac{\nu}{\alpha}+1)} \int_0^\infty d\eta\ \eta^{\nu/\alpha}\ \frac{\sum_{n=0}^{\infty} [n]_A z^n e^{-[n]_A \eta}}{\sum_{n=0}^{\infty} z^n e^{-[n]_A \eta}} \tag{3.11a}$$

$$\begin{aligned} y_q(z) &= z\partial_z Y_q(z) \\ &= \frac{1}{\Gamma(\frac{\nu}{\alpha}+1)} \int_0^\infty d\eta\ \eta^{\nu/\alpha} \left[\frac{\sum_{n=0}^{\infty} [n]_A n z^n e^{-[n]_A \eta}}{\sum_{n=0}^{\infty} z^n e^{-[n]_A \eta}} \right. \\ &\quad \left. - \frac{\left(\sum_{n=0}^{\infty} [n]_A z^n e^{-[n]_A \eta}\right)\left(\sum_{n=0}^{\infty} n z^n e^{-[n]_A \eta}\right)}{\left(\sum_{n=0}^{\infty} z^n e^{-[n]_A \eta}\right)^2} \right]\ . \end{aligned} \tag{3.11b}$$

3.1 High temperature (low density) approximation

In this subsection, we investigate the system in the high-temperature limit for high q, which, as we shall show in the next section, means $q > 3$. In that limit, the series in functions $Y_q(z)$ and $y_q(z)$ can be approximated by their first three terms ($n = 0, 1, 2$):

$$Y_q(z) \cong \frac{1}{\Gamma(\frac{\nu}{\alpha}+1)} \int_0^\infty d\eta\ \eta^{\nu/\alpha} \left[\frac{z e^{-\eta} + (1+q^2) z^2 e^{-(1+q^2)\eta} + \cdots}{1 + z e^{-\eta} + z^2 e^{-(1+q^2)\eta} + \cdots} \right]\ , \tag{3.12a}$$

$$\begin{aligned} y_q(z) &\cong \frac{1}{\pi(\frac{\nu}{\alpha}+1)} \int_0^\infty d\eta\ \eta^{\nu/\alpha} \left[\frac{z e^{-\eta} + 2(1+q^2) e^{-(1+q^2)\eta} + \cdots}{1 + z e^{-\eta} + z^2 e^{-(1+q^2)\eta} + \cdots} \right. \\ &\quad \left. - \frac{(z e^{-\eta} + (1+q^2) z^2 e^{-(1+q^2)\eta} + \cdots)(z e^{-\eta} + 2 z^2 e^{-(1+q^2)\eta} + \cdots)}{(1 + z e^{-\eta} + z^2 e^{-(1+q^2)\eta} + \cdots)^2} \right]\ . \end{aligned} \tag{3.12b}$$

Analogously to the case of one degree of freedom analysed in the previous section, when $q = \infty$ only the terms $n = 0, 1$ in the series survive and we have a fermionic ideal quantum gas.

If the fugacity z is assumed small compared to one, the denominators in $Y_q(z)$ and $y_q(z)$ above can be approximated according to:

$$\left(1 + z e^{-\eta} + z^2 e^{-(1+q^2\eta)}\right)^{-m} \cong (1 + z e^{-\eta})^{-m} - m e^{-(1+q^2)\eta} (1 + z e^{-\eta})^{-m-1} + \cdots\ , \tag{3.13}$$

where $m = 1 (m = 2)$ in expression (3.12.a) ((3.12.b)). Performing the integrations and keeping terms up to the third order in z, we obtain

$$P = \frac{\Lambda^{-\nu}}{\beta} z[1 + F_1 z + F_2 z^2 + O(z^3)] \, , \tag{3.14}$$

and

$$n = \Lambda^{-\nu} z[1 + 2zF_1 + 3z^2 F_2 + O(z^3)] \, , \tag{3.15}$$

where the coefficients F are given by:

$$\begin{aligned} F_1 &= \frac{-1}{2^{\frac{\nu}{\alpha}+1}} + \frac{1}{(1+q^2)^{\nu/\alpha}}, \\ F_2 &= \frac{1}{3^{\frac{\nu}{\alpha}+1}} - \frac{1}{(2+q^2)^{\nu/\alpha}}, \end{aligned}$$

Inverting (3.15) we can write z as a power series in $(n\Lambda^\nu)$:

$$z = n\Lambda^\nu - 2F_1 (n\Lambda^\nu)^2 + (8F_1^2 - 3F_2)(n\Lambda^\nu)^3 + \cdots \, . \tag{3.16}$$

Substituting (3.16) in expression (3.14) for the pressure, we finally obtain the virial expansion for the equation of state of our quantum q-gas in the large q and small fugacity limits:

$$\begin{aligned} P &= \frac{n}{\beta} \left[1 - \left(\frac{-1}{2^{\frac{\nu}{\alpha}+1}} + \frac{1}{(1+q^2)^{\nu/\alpha}} \right) (n\Lambda^\nu) + 2 \left(\frac{1}{2^{\frac{2\nu}{\alpha}+1}} + \right. \right. \\ &- \left. \left. \frac{1}{3^{\frac{\nu}{\alpha}+1}} - \frac{2}{(2+2q^2)^{\nu/\alpha}} + \frac{1}{(2+q^2)^{\nu/\alpha}} \right) (n\Lambda^\nu)^2 + \cdots \right] \, . \end{aligned} \tag{3.17}$$

The virial expansion (3.17) deserves some comments. It clearly shows that for infinite deformation our q-gas behaves, as expected, exactly like an ideal Fermi-gas. Also, it can be seen that for finite values of q, the pressure is reduced with respect to the ideal Fermi-gas and that this difference is inversely proportional to the deformation. Finally, as is corroborated by (3.16), these results are valid for $n\Lambda^\nu << 1$, which means high-temperature (or low density) approximation.

3.2 Bose-Einstein condensation

The study of Bose-Einstein condensation phenomenon requires some precaution. As usual, when $z \to 1$ we have to take into account the zero-point energy and single out its contribution in expressions (3.11). In addition, the partition function (3.4) clearly shows that when $\omega_i = 0$ the effect of the deformation is cancelled. Therefore, the series in (3.11) cannot be approximated by a polynominal for the zero energy level and the series will only converge if $z < 1$. Thus, the chemical potential has to be non-negative, as it is the case for the usual non-deformed ideal gas.

Let us now consider lower temperatures. Keeping n constant, $n\Lambda^\nu$ increases and from (3.16) so does z. When z reaches 1, the temperature T attains its critical value T_c^q, defined by

$$n\Lambda_c^\nu = y_q(1) \, , \tag{3.18}$$

which can be expressed as:

$$T_c^q = \frac{\gamma \Gamma^{\alpha/\nu}(\frac{\nu}{2}+1) h^\alpha n^{\alpha/\nu}}{k\pi^{\alpha/2}\Gamma^{\alpha/\nu}(\frac{\nu}{\alpha}+1) y_q^{\alpha/\nu}(1)} \; . \tag{3.19}$$

When $T \to 0$, as the ground state is not affected by deformation, there will be an accumulation of particles in this state: Bose-Einstein condensation is present in our deformed system. Comparing T_c to the critical temperature for non-deformed gases of the same density n, we find

$$\frac{T_c^q}{T_c} = \left(\frac{2.61}{y_q(1)} \right)^{\alpha/\nu} \; . \tag{3.20}$$

Figure 1 shows T_c^q/T_c as a function of the deformation parameter q for $\nu/\alpha = 3/2,\ 2$ and 3, and we can see that as ν/α decreases the deformed critical temperature T_c^q increases with respect to the non-deformed case.

As we have already mentioned, in the vicinity of T_c^q we have to take into account the zero-point energy and single out its contribution in (3.11). Therefore the expressions for P and n become:

$$P(T,z) = \beta^{-1}[-V^{-1}\ln(1-z) + \Lambda^{-\nu} Y_q(z)] \; , \tag{3.21a}$$

$$n(T,z) = \frac{1}{V}\,\frac{z}{1-z} + \Lambda^{-\nu} y_q(z) \; , \tag{3.21b}$$

where as usual the extra terms only contribute when $z \simeq 1$. Similarly to the non-deformed case the first term in the right-hand side of (3.21.a) can always be neglected and the basic equations are:

$$P(T,z) = \beta^{-1}\Lambda^{-\nu} Y_q(z) \; , \tag{3.22a}$$

$$n(T,z) = \frac{1}{V}\,\frac{z}{1-z} + \Lambda^{-\nu} y_q(z), \tag{3.22b}$$

where the first term on the right-hand side of (3.22.b), which is due to the contribution of the zero energy, is relevant only for $T \le T_c^q$. In this region z remains equal to one, as in the usual case.

The specific heat per particle C_V is defined as

$$\frac{C_V}{k} = \frac{1}{kn}\left.\frac{\partial \tilde{\epsilon}}{\partial T}\right|_n \; , \tag{3.23}$$

where $\tilde{\epsilon}$ is the energy density (internal energy per volume). From the thermodynamic definitions of $\tilde{\epsilon}$ and of the entropy density one has

$$\tilde{\epsilon} = n\mu - \beta \left.\frac{\partial P}{\partial \beta}\right|_\mu - P \; , \tag{3.24}$$

with P given by (3.22.a). We can easily find that

$$\tilde{\epsilon} = \frac{\nu}{\alpha} P \; , \tag{3.25}$$

thus generalizing the standard result for $q = 1$.

To obtain C_V we have to compute $\partial_T \tilde{e}|_n \equiv (\partial\tilde{e}/\partial T)|_n$ which from (3.25) can be obtained in the two regimes from

$$\begin{aligned} P(T) &= \beta^{-1}\Lambda^{-\nu} Y_q[z(T)] \qquad T > T_c \\ P(T) &= \beta^{-1}\Lambda^{-\nu} Y_q(1) \qquad\quad T \leq T_c \ . \end{aligned} \tag{3.26}$$

Now, for $T > T_c^q$

$$\partial_T \tilde{e}|_n = \frac{\nu}{\alpha}\left(\frac{\nu}{\alpha}+1\right) k\Lambda^{-\nu} Y_q(z) + \frac{\nu}{\alpha} kT\Lambda^{\nu} Y_q'(z)\partial_T z|_n \ , \tag{3.27}$$

where $Y_q'(z) = \partial_z Y_q(z)$. From $y_q(z) = \Lambda^\nu n$, we have:

$$\partial_T z|_n = -\frac{\nu}{\alpha}\ \frac{\Lambda^\nu T^{-1} n}{y_q'(z)} \ . \tag{3.28}$$

Finally substituting (3.28) in (3.27) we get

$$\frac{C_V}{k} = \frac{\nu}{\alpha}\left(\frac{\nu}{\alpha}+1\right)(\Lambda^\nu n)^{-1} Y_q(z) - \left(\frac{\nu}{\alpha}\right)^2 \frac{y_q(z)}{z y_q'(z)} \ . \tag{3.29}$$

For $T < T_c^q$ it is trivial to find

$$\frac{C_V}{k} = \frac{\nu}{\alpha}\left(\frac{\nu}{\alpha}+1\right)(\Lambda^\nu n)^{-1} Y_q(1) \ . \tag{3.30}$$

Clearly equations (3.29) and (3.30) also generalize the usual $q = 1$ ideal bosonic gas specific heat.

4 Analysis of the Results and Concluding Remarks

We are now going to analyse our numerical results. Tables I to VI show values of $Y_q(1)$, $y_q(1)$ and $y_q'(1)$ for $q = 1.5,\ 4.5$, when $\nu/\alpha = 3/2,\ 2,\ 3$ considering distinct values of the upper limit of the sums in (3.11). The integrals always converge in the cases we have studied and as q grows, less terms are necessary in the sums in order to reach a given accuracy.

The integral $y_q(z)$ can be rewritten as

$$y_q(z) = \frac{\nu}{\alpha}\ \frac{1}{\Gamma(\frac{\nu}{\alpha}+1)}\int_0^\infty dx\ x^{\frac{\nu}{\alpha}-1}\ \frac{\sum_{n=0}^{\infty} n z^n e^{-[n]_A x}}{\sum_{n-0} z^n e^{-[n]_A x}} \ , \tag{4.1}$$

and for $\nu/\alpha = 1$ the integral diverges for $z = 1$. Thus, similarly to the non-deformed case we do not have Bose-Einstein condensation for $\nu/\alpha = 1$.

In Figures 2 and 3 we present the chemical potential and the specific heat as functions of the temperature for $\frac{\nu}{\alpha} = \frac{3}{2}, 2, 3$ and for two values of q, $q = 1.05$ and 4.5. Figures 2

show that as ν/α increases, the effect of the deformation is less relevant. The specific heat for our deformed q-gas has a very interesting feature, exhibiting a discontinuity known as λ-point transition in all cases. This behaviour is different from the usual bosonic ideal gas which has a cusp singularity at $T = T_c$. The comparison of curves for a very small ($q = 1.05$) and a very large ($q = 4.5$) values of q shows that as q decreases the discontinuity diminishes and disappears at $q = 1$, becoming then a cusp singularity.

We have shown that for $\frac{\nu}{\alpha} > 1$ a deformed q-gas has the Bose-Einstein condensation phenomenon with the specific heat exhibiting a λ-point discontinuity. The critical temperature being higher than for the bosonic ideal gas, we see that the presence of deformation favours the Bose-Einstein condensation phenomenon.

Based on the relation between q-oscillators and quantum algebras (see section 2), we expect that physical systems with quantum group symmetry can exhibit the Bose-Einstein condensation phenomenon and also that, as in the case of the ideal quantum q-gas, the deformation parameter favours and controls this phenomenon. As the Bose-Einstein condensation phenomenon is relevant in superconductivity and quantum symmetries increase the critical temperature through the deformation parameter, we further expect that quantum symmetries can be important in understanding high-T_c superconductivity.

We also believe that the role played by the deformation parameter in favouring the Bose-Einstein condensation phenomenon is a general feature of deformed systems, i.e. quantum symmetries favour criticality.

It is also interesting to point out the similarity between the Generalized Statistical Mechanics [24] and the results we have found concerning the critical behaviour of the system [25]. In particular in ref. [25], T_c increases as q_s (the statistical parameter introduced to generalize the statistical mechanics) increases.

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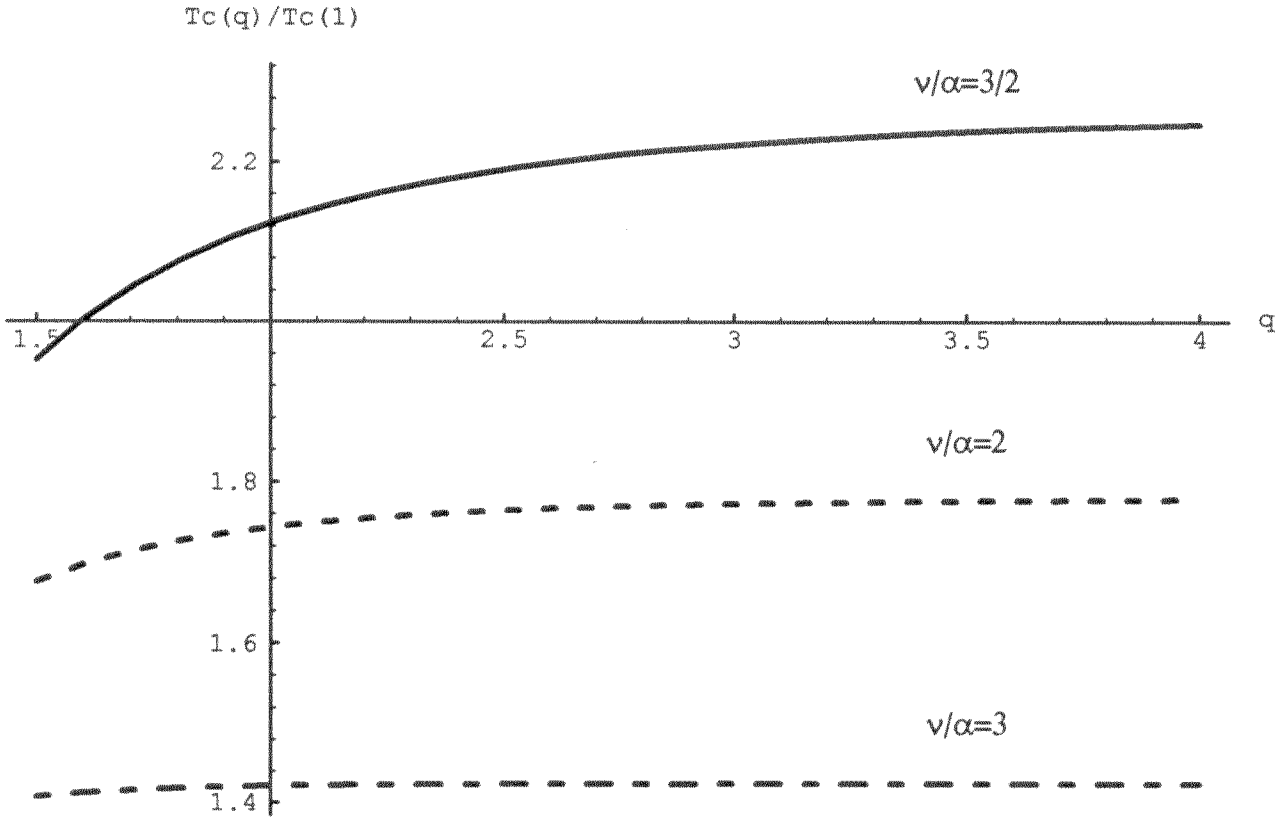

FIGURE 1

Figure 1 T_c^q/T_c behaviour as a function of the deformation parameter q for $\frac{\nu}{\alpha} = \frac{3}{2}, 2, 3$, where T_c is the critical temperature for the non-deformed case, ν is the spatial dimension and α is a parameter relative to the dispersion law (3.6).

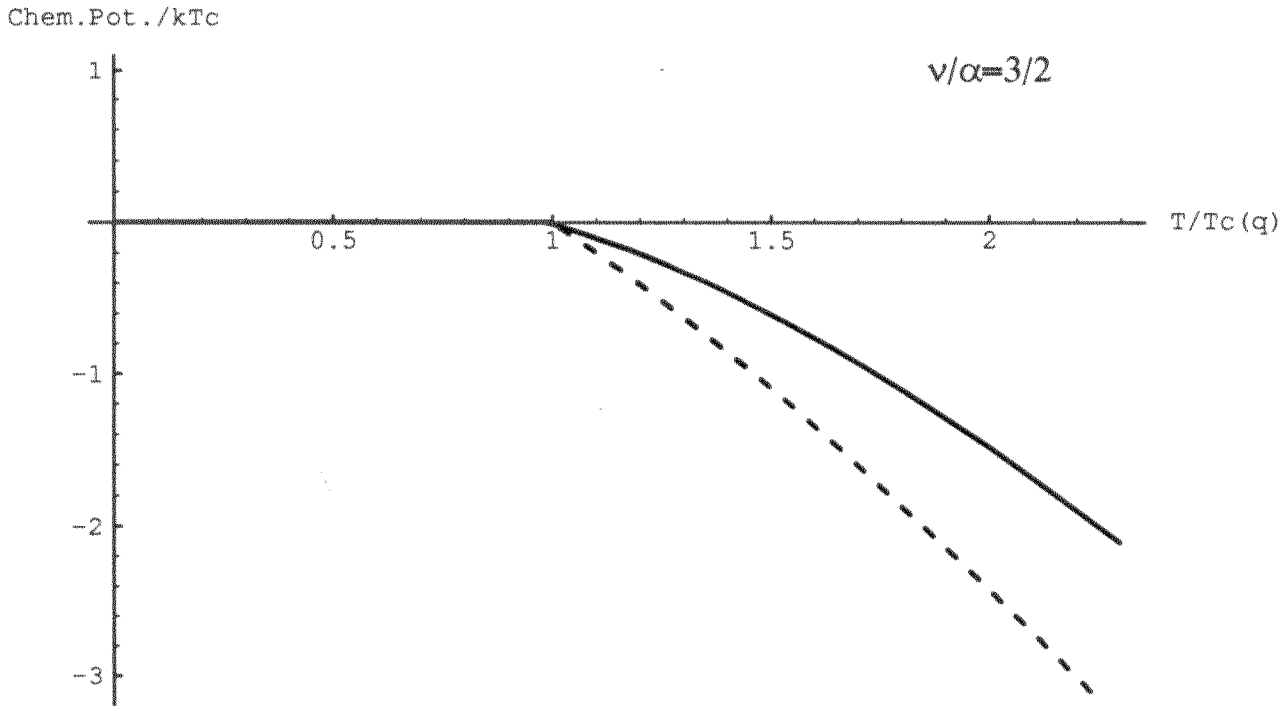

FIGURE 2-a

Figures 2 μ/kT_c behaviour as a function of the T/T_c^q for three values of ν/α where ν is the spatial dimension and α is a parameter relative to the dispersion law (3.6): (a) $\frac{\nu}{\alpha} = \frac{3}{2}$; (b) $\frac{\nu}{\alpha} = 2$; (c) $\frac{\nu}{\alpha} = 3$. The dashed line corresponds to $q = 4.5$ and the continuous line to $q = 1.05$.

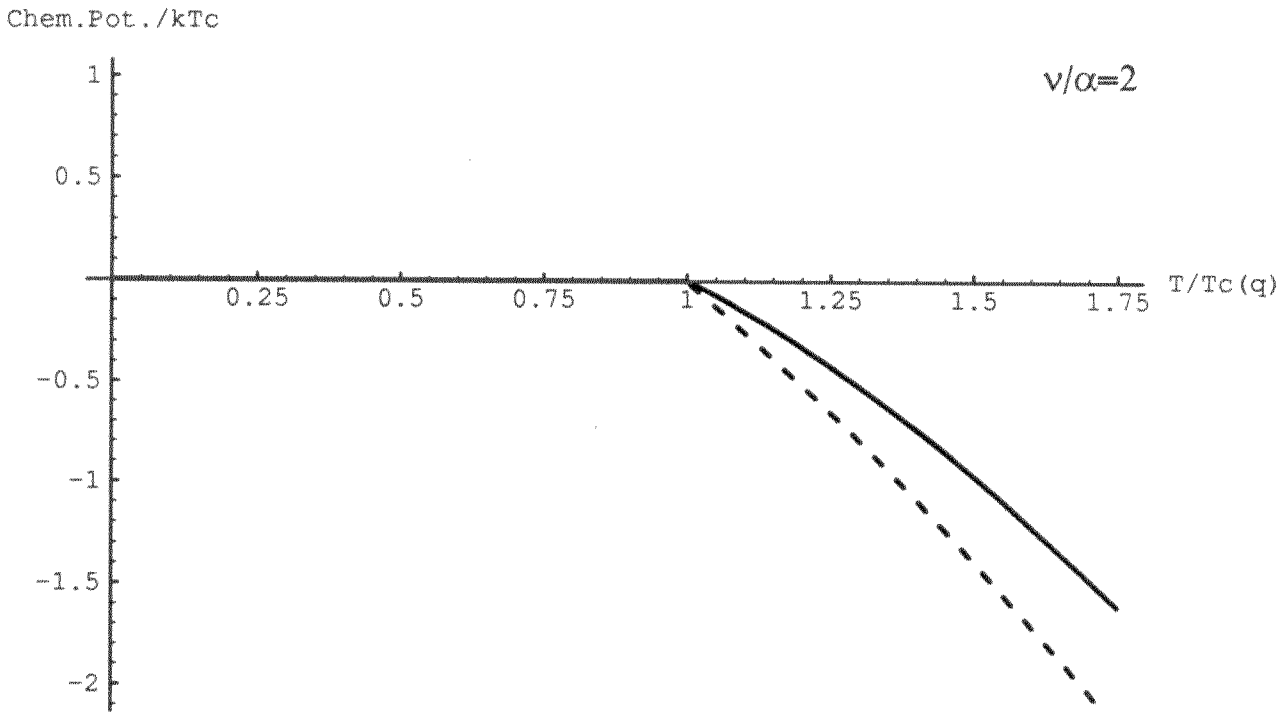

FIGURE 2-b

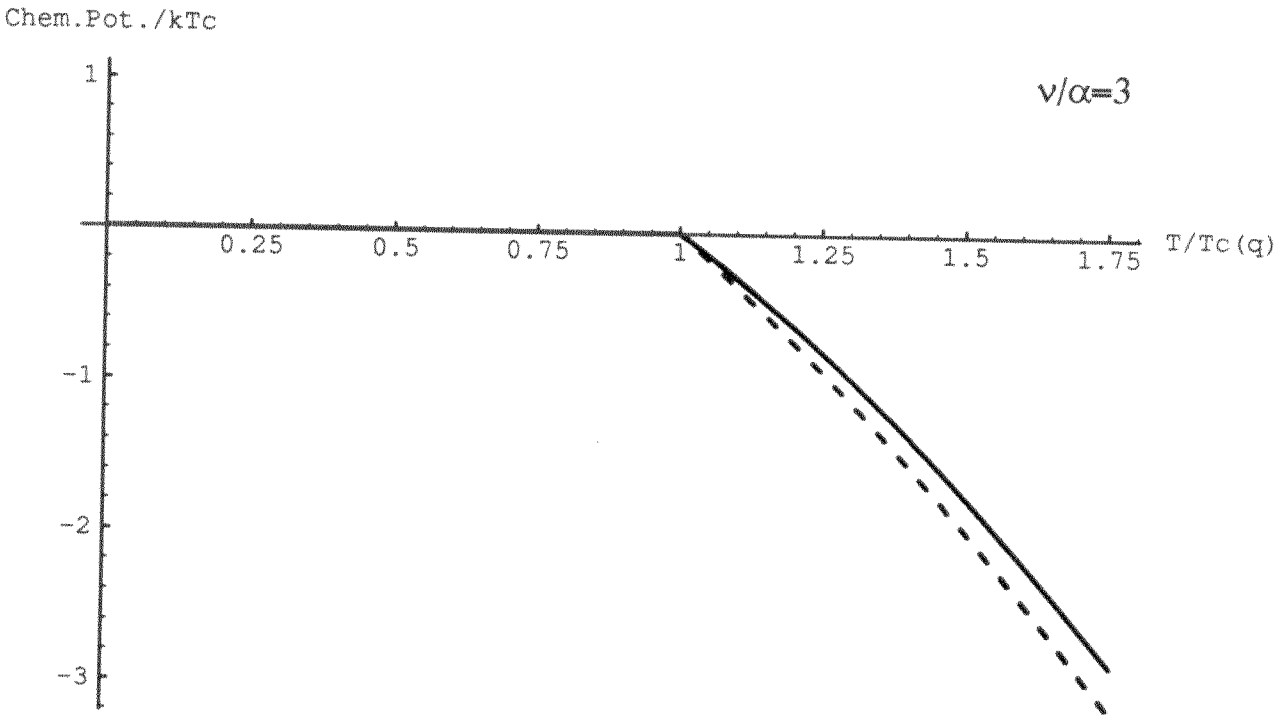

FIGURE 2-c

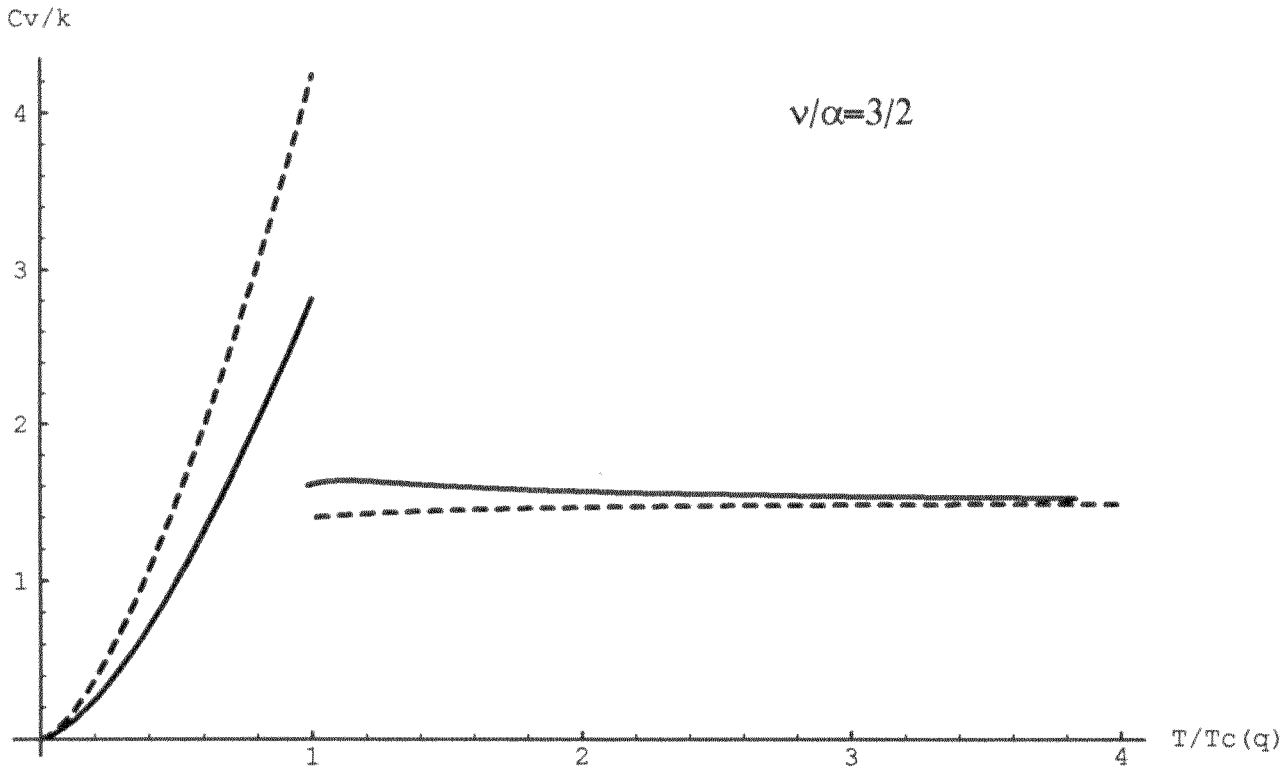

FIGURE 3-a

Figures 3 C_V behaviour as a function of the T/T_c^q for three values of ν/α where ν is the spatial dimension and α is a parameter relative to the dispersion law (3.6): (a) $\frac{\nu}{\alpha} = \frac{3}{2}$; (b) $\frac{\nu}{\alpha} = 2$; (c) $\frac{\nu}{\alpha} = 3$. The dashed line corresponds to $q = 4.5$ and the continuous line to $q = 1.05$.

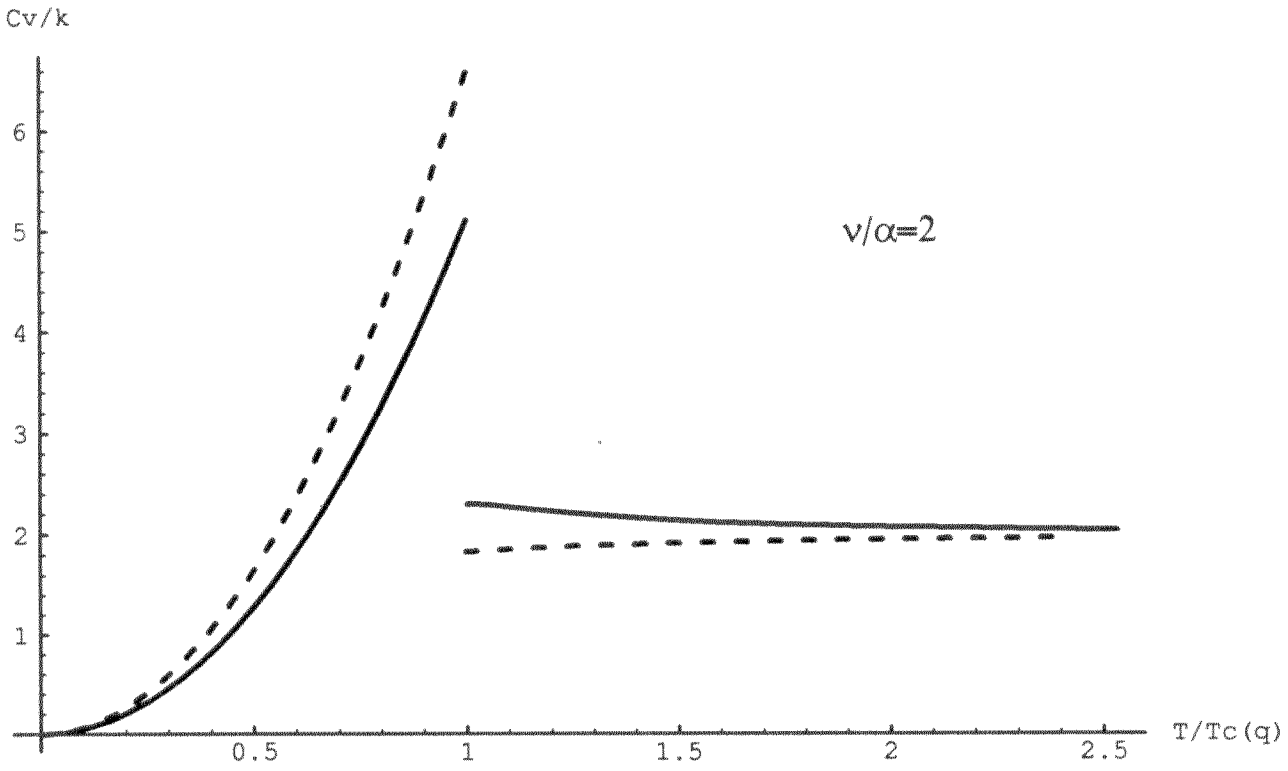

FIGURE 3-b

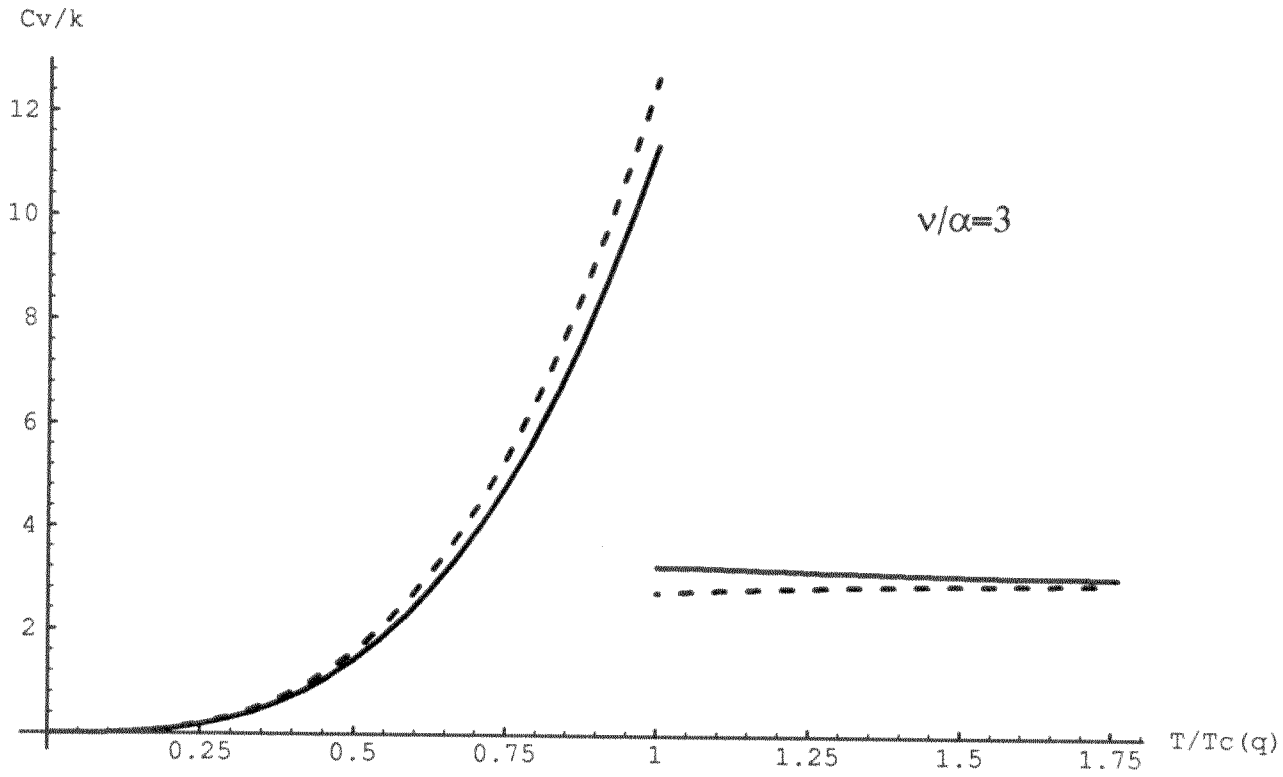

FIGURE 3-c

Table Captions

Table 1 For $q = 1.5$ and $\frac{\nu}{\alpha} = \frac{3}{2}$, in order to have three significant figures in $Y_q(1)$, $y_q(1)$ and $y'_q(1)$ we must keep respectively 6, 7 and 9 terms in the sums. The accuracy of the software Mathematica employed, which is 10^{-7}, is attained respectively with 12 and 14 terms.

Table II For $q = 1.5$ and $\frac{\nu}{\alpha} = 2$, in order to have three significant figures in $Y_q(1)$, $y_q(1)$ and $y'_q(1)$, we must keep respectively 4,5 and 6 terms in the sums. The accuracy of the software Mathematica employed, which is 10^{-7}, is attained respectively with 9 and 10 terms.

Table III For $q = 1.5$ and $\frac{\nu}{\alpha} = 3$, in order to have three significant figures in $y_q(1)$, $y_q(1)$ and $y'_q(1)$, we must keep 4 terms in the sums. The accuracy of the software Mathematica employed, which is 10^{-7}, is attained respectively with 6 and 8 terms.

Table IV For $q = 4.5$ and $\frac{\nu}{\alpha} = \frac{3}{2}$, in order to have three siginificant figures in $Y_q(1)$, $y_q(1)$ and $y'_q(1)$, we must keep respectively 3 and 4 terms in the sums. The accuracy of the software Mathematica employed, which is 10^{-7}, is attained with 5 terms.

Table V For $q = 4.5$ and $\frac{\nu}{\alpha} = 2$, in order to have three significant figures in $Y_q(1)$, $y_q(1)$ and $y_q(1)$, we must keep 3 terms in the sums. The accuracy of the software Mathematica employed, which is 10^{-7}, is attained with 4 terms.

Table VI For $q = 4.5$ and $\nu/\alpha = 3$, the system has a fermionic behaviour ($n = 0, 1$) within an accuracy of at least 10^{-4}; the software Mathematica accuracy is attained with 3 terms.

Table VII Comparison of numerical results for $y_q(z)$, $y_q(z)$ and $y'_q(z)$ for $q = 4$, $\frac{\nu}{\alpha} = \frac{3}{2}$ when z is 1 and 0.2. The functions converge to the value within the Mathematica software accuracy more rapidly when $z = 0.2$, as expected.

TABLE I

$$q = 1.5 \quad , \quad \frac{\nu}{\alpha} = \frac{3}{2}$$

n	4	5	6	7	8	11	12	13	14
$Y_{1.5}(1)$	0.982873	0.983633	-	-	-	0.983875	0.983875		
$y_{1.5}(1)$	-	0.955703	0.956378			0.956645	0.956645		
$y'_{1.5}(1)$				0.879858	0.880061			0.880154	0.8801

TABLE II

$$q = 1.5 \quad , \quad \frac{\nu}{\alpha} = 2$$

n	2	3	4	5	8	9	10
$Y_{1.5}(1)$	0.958882	0.965032	-	-	0.965988	0.965988	-
$y_{1.5}(1)$	-	0.926074	0.928304	-	0.928787	0.928787	-
$y'_{1.5}(1)$	-		0.848302	0.849383	-	0.849669	0.849669

TABLE III

$$q = 1.5 \quad , \quad \frac{\nu}{\alpha} = 3$$

n	2	3	5	6	7
$Y_{1.5}(1)$	0.966901	0.967740	0.967790	0.967790	-
$y_{1.5}(1)$	0.934764	0.936702	0.936847	0.936847	-
$y_{1.5}(1)$	0.873327	0.877330	-	0.877706	0.877706

TABLE IV

$$q = 4.5 \quad , \quad \frac{\nu}{\alpha} = \frac{3}{2}$$

n	1	2	3	4	5
$Y_{4.5}(1)$	0.867200	0.872085	-	0.872122	0.872122
$y_{4.5}(1)$	0.765149	0.772049	-	0.772119	0.772119
$y_{4.5}(1)$	-	0.612900	0.613005	0.613006	0.613006

TABLE V

$$q = 4.5 \quad , \quad \frac{\nu}{\alpha} = 2$$

n	1	2	3	4
$Y_{4.5}(1)$	0.901543	0.902638	0.902640	0.902640
$y_{4.5}(1)$	0.822467	0.824053	0.824056	0.825056
$y'_{4.5}(1)$	0.693147	0.695079	0.695084	0.695084

TABLE VI

$$q = 4.5 \quad , \quad \frac{\nu}{\alpha} = 3$$

n	1	2	3
$Y_{4.5}(1)$	0.947033	0.947087	0.947087
$y_{4.5}(1)$	0.901543	0.901624	0.901624
$y'_{4.5}(1)$	0.822467	0.822572	0.822572

TABLE VII

$$q = 4 \ , \ \frac{\nu}{\alpha} = \frac{3}{2} \ , \ z = 0.2$$

	2	3	4	5
$Y_4(0.2)$	0.193877	0.193878	0.193878	0.193878
$y_4(0.2)$	0.188102	0.188106	0.188106	0.188106
$y_4'(0.2)$	0.885987	0.886041	0.886041	0.886041
				z=1
$Y_4(1)$	0.874085	0.874157	0.874158	0.874158
$y_4(1)$	0.774906	0.775046	0.775048	0.775048
$y_4'(1)$	0.616266	0.616478	0.616482	0.616482

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