ON JACOBI FIELDS(*)

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ABSTRACT

We define curves on a Riemannian manifold as integrals of generalized Jacobi fields. We show that the force term that deviates the trajectory from the geodesic motion can be constructed as a functional of the metric tensor.

These curves can be interpreted as particles (observers) coupled non-minimally with gravitation which can provide a class of residual observers for the inevitable singularity - as shown in the text.

1. Introduction

The interest on Jacobi fields rests - for most relativists - on its intimate connection with tidal forces, the focusing effect of gravity and the discussion of singu -

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larity regions. In this paper we use Jacobi fields, defined in a 4-dim Riemannian manifold, to introduce a class of accelerated curves (actually a class of congruence of accelerated curves) which are integral curves of Jacobi fields in the following sense: the connecting vectors of two neighbouring curves is a Jacobi field. This way of introducing curves appears to be natural and straightforward, and the mathematical properties of these accelerated curves are discussed. The conformal equivalence to geodesics of some classes of accelerated curves follows naturally. Also accelerations (or forces) can be considered as being of purely metrical origin because they can always be expressed as a series of terms depending only on metrical functions.

Geodesics are a special case of such curves. The special role of geodesics among the whole class of continuous curves in M_4 is a consequence of Einstein's theory: it is the trajectory of particles interacting solely with and minimally coupled to gravitation. In the past years a lot of work has been done on studying local and global properties of geodesics, and some embarassing results have appeared (e.g. the singularity theorems [1]). This gave us the suspicion that probably Einstein's theory is not the final answer to the gravity problem. Modifications of the equations of motion of the field ($g_{\mu\nu}$) and some alternatives of the interactions of particles with gravitation have been proposed (e.g. Cartan theory of spin-spin contact gravitational interaction, non-minimal coupling of interacting fields, etc.). In this vein, we should be prepared to face the problem of gravity being able to accelerate particles, in some cases, diminishing the focusing of them and producing possible residual observers to the classical singularity.

2. Definitions

Let M_4 be a four-dimensional Riemannian manifold with a metric connection structure on it. This connection defines naturally the operation of covariant differentiation $\vec{\nabla}$ of vectors and tensors in M_4. A curve Γ on M_4 is a map γ of an interval I of R^1 into M_4. Let $\gamma(s_0)$ represent a generic point of Γ, $s_0 \in I$. The tangent vector to the curve at s_0 will be $\vec{V} = (\frac{\partial}{\partial s})_{s=s_0}$ which can be described by its components V^α in the coordinate basis $(\frac{\partial}{\partial \chi^\alpha})_{s_0}$. A curve Γ is by definition a geodesic if the covariant derivative of the tangent vector $\vec{V}$ along the curve,

$$V^\alpha \nabla_\alpha V^\beta \equiv \frac{D}{Ds} V^\beta \qquad (1)$$

is parallel to V^β, namely

$$V^\alpha \nabla_\alpha V = f V^\beta$$

where f is a function which can be made zero by a reparametrization of the curve. The new parameter thus obtained is called an afinne parameter and the geodesic equation gives

$$\frac{D}{Ds} V^\beta = V^\alpha \nabla_\alpha V^\beta = 0 \qquad (2)$$

In what follows, all the congruence of curves [2] are parametrized with the same parameter s, at least on a small compact domain.

A vector $\vec{\pi}$ which connects points on two infinitesimally neighbouring curves of a congruence, with equal values for the parameters s, is called a connecting vector. We call a Jacobi field (JF) along Γ any connecting vector $\vec{\pi}$ which satisfies the equation

$$\frac{D^2\pi^\alpha}{Ds^2} + R^\alpha{}_{\mu\nu\rho} V^\mu V^\rho \pi^\nu = 0 \qquad (3)$$

in which $\frac{D}{Ds}\pi^\alpha$ is given by (1). The curvature tensor is defined by

$$\nabla_{[\rho} \nabla_{\nu]} \ell_\mu \equiv (\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho) \ell_\mu = R_{\mu\varepsilon\nu\rho} \ell^\varepsilon$$

for an arbitrary vector $\vec{\ell}$. The second term of equation (3) represents, in Einstein's gravity, tidal forces experienced by neighbouring particles moving on a congruence of geodesics.

We call <u>integral curves</u> of (3) the congruence of curves such that the connecting vector is a Jacobi field along the congruence, that is, the connecting vector of two neighbouring curves of the congruence is a solution of (3).

For Ricci-flat space-times Weyl conformal tensor [3] $C_{\alpha\rho\mu\nu}$ reduces to the curvature tensor $R_{\alpha\beta\mu\nu}$. Now it is well known that we can separate Weyl tensor into two symmetric trace-free tensors $E_{\alpha\beta}$, $H_{\alpha\beta}$ with respect to a congruence (time-like or null) whose tangent vector is $\vec{V}$

$$E_{\alpha\beta} = - C_{\alpha\rho\beta\sigma} V^\rho V^\sigma \qquad (4a)$$

$$H_{\alpha\beta} = C_{\alpha*\rho\beta\sigma} V^\rho V^\sigma \qquad (4b)$$

(4a) and (4b) are called respectively the electric and magnetic part of Weyl tensor.

Equations (4) and (3) tell us that the evolution of the connecting vector depends only on the electric part of Weyl tensor, and linearly.

The most simple and direct way of generalizing equation (3) is by assuming that non-linear terms in $E_{\alpha\beta}$ and/or $H_{\alpha\beta}$ can appear in the RHS of this expression. We are thus led to define: a generalized Jacobi field (GJF) $\vec{Z}$ is such that it satisfies the equation

$$\frac{D^2 Z^\alpha}{Ds^2} = N^\alpha{}_\beta \; Z^\beta \tag{5}$$

where $N^\alpha{}_\beta$ is a polinomial function of the curvature tensor. We call attention to the fact that we can also introduce in equation (5) a new tensor $\tilde{N}^\alpha{}_\beta$, not reducible to geometry but which can be related to it by some dynamical equation. This non-geometric way of introducing GJF will not be pursued here. The final aim of setting up equation (5) is to be able to describe accelerated curves representing non-minimal coupling of particles with gravitation as we shall see in the next section.

It seems worthwhile to remark that we can use the projection field $\vec{Z}_\perp$ when setting the equation of GJF. Indeed using the projection tensor

$$h_{\alpha\beta} \equiv g_{\alpha\beta} - V_\alpha V_\beta$$

we define the projection $\vec{Z}_\perp$ orthogonal to the tangent vector $\vec{V}$ as

$$Z^\alpha{}_\perp = h^\alpha{}_\beta \; Z^\beta$$

A straightforward calculation shows that the distinction between both cases depends quadratically on the acceleration $a = \frac{D\vec{v}}{Ds}$, when the associated congruence is not geodesic. We find

$$h^{\alpha}{}_{\beta} \frac{D}{Ds} h^{\beta}{}_{\delta} \frac{D}{Ds} (h^{\delta}{}_{\rho} Z^{\rho}) = h^{\alpha}{}_{\beta} \frac{D^2}{Ds^2} Z^{\beta} - 2a^{\alpha} a_{\rho} Z^{\rho} \qquad (7)$$

The importance of the projection $\vec{Z}_{\perp}$ is due to the interpretation of $\frac{D^2}{Ds^2} \vec{Z}_{1}$ as the relative acceleration of neighbouring points. We will use $\vec{Z}$ instead of $\vec{Z}_{\perp}$ in our equations but from (7) we can easily translate all our results to the corresponding projected expression.

3. Accelerated Curves

Equation (5) tells us that in general the paths Γ of the associated congruence will have a non-null accelerated vector. In order to make explicit the relationship between the acceleration a^{α} and the tensor $N^{\alpha}{}_{\beta}$, let us make the construction of the GJF $\vec{Z}$ from the value of $\vec{Z}$ at a given point $P \in \Gamma$ by Lie-transport along Γ. We define

$$\underset{\vec{V}}{£} \vec{Z} \equiv \frac{D\vec{z}}{Dv} - \frac{D\vec{v}}{Dz} = 0 \qquad (8)$$

where $\frac{D}{Dm}$ stands for the derivative in $\vec{M}$ direction,

$$\frac{D}{Dm} \equiv M^{\alpha} \nabla_{\alpha}$$

From equations (5) and (8) we find

$$\nabla_{\mu} a^{\alpha} = N^{\alpha}{}_{\mu} - R^{\alpha}{}_{\rho\mu\nu} V^{\rho} V^{\nu} \qquad (9)$$

The behaviour of the acceleration is governed by $N^{\alpha}{}_{\mu}$ and the curvature tensor. Assuming the geometrical origin of $N^{\alpha}{}_{\mu}$ we develop it in powers of the curvature tensor and its dual. In Ricci-flat manifolds we can distinguish 3 classes of polynomials:

(a) n-type Electric field: defined by power of order n of $E^{\mu}{}_{\nu}$,

$$\underset{(n)}{E}{}^{\alpha}{}_{\beta} = k \underbrace{E^{\alpha}{}_{\rho} \; E^{\rho}{}_{\sigma} \; \ldots \; E^{\lambda}{}_{\beta}}_{n\text{-terms}} \tag{10a}$$

k a constant. A particular case of this class is n = 1, k=-1, when the curves are minimally coupled to gravitation. And since we are assuming that acceleration cannot be of non-geometrical origin, they must be unaccelerated (geodesics).

(b) q-type Magnetic field: defined by powers of order q of $H^{\mu}{}_{\nu}$,

$$\underset{(q)}{H}{}^{\alpha}{}_{\beta} = k' \underbrace{H^{\alpha}{}_{\rho} \; H^{\rho}{}_{\sigma} \; \ldots \; H^{\lambda}{}_{\beta}}_{q\text{-terms}} \tag{10b}$$

(c) n-Electric and q-Magnetic field: defined by powers of order n of $\underset{\sim}{E}$ and of order q of $\underset{\sim}{H}$,

$$\underset{(n,q)}{M}{}^{\alpha}{}_{\beta} = K'' \underbrace{E^{\alpha}{}_{\rho} \; E^{\rho} \; \ldots \; E^{\lambda}{}_{\mu}}_{n\text{-terms}} \; \underbrace{H^{\mu}{}_{\nu} \; \ldots \; H^{\varepsilon}{}_{\beta}}_{q\text{-terms}} \tag{10c}$$

Due to the properties of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ (of eqs (4)) we have

$$\begin{aligned}
&\underset{(n)}{E}{}^{\alpha\beta} = \underset{(n)}{E}{}^{\beta\alpha} && \underset{(q)}{H}{}^{\alpha\beta} = \underset{(q)}{H}{}^{\beta\alpha} \\
&\underset{(n)}{E}{}^{\alpha\beta} V_{\beta} = 0 && \underset{(q)}{H}{}^{\alpha\beta} V_{\beta} = 0 \\
&\underset{(n,q)}{M}{}^{\alpha\beta} V_{\beta} = 0 && \underset{(n,q)}{M}{}^{\alpha\beta} \neq \underset{(n,q)}{M}{}^{\beta\alpha} \\
&\underset{(1)}{E}{}^{\alpha}{}_{\alpha} = 0 && \underset{(1)}{H}{}^{\alpha}{}_{\alpha} = 0
\end{aligned} \tag{11}$$

Calling the force that accelerates the particle as F_α (we normalize mass to unity),

$$\frac{Dv^\alpha}{Ds} \equiv V^\beta \nabla_\beta V^\alpha = F^\alpha \tag{12}$$

we have from (9)

$$\nabla_\mu F_\alpha = N_{\alpha\mu} + E_{\alpha\mu}$$

For classes (a) and/or (b) the following properties hold:

(I) $F_{\alpha|\mu} - F_{\mu|\alpha} = 0$ which implies $F_\alpha = \nabla_\alpha \emptyset$

(II) $\nabla_\alpha \nabla^\alpha \emptyset = 0$

(III) $V^\beta \nabla_\beta F_\alpha = 0$

It can be shown (4) that the existence of the function $\emptyset$ for classes (a) and (b) defines a conformal transformation which maps the accelerated curves (12) into geodesics. Thus for cases purely electric and purely magnetic n-type fields, the accelerated congruence is conformally equivalent to a congruence of geodesics.

The lack of simmetry of $\underset{(n,q)}{M}{}^{\alpha\beta}$ implies that there is not such a potential function $\emptyset$ in class (c). Property (III) says that all curves of purely electric, and/or magnetic type have constant acceleration. A similar result holds for curves defined by $\underset{(n,q)}{M}$. So the whole class of curves defined by polynomials of $\underset{\sim}{E}$ and $\underset{\sim}{H}$ have constant acceleration.

4. The Optical Parameters of the congruence and the Focusing Effect

The geometrical behaviour of a congruence of curves with tangent field $\vec{V}(x)$ can be described by two tensors: the rotation tensor

$$\omega_{\alpha\beta} \equiv \nabla_{[\alpha} V_{\beta]}$$

and the general shear tensor

$$\theta_{\alpha\beta} \equiv \nabla_{(\alpha} V_{\beta)}$$

Instead of using the symmetrical tensor $\theta_{\alpha\beta}$, we deal with its irreducible parts, the (trace-free) shear tensor $\sigma_{\alpha\beta}$ and the expansion parameter

$$\theta = \theta_{\alpha\beta}\, g^{\alpha\beta}$$

By making use of the projection tensor $h_{\alpha\beta}$.

We can write

$$\sigma_{\alpha\beta} = \frac{1}{2}\, h^{\lambda}{}_{[\alpha}\, h^{\varepsilon}{}_{\beta]}\, \nabla_{\varepsilon} V_{\lambda} - \frac{1}{3}\,\theta\, h_{\alpha\beta} \qquad (13)$$

$$\omega_{\alpha\beta} = \frac{1}{2}\, h^{\lambda}{}_{[\alpha}\, h^{\varepsilon}{}_{\beta]}\, \nabla_{\varepsilon} V_{\lambda} \qquad (14)$$

From Bianchi's identities [2] we can obtain the equations of evolution for θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$. After a lenghty calculation we find [2] :

$$h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, \frac{D}{Ds}\,\omega_{\mu\nu} - \frac{1}{2}\, h_{[\alpha}{}^{\mu}\, h_{\beta]}{}^{\nu}\, \nabla_{\nu}\, a_{\mu} + \frac{2}{3}\,\theta\,\omega_{\alpha\beta} + \sigma^{\mu}_{[\alpha}\,\omega_{\beta]\mu} = 0 \qquad (15)$$

$$\frac{D}{Ds}\theta - \nabla_{\alpha}\, a^{\alpha} + \sigma^{2} + \frac{\theta^{2}}{3} + 2\omega^{2} - R_{\mu\nu}\, V^{\mu} V^{\nu} = 0 \qquad (16)$$

$$\left(\frac{D}{Ds}\,\sigma_{\mu\rho}\right) h_{\alpha}{}^{\mu} h_{\beta}{}^{\rho} + a_{\alpha} a_{\beta} - \frac{1}{2}\, h_{\alpha}{}^{\mu} h_{\beta}{}^{\rho}\, \nabla_{(\rho} a_{\mu)} + \sigma_{\alpha\mu}\sigma^{\mu}{}_{\beta} + \frac{2}{3}\,\theta\,\sigma_{\alpha\beta} +$$

$$- \omega_\alpha \omega_\beta + \frac{1}{3} h_{\alpha\beta} \left[\nabla_\alpha a^\alpha + \omega^2 - \sigma^2 \right] + E_{\alpha\beta} - \frac{1}{2} \pi_{\alpha\beta} = 0 \qquad (17)$$

Some properties of these equations can be analysed for three distinct classes of paths:

(i) H-poles: when $N^\alpha{}_\mu = H^\alpha{}_\mu$. From the properties of $H^\alpha{}_\mu$ we find

$$\nabla_\alpha F^\alpha = 0 \qquad (18a)$$

$$F_\alpha = \Lambda_{|\alpha} \qquad (18b)$$

Equation (18a) implies that there is no contribution from the force term to the focusing effect of gravity. Indeed, the existence of conjugate points on path Γ will depend - similarly as in geodesic curves - on the sign of the scalar $R_{\mu\nu} V^\mu V^\nu$. If the metric is such that $R_{\mu\nu} V^\mu V^\nu \leq 0$ then the equation of expansion for the accelerated H-pole will give $\dot{\theta} + \frac{\theta^2}{3} \leq 0$ (for $\sigma = \omega = 0$). This implies the existence of conjugate points (in which GJF $\vec{Z}$ is null).

The equation of rotation $\omega_{\mu\nu}$ depends only on the curl of the acceleration. From (18b) we can conclude that the acceleration induced on a particle which is non-minimally coupled with gravitation - given by the presence in the equation of congruence deviation of a term linear in the magnetic part of the Weyl tensor - is not able to modify directly the creation or destruction of vorticity. This means that if the vorticity is null at some point of the path, it will be null along the path irrespectively of how big the acceleration becomes. The effect of non-null acceleration on the shear σ comes only from the second and third terms of expression (16).

(ii) case $N_{\alpha\mu} = \sum_{a=1}^{A} e_a \, E_{(a)\alpha\mu} + \sum_{b=1}^{B} h_b \, H_{(b)\alpha\mu}$

By equation (9)

$$a^{\alpha} = \nabla^{\alpha} \phi \tag{19a}$$

$$\nabla_{\alpha} a^{\alpha} \equiv \Box \phi = \sum_{a} e_a E^{\alpha}_{(a)}{}_{\alpha} + \sum_{b} h_b H^{\alpha}_{(b)\alpha} \tag{19b}$$

The convergence of the congruence (equation of evolution of θ) depends on the sign of $\Box \emptyset$. If $\Box \emptyset$ is positive definite, we can apply the same reasoning as in Raychaudhuri's equation for geodesics - showing the focusing effect of gravity. In this case the role of the acceleration is only to diminish (without being able to eliminate) the converging power of gravity, by enlarging the distance of focal points. Due to (19) there is no effect on the evolution of vorticity and the effect on shear comes only from the second and third terms of (16).

(iii) the general case (from (10c))

$$N_{\alpha\mu} = \sum_{n,q} m(n,q) \; M_{(n,q)}{}_{\alpha\mu}$$

The contribution of acceleration to the focusing effect of gravity on these curves will be given by

$$\nabla_{\alpha} a^{\alpha} = \sum_{n,q} m(n,q) \; M^{\alpha}_{(n,q)}{}_{\alpha} \tag{20}$$

As in case (ii) the positivity of (20) will diminish the focusing power of gravity on the trajectories. Also by choosing suitable coefficients m(n,q) these accelerated trajectories can be focused at any desired rate.

In conclusion, if we can have matter coupled non-minimally to gravitation as in (ii) and (iii) the problem of singularity will have another view, because residual observers can still survive the inevitable singularity.

REFERENCES

[1] HAWKING, S.W. & ELLIS, G.F. - "The Large Scale Structure of Space-Time". Cambridge University Press. See also references therein.

[2] ELLIS, G.F. - "Relativistic Cosmology", in R. Sachs(ed.), General Relativity and Cosmology, Rendiconti S.I.F. , XLVIII Corso, Academic Press, New York.

[3] EISENHART, L.P. - Riemannian Geometry, Princeton, N.J.

[4] NOVELLO, M.; GALVÃO,C.A.P.; SOARES, I.D. ; SALIM, J. M. - "Electric and Magnetic Gravitational Monopoles". To be published in J.Phys., 1976.