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SYMPLECTIC PROJECTOR IN CONSTRAINED SYSTEMS

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Abstract

In this work we develop a geometric formalism for constrained Hamiltonian systems. Using a symplectic projector, we can write the Dirac bracket of two functions as the Poisson bracket of the projected function on a submanifold defined by a local basis of 1-forms. This approach yields in a natural way all properties associated to a generalized symplectic structure.

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I INTRODUCTION

The method of generalized canonical quantization of constrained systems has been discussed along two main lines namely a) emphasizing its geometric character[1] and b) developing algebraic methods involving the structure of the Poisson brackets.[2] The Lagrange multiplier method is applied as a starting point in both approaches. This method, however, is disadvantageous for quantization, since the conjugated momentum of the multiplier is zero. Takahashi[3] has developed a technique that was generalized by Schwartz[4] to several independent linear suplementary conditions.

In his technique neither lagrange multipliers nor the elimination of coordinate are required. In the same spirit Amaral,[5] developed a geometric approach by introducing a projector in the space of configuration-velocities. Here the supplementary condition need not be restricted to a homogeneous equation of the first degree in velocities. All we need is to have a local vector space generated by the constraints. The vector normal to a hypersurface defined in the configuration-velocities space is generated by the gradient in velocities.[6] With the local vector we can construct the projector on the hypersurface where the virtual displacement must be restricted. In this manner, a generalized variational principle can be extended to a non-holonomic system. For an interesting apllication of this method to a classical particle with spin see Amorim.[7]

In the present work we extend the projector method enabling us to handle in a very simple way the dynamical variables and the Poisson brackets associated to the constrained system.

Constraints of the second class define a system which is the local basis of the vector space of forms (ideal) in the exterior algebra. This simple observation enables us to construct the symplectic projetor on the submanifold complementary to the local basis. The infinitesimal variations of the observables must belong to this submanifold. As a consequence the Dirac bracket is written as a function of the matrix elements of the projector, which are the Casimir invariant functions.

In sec.II we briefly describe the method in the configuracional space, whereas sec.III is devoted to discuss the projector in the framework of the symplectic geometry. In sec IV we make some remarks on the structure of the theory.

II - PROJECTORS IN THE CONFIGURATION SPACE

Let us consider a system described by a Lagrangean $L = L(x_\mu, \dot{x}_\mu, t)$ together with a set of T independent constraints:

$$\phi^i(x_\mu, \dot{x}_\mu) = 0 \qquad i = 1,\ldots,T \qquad \mu = 1,\ldots,n \quad . \tag{1}$$

These constraints define a hypersurface in the configuration - velocities space. The T normal vectors to the hypersurface are described by a vectorial basis$\{ \mid e^{\nu} > \}$ that is :

$$|e^{i}\rangle = \partial_{\dot{\nu}}\phi^{i}|\, e^{\nu}\rangle \qquad i = 1,\ldots,T \quad \nu = 1,\ldots,n \quad . \qquad (2)$$

Here and in what follows $\partial_{\dot{\nu}} \equiv \partial/\partial\dot{x}_{\nu}$,and $\partial_{\nu} \equiv \partial/\partial x_{\nu}$; we adopted the sum convention over repeated indices.

The vectors $|e^{i}\rangle$ span a local geometry whose metric is non singular:

$$\langle\, e^{i}|\, e^{j}\,\rangle = g^{ij} \qquad (3)$$

Following Whittaker[8] we can consider a generalized variational principle extended to a non-hollonomic case. The constraints must be imposed on the virtual displacement and not on the trajectories.[9] We can do this by means of a projector on the hypersurface orthogonal to a local basis. The projector operator is a symmetric operator in the configuration-velocities space.[10]

$$P = I - g_{jk}\ |e^{k}\rangle\langle\, e^{j}| \qquad (4)$$

The matrix elements of P in the basis$\{ \, | e^{\nu}\rangle \, \}$ are :

$$\langle e^{\mu}|P|e^{\nu}\rangle = P^{\mu\nu} = g^{\mu\nu} - g_{ij}g^{\mu\rho}\partial_{\rho}\phi^{i}\partial_{\alpha}\phi^{j}g^{\alpha\nu} \qquad (5)$$

where $g^{\mu\nu} = \langle e^{\mu}|e^{\nu}\rangle$ is the global metric of the configuration space. The componentes of the virtual displacement on the hypersurface are:

$$\delta x^{\mu *} = P^{\mu\rho}\delta x_{\rho} \qquad . \qquad (6)$$

Making the variation of the action along $\delta x^{\mu *}$ and taking into account that δx_{μ} is arbitrary we have the vector Euler-Lagrange E_{ν} of the unconstrained system projećted on the hypersurface:

$$P^{\rho\nu}E_{\nu} = 0 \qquad . \qquad (7)$$

This system of equation together with the constraint conditions (1) are the equations of motion of the system.[11]

III PROJECTOR IN THE SYMPLECTIC SPACE

Let the 2n variables of the R^{2n} space be denoted by ξ^{ν}. In this space a symplectic metric is defined by a regular 2-form :

$$d\xi^{\mu} \wedge d\xi^{\nu} = \epsilon^{\mu\nu} \qquad . \qquad \nu,\mu = 1,\ldots,2n \qquad . \tag{8}$$

Here the set $\{d\xi^{\mu}\}$ is a local free basis of 1-forms. The Poisson bracket of two functions $f(\xi)$, $h(\xi)$ assumes the form,

$$\{ f,h \} = \epsilon^{\mu\nu}\partial_{\mu}f\partial_{\nu}h = df \wedge dh \tag{9}$$

Let us consider a submanifold of the fase space determined by a set of T independent constraints of the second classe. Due to these constraints the Poisson bracket is defined only on a manifold of Dim = 2n - T (even), by a singular metric. We have the singular generalized Poisson bracket

$$\{ f,g \} = P^{\mu\nu}\partial_{\mu}f\partial_{\nu}g \qquad . \tag{10}$$

with *det* P = 0.

The constraints define a set of independent 1-forms,

$$d\phi^{i} = \partial_{\nu}\phi^{i} d\zeta^{\nu} \qquad i = 1 \ldots T \quad . \tag{11}$$

Thus, we have a symplectic local metric :

$$g^{ij} = d\phi^{i} \wedge d\phi^{j} \equiv \{\phi^{i}, \phi^{j}\} \tag{12.a}$$

whose inverse is

$$g_{ij} = d\phi_{i} \wedge d\phi_{j} \equiv \{\phi_{i}, \phi_{j}\} \tag{12.b}$$

and

$$d\phi_{i} \wedge d\phi^{j} \equiv \{\phi_{i}, \phi^{j}\} = \delta_{i}^{j} \quad . \tag{12.c}$$

These relations are not new constraints because $d^{2}\phi^{i} = 0$. Thus, the constraints define an ideal of Dim T (T even due the antissimetry of g_{ij}).

Let us now extend the concept of projector to the symplectic geometry. The projector is defined by a bilinear form:

$$S = d\phi_{i} \otimes d\phi^{i} = g_{ik} d\phi^{k} \otimes d\phi^{i} \quad . \tag{13}$$

The projector on the manifold allowed by the constraints is the complementary :

$$P = I - S \qquad , \qquad (14)$$

where I is a identity matrix of Dim = 2N. From (13) it is easy to verify the properties:

$$P \wedge P \equiv P^2 = P \quad , \quad P^T = P \quad , \quad dP = 0 \qquad (15)$$

In free coordenates P has the components:

$$P^{\mu\nu} = d\zeta^{\mu} \wedge P \wedge d\zeta^{\nu} = \epsilon^{\mu\nu} - (d\zeta^{\mu} \wedge d\phi_i) \otimes (d\phi^i \wedge d\zeta^{\nu}) \qquad (16)$$

or

$$P^{\mu\nu} = \epsilon^{\mu\nu} - g_{ij}\epsilon^{\mu\rho}\partial_{\rho}\phi^i\partial_{\alpha}\phi^j\epsilon^{\alpha\nu} \qquad (17)$$

where we have used the definition (9). We observe that theses functions (17) are the Casimir invariant functions as was shown by Ruggeri,[12] they have the same structure of (5). The projector in the symplectic case has a representation in a degenerated 2-form:

$$P = P^{\mu\nu} d\zeta_{\mu} \wedge d\zeta_{\nu} \tag{18}$$

Here $d\zeta_{\mu} \wedge d\zeta_{\nu} = \varepsilon_{\mu\nu}$ is the inverse of the fundamental Poisson bracket:

$$\varepsilon_{\mu\rho}\varepsilon^{\rho\nu} = \delta^{\nu}_{\mu} \tag{19}$$

It is easy to verify that : TrP = 2n - T.

Consider now two functions A and B of the symplectic space. Their variations, acording to the Whittaker principle, must be projected on the submanifold of Dim 2N -T. That is :

$$dA^{*} = P \wedge dA \quad , \quad dB^{*} = P \wedge dB \tag{20}$$

Where $dA = \partial_{\mu}A \; d\zeta^{\mu}$ taking into account the properties (16) and (19) we can write :

$$dA^{*} \wedge dB^{*} = dA \wedge P \wedge dB = P^{\mu\nu}\partial_{\mu}A\partial_{\nu}B \tag{21}$$

From definiton (13) and (14) we can see that :

$$dA^{*} \wedge dB^{*} = \{A, B\}^{*} \tag{22}$$

With :

$$\{A,B\}^{*} = \{A,B\} - \{A,\phi^{k}\}\, g_{ki}\{\phi^{i}, B\} \qquad (23)$$

Since P is idempotent we have:

$$\{A,B\}^{*} = \{A^{*},B^{*}\} = \{A^{*},B\} = \{A,B^{*}\} \qquad (24)$$

It may be noticed that the involution relation between the second class constraint and the fundamental coordinates compatible with the constraints assumes the simple form:

$$d\phi^{i} \wedge d\xi^{\nu *} = 0 \qquad (25)$$

since that $P \wedge d\phi^{i} = 0$.

From (21) the Hamilton equation of motion has the form, in each instant:

$$\dot{\xi}^{\mu *} = d\xi \wedge P \wedge dH = \{\xi, H\}^{*} \quad . \qquad (26)$$

We conclude from this that the second class constraints are constants of the motion simply because $P \wedge d\phi^{i} = 0$, and all symmetries of the system must belong to the manifold of dimension T. This result points out that we may write the symmetries as linear combination of the second class constraints. Theelements of the group of symmetry g has the following variation: $d^{\perp}g = S\wedge dg$; $d^{\parallel}g = P\wedge dg = 0$ in each submanifold.

IV FINAL REMARKS

We make a remark concerning the definition of projectors. A projection is defined in a linear manifold by the property $P^2=P$. Any idempotent operator is a projection and reciprocally.[19] The projection in the configuration space in which we have a positive-definite inner product is called orthogonal projection and is associated to a bilinear symmetric form. In the symplectic space in which we have a skew-symmetric external product the projector is associated to a skew-symmetric a bilinear form.

Finally, one observe that if the projector 2-form is exact one may write $P = d\alpha$, where $\alpha = A^{\mu}d\zeta_{\mu}$ defined only on the submanifold. In this case the Dirac braket may be write as:

$$\{\zeta^{\mu},\zeta^{\nu}\}^{*} = \partial_{\mu}A^{\nu}-\partial_{\nu}A^{\mu} \tag{27}$$

They bracket appears in this case as a non singular generalized Poisson bracket in the reduced space of Dim 2n-T. If the projector 2-form is not exact we have a non trivial topology. The study of the singularities of the projector will be object of another paper.

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