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Parafermionic Reductions of WZW Model

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Abstract

We investigate a class of conformal Non-Abelian-Toda models representing a noncompact $SL(2,R)/U(1)$ parafermionions (PF) interacting with a specific abelian Toda theories and having a global $U(1)$ symmetry. A systematic derivation of the conserved currents, their algebras and the exact solution of these models is presented. An important property of this class of models is the affine $SL(2,R)_q$ algebra spanned by charges of the chiral and antichiral nonlocal currents and the $U(1)$ charge. The classical (Poisson Brackets) algebras of symmetries VG_n of these models appears to be of mixed PF-WG_n type. They contain together with the local quadratic terms specific for the W_n-algebras the nonlocal terms similar to the ones of the classical PF-algebra. The renormalization of the spins of the nonlocal currents is the main new feature of the quantum VA_n-algebras. The quantum VA_2-algebra and its degenerate representations are studied in detail.

Key-words: Non-Abelian-Toda models; Integrable models; Parafermionic algebras; q-deformed algebras .

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1 INTRODUCTION

The identification of 2-dimensional critical phenomena as conformal minimal models of the (extended) Virasoro algebra [1] provide powerful algebraic tools for calculation of the critical exponents. Within this framework the long standing problem of classification of the universality classes in two dimensions is reduced to the problem of exhausting all the extensions of the Virasoro algebra. The first part of the list of extensions of the Virasoro algebra contains the well known Lie algebraic lower spin extensions ($s \leq 2$) including the conformal current algebra ($s = 1$), the $N = 1, 2, 3, 4$ super Virasoro algebras ($s \leq 3/2$), etc. An important step in completing this list was proposed by Zamolodchikov and Fateev [2], [3] and [4]. They observe that for describing the critical behavior of a large class of statistical mechanical systems one has to consider *two new types of non-Lie algebraic extensions.* The first one include together with the stress tensor ($s = 2$) a new set of local higher spin ($s = 3, 4, \cdots, N$) currents which close an associative algebra of quadratic relations known as W_N-algebra [14]. The second one represents a *nonlocal extension* of the Virasoro algebra with a set of fractional spins ($s = \frac{k(N-k)}{N}, \quad k = 1, 2, \cdots, N-1$) nonlocal currents–the Z_N-parafermionic algebra (PF) (see ref. [6] for a generalized PF algebras).

An universal method for deriving all these algebras as well as the (classical) Lagrangean of models with such symmetries consists in considering the *gauged G/H- WZW models* and their algebras of symmetries [7],[8], [9]. For example the W_n-algebra appears from the $SL(n, R)$ current algebra by gauging the nilpotent subalgebras $N^{\pm}(n)$ [7],[9]. The parafermionic algebras arises when the Cartan subalgebra of $SU(n)$ is gauged away [3], [6]. The natural question to ask is *whether gauging another subalgebras*, say, of mixed type $U(1) \oplus N^{+}(n-1) \oplus N^{-}(n-1)$ (i.e. a part of the nilpotent subalgebras $N^{\pm}(n)$ with only one Cartan generator) one produces a *new type of extensions* of Virasoro algebra different from W and PF algebras. This question was raised in a slightly different form by Gervais and Saveliev [10], studying the symmetries of the (classical)B_n nonabelian Toda theories (NA). An explicit form of such (classical) nonlocal and non–Lie algebra (called V algebra), unifying together the nonlocal PF-currents and the local WB_{n-1}currents was constructed in ref [11] for the case of B_2-NA-Toda model.

In our recent paper [12] we have found the classical and quantum algebras of symmetries of the first few models of the family of A_n -NA-Toda theories. The present paper is devoted to the systematic construction of the (classical) $VG_n^{(j,1)}$-algebras ($G_n = A_n, B_n, C_n \quad or D_n$) of mixed $PFA_1 - WG_{n-1}$ type and their quantization. They arise as algebras of the conserved currents of the simplest family of $G_n^{(j,1)}$–NA-Toda theories representing a noncompact $SL(2, R)/U(1)$ PF's interacting with $G_{n-j} \otimes G_{j-1}$ abelian Toda model ($G_0 = 1 \;\; j = 1, \cdots n$). This family of NA-Toda theories can be obtained as a specific (parafermionic) reduction of the G_n-WZW model imposing a set of constraints similar to these ones leading to standard G_n abelian Toda model, but with one of the constraints $J_{-\alpha_j} = \mu_j$ (and $\bar{J}_{\alpha_j} = \bar{\mu}_j$) removed ($j = 1, \cdots, n$ fixed),i.e. we *leave unconstrained* one of the simple negative (positive) root current. This changes the group of the residual gauge transformations and contrary to the abelian Toda case the transformation responsible for the vanishing of the Cartan subalgebra current $J_{\lambda_j H}$ (j-fixed) does not exist. We therefore require the *additional constraint* $J_{\lambda_j H} = \bar{J}_{\lambda_j H} = 0$. The main consequence of this *parafermionic constraint* is that two of the chiral (antichiral) conserved current $V_j^+ = J_{\alpha_j}$ and $V_j^- = J_{-\alpha_1-\cdots\alpha_n}$ of the reduced model ($G_n^{(j,1)}$ -NA-Toda) appears to be *nonlocal (PF-type) currents.* The simplest representative of this family of NA-Toda models $G_n^{(1,1)}$ is given by the lagrangean

$$\begin{aligned}\mathcal{L}_n^{(1)} &= -\frac{k}{2\pi}\{\tilde{\eta}_{ik}g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_k + \frac{e^{k_{12}\phi_1}}{\Delta}g^{\mu\nu}\partial_\mu\psi\partial_\nu\chi \\ &- (\frac{2}{k})^2\sum_{i=1}^{n-1}\frac{2}{\alpha_i^2}e^{-\tilde{k}_{ij}\phi_j} - \frac{e^{k_{12}\phi_1}}{2\Delta}k_{12}\epsilon_{\mu\nu}\partial_\mu\phi_1(\psi\partial_\nu\chi - \chi\partial_\nu\psi)\}\end{aligned} \tag{1.1}$$

where $\Delta = 1 + \frac{1}{2\mathcal{K}_{11}}e^{k_{12}\phi_1}\psi\chi$, ($k_{12} = -1$ for A_n, B_n, C_n, D_n except for B_2 where $k_{12} = -2$; $\mathcal{K}_{11} = \frac{n}{n+1}$ for A_n and $\mathcal{K}_{11} = 1$ for B_n). The corresponding equations of motion of (1.1) can be also derived by a slight modification of the Leznov-Saveliev method starting with an appropriate grading operator $Q = \sum_{i=2}^{n} 2\frac{\lambda_i H}{\alpha_i^2}$, (see Sect. 2 and app. A).

Apart from the two *nonlocal chiral currents* $V^\pm$ (of spin $s^\pm = \frac{(n+1)}{2}$ for A_n, $s^\pm = n$ for B_n and C_n, $s^\pm = n-1$ for D_n) each of these models exhibit other $n-1$ *local chiral* (antichiral) currents $W_{n-l+2}, \quad l = 2, \cdots, n$ of spins $s_l = n-l+2$ for A_n; of spins ($s_i = 2, 4, \cdots 2(n-i+1)$ for B_n and C_n and of spins $s_l = 2, 4, \cdots 2(n-2), n$ for D_n.The *nonchiral* $U(1)$ current

$$J_\mu = -\frac{k}{4\pi}\frac{(\psi\partial_\mu\chi - \chi\partial_\mu\psi - k_{12}\psi\chi\partial_\mu\phi_1)}{\Delta}e^{k_{12}\phi_1} \tag{1.2}$$

generating *global* $U(1)$ gauge transformations $\phi_i' = \phi_i$, $\psi' = \psi e^\alpha$ and $\chi' = \chi e^{-\alpha}$ (α constant) completes the list of conserved currents for the $G_n^{(1,1)}$ model.

One of the *basic results* presented in this paper is the explicit form of the Poisson brackets (PB's) algebra $VA_n^{(1,1)}$ of the conserved currents of $A_n^{(1,1)}$ -NA Toda theory. This new algebra appears to be a *natural unification* of the main features of W- and PF-algebras. As it is shown in Sect. 3, the PB's of the local currents are quite similar to the classical WA_{n-1} algebra but including new terms in the quadratic part - the product V^+V^- and its derivatives. The local and nonlocal currents obey PB's with quadratic terms in the form $V^\pm W_k$ (and its derivatives). Finally the PB-algebra of the nonlocal currents contains specific *nonlocal quadratic* terms (see sect 3):

$$\begin{aligned}\{V^\pm(\sigma), V^\pm(\sigma')\} &= -\frac{n+1}{nk^2}\epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') \\ \{V^+(\sigma), V^-(\sigma')\} &= \frac{n+1}{nk^2}\epsilon(\sigma-\sigma')V^+(\sigma)V^-(\sigma') + (\frac{k}{2})^{n-1}\partial_{\sigma'}^n\delta(\sigma-\sigma') \\ &-\sum_{s=0}^{n-2}(\frac{k}{2})^{s-1}W_{n-s}(\sigma')\partial_{\sigma'}^s\delta(\sigma-\sigma')\end{aligned} \tag{1.3}$$

where $\epsilon(\sigma) = $ sign (σ) . Our *main observation* is that the *nonlocal terms* in (1.3) (those with $\epsilon(\sigma)$) are of *PF-type.* One obstacle of such identification is the *discrepancy* between the *fractal spins* of the PF currents and the (half) *integer spins* of the nonlocal currents $V^\pm$. The precise statement is that the *semi-classical limit* $N \to \infty$ of the operator product expansion (OPE) algebra of certain W–reduced G_n parafermions Ψ_α of spins $s_\alpha = \frac{n+1}{2} - \frac{\alpha^2}{2N}$ coincides with the PB's -algebra (1.3) of $V^\pm$, ($s^\pm = \frac{n+1}{2}$). The Z_N-parafermions provide the simplest example ($G = A_1$, $n = 1$) of quantization of $V^\pm$ and their classical algebra (1.3). We identify the quantized $V^\pm$ with the PF currents Ψ_1 and $\Psi_1^\dagger$ of spins $s^\pm = 1 - \frac{1}{N}$ namely, $V^+ = \frac{1}{\sqrt{N}}\Psi_1$, $V^- = \frac{1}{\sqrt{N}}\Psi_1^\dagger$. This way we impose the OPE's of $V^\pm$ to be of the form (see sect. 3 of ref. [3]):

$$\begin{aligned} V^\pm(1)V^\pm(2) &= \frac{N-1}{(N)^{3/2}}(z_{12})^{\Delta_2^\pm - 2\Delta_1^\pm}\left(V_2^\pm(2) + O(z_{12})\right) \\ V^-(1)V^+(2) &= \frac{1}{N}(z_{12})^{-2\Delta_1^\mp}\left(I + \frac{2\Delta_1}{c}T(2)z_{12}^2 + O(z_{12}^3)\right) \end{aligned} \tag{1.4}$$

where $\Delta_2^\pm = 2 - \frac{4}{N}$, $\Delta_1^\pm = 1 - \frac{1}{N}$, $c = 2\frac{N-1}{N+2}$, $N = 2, 3, \cdots$. Next we define the classical PB's as certain limit of the OPE's (1.4):

$$\{V^a(1), V^b(2)\} = lim_{N\to\infty}(-\frac{iN}{2\pi})(V^a(1)V^b(2) - V^b(2)V^a(1)) \tag{1.5}$$

where $a, b = \pm 1$. The last step is to verify that the $lim_{N\to\infty}$ of (1.4) indeed reproduces the PB's algebra (1.3) with $n = 1$. One can also derive the OPE's (1.4) and the renormalization of the spins of $V^\pm$, $\Delta^{quant} = \Delta^{class} - \frac{1}{N}$ following the procedure of the quantum hamiltonian reduction [5, 3]. Starting with the bosonized form of the $SL(2)$ current algebra (see for instance [24]) and imposing the constraints $J_3 = 0$, one obtain the free field representation of the PF-currents. The OPE's (1.4) as well as the new (anomalous) dimensions of $V^\pm$appears as a simple consequence of this construction.

The purpose of this discussion of the parafermionic properties of the nonlocal currents $V^\pm$ is to point out that their *quantization requires deep changes in the classical algebraic structure* (1.3) namely, i) renormalization of the bare spins $s_{cl}^\pm = \frac{n+1}{2}$ to $s_q^\pm = \frac{n+1}{2}(1 - \frac{1}{2k+n+1})$; ii)Breaking the global $U(1)$ symmetry to some discrete group; iii) The quantum counterpart of the PB's of the charges $Q^\pm_{m+\frac{(1+l)(n+1)}{2(2k+n+1)}}$ ($m \in Z, l = 1, \cdots, 2k+n$) of $V^\pm$ are the so called *PF commutators* (an infinite sum of bilinears of the charges) (see sect.4 of ref [3]). The quantization of the local currents W_p is similar to the one of the classical WG_{n-1}-algebras. It consists in the familiar substitution $i\hbar\{\ \}_{PB} = [\ ]_{comm}$ followed by certain changes in the structure constants and of the central charge. No spin renormalization and PF type commutators are required in this case. All these new features of the quantum VA_n-algebras we shall demonstrate in Sect. 9 on the example of the quantization of the $VA_2^{(1,1)} \equiv V_3^{(1,1)}$. The quantum counterparts of the PB's algebra (1.3) appears to be the "parafermionic commutation relations " (9.16) and (9.17). The method we are using allows us to find also the *anomalous dimensions* of the "completely degenerate " representations of the quantum $V_3^{(1,1)}$-algebra.

The two chiral algebras $VG_n^{(1,1)}$ and $\bar{V}G_n^{(1,1)}$ together with the nonchiral $U(1)$ current (1.2) of charge $Q_0 = \int J_0 d\sigma$

$$\begin{aligned} \{Q_0, V^\pm(\sigma),\} &= \pm V^\pm(\sigma), \qquad \{Q_0, W_p(\sigma),\} = 0 \\ \{Q_0, \bar{V}^\pm(\sigma),\} &= \pm\bar{V}^\pm(\sigma), \qquad \{Q_0, \bar{W}_p(\sigma),\} = 0 \end{aligned} \tag{1.6}$$

$p = 2, 3, \cdots, n$ *do not exhaust* all the symmetries of the $G_n^{(1,1)}$ -NA Toda model (1.1). It turns out that certain charges of the *chiral nonlocal* currents , $Q^+ = \int V^+ d\sigma$ and $Q^- = \int \sigma^{n-1} V^- d\sigma$ have nonvanishing equal time PB's with the *antichiral nonlocal* charges $\bar{Q}^- = \int \bar{V}^- d\sigma$ and $\bar{Q}^+ = \int \sigma^{n-1} \bar{V}^+ d\sigma$. They are linked by the *topological charge*

$$H_0 = \frac{k}{2\pi} \int_{-\infty}^{\infty} \partial\varphi d\sigma = -Q_0 + \frac{k}{2\pi} \mathcal{K}_{11} k_{12} \int_{-\infty}^{\infty} \partial\phi_1 d\sigma \tag{1.7}$$

in the following algebra

$$\{Q^\pm, \bar{Q}^\mp\} = \pm \frac{k\pi}{2} \int_{-\infty}^{\infty} d\sigma \partial e^{\pm \frac{1}{\mathcal{K}_{11}} \varphi}, \qquad \{Q^\pm, \bar{Q}^\pm\} = 0 \tag{1.8}$$

Simple redefinitions of the nonlocal charges (see Sect. 4) allows to rewrite the algebra (1.6) and (1.8) in the standard form of the *affine q-deformed $SL(2,R)$ PB'S algebra* of level zero [27, 41]. Note that the deformation parameter $q = e^{-\frac{2\pi}{k} \frac{1}{\mathcal{K}_{11}}}$ is a function of the *classical (bare) coupling constant.* It is important to mention that the $SL(2,R)_q$ PB's algebra appears in the NA-Toda theories (1.1) as *algebra of the (Noether) symmetries of their equations of motion* and leaves invariant the NA-Toda hamiltonian. One has to distinguish this classical q-deformed $\hat{S}L(2,R)_q$ PB algebra (generated by the nonlocal conserved currents) from the *Poisson-Lie group $G_n(r)$* of the monodromy matrices $M \in G_n$ satisfying together with the standard group multiplication laws the Sklyanin PB's algebra [28],

$$\{M \overset{\otimes}{,} M\} = \frac{-2\pi}{k} [r, M \otimes M] \tag{1.9}$$

where r is the classical r-matrix. The latter appears in G_n WZW models [8], [13], [15] and the abelian G_n-Toda theories (as well as in a large class of integrable models [16]) as the right hand transformation that leaves invariant the Poisson structure of the corresponding models. Some preliminary results concerning the $G_n(r)$-algebra generated by the classical monodromy matrices of $G_n^{(1,1)}$ -NA-Toda models are given in Sect. 4.

The Poisson-Lie groups are known to be the classical analog of the quantum group $U_q(G_n)$ encoded in the quantum exchange algebra [17],[8],[18],[19]. One might wonder which is the *quantum counterpart of the classical $\hat{S}L(2,R)_q$ PB's algebra.* Partial answer to this question is given by the simplest example of $n = 1$, $N = 2$ (the Z_2 PF, i.e. critical Ising model). The quantum nonlocal charges Q^+ and $\bar{Q}^-$ coincide with the Ramond sector of zero modes ψ_0 and $\bar{\psi}_0$ of the (chiral) Ising fermions. It turns out that their *commutator* is proportional to the fermion parity operator Γ

$$[\psi_0, \bar{\psi}_0] = i\alpha\Gamma, \quad \Gamma\psi_0\Gamma^{-1} = -\psi_0, \quad \Gamma\bar{\psi}_0\Gamma^{-1} = -\bar{\psi}_0 \tag{1.10}$$

and $\Gamma^2 = 1$. This algebra appears to be the quantum analog of (1.8) for this particular case. It is important to note that the *nonvanishing* commutator of the left and right fermionic zero modes is not in contradiction with the *holomorphic factorization* of the critical Ising model. What is crucial for this factorization is that the *anticommutator* $[\psi_m, \bar{\psi}_n]_+ = 0$, $n, m \in Z$ indeed *vanishes.*

To make the discussion of the $SL(2,R)_q$ symmetries of the $G_n^{(1,1)}$ NA-Toda models complete, we have to demonstrate that classical solutions with nontrivial topological charge $H_0 \neq 0$ (i.e. $\varphi(\infty,0) \neq \varphi(-\infty,0)$) do exist. We derive in Section 5 the general solution of (1.1) in a simple and explicit form, appropriate for the analysis of these asymptotics. Our construction is the NA-Toda analog of the Gervais-Bilal's [20] solution of the abelian Toda models. It is based on the *important observation* that the fields ψ, χ and ϕ_i, $i = 1, \cdots, n-1$ of the NA-Toda theory (1.1) can be realized in terms of the corresponding abelian Toda fields φ_A, $A = 1, \cdots n$ and the chiral nonlocal currents V^+ and $\bar{V}^-$ considered as independent variables. The origin of this transformation of the solutions of the $G_n^{(1,1)}$ NA-Toda into those of the G_n abelian Toda (and vice-versa) is in the fact that both can be realized as gauged G_n/H_i-WZW models with $H_1^{\pm} = U(1) \otimes N^{\pm}(n-1)$, $H_2^{\pm} = N^{\pm}(n)$. Therefore the transformation we have found can be identified as (field dependent) gauge transformations $h(V^+) \otimes \bar{h}(\bar{V}^-) \in G_n \otimes G_n$ that maps G/H_1-WZW into G/H_2-WZW, $g_1 = \bar{h} g_2 h$, $g_i \in G_n/H_i$. This provides us with a powerful method for explicit construction of these transformations. Consider a set of constraints, gauge fixing conditions and remaining currents which define the reduction of G_n WZW model to $G_n^{(1,1)}$ NA-Toda. For $G_n = A_n$,

$$J^{NA} = V^+ E_{-\alpha_1} + \sum_{i=2}^{n} E_{-\alpha_i} + V^- E_{\alpha_1+\alpha_2+\cdots+\alpha_n} + \sum_{i=2}^{n} W^{NA}_{n-i+2} E_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_n} \tag{1.11}$$

(similar for $\bar{J}^{NA}$) and those leading to the abelian Toda are

$$J^{A} = \sum_{i=1}^{n} E_{-\alpha_i} + \sum_{i=1}^{n} W^{A}_{n-i+2} E_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_n} \tag{1.12}$$

The transformations $h(V^+)$, $\bar{h}(\bar{V}^-)$ that maps (1.11) into (1.12) satisfy the following system of first order differential equations

$$(J^A + \frac{k}{2}\partial)h^{-1} = h^{-1}J^{NA} \qquad (\bar{J}^{NA} - \frac{k}{2}\bar{\partial})\bar{h}^{-1} = \bar{h}^{-1}\bar{J}^{A} \tag{1.13}$$

We present in Sect. 6 the explicit form of the solutions of eqns. (1.13) in the A_n case i.e. h, $\bar{h} \in SL(n+1,R)$.

The fact that one can *connect all the coset models* obtained from a given G_n-WZW model (i.e. all the hamiltonian reductions of G_n-WZW) by *specific (current dependent) G_n gauge transformations* leads to important consequences concerning the symmetry structure of the $G_n^{(1,1)}$-NA-Toda models. As one can see from eqns. (1.11-1.13) the transformation $h(V^+)$ gives as a byproduct the explicit constructions of the currents W_p^A, $p = 2, \cdots n+1$ in terms of the conserved currents $V^{\pm}$ and W_i^{NA}, $i = 2, \cdots, n$ of the $G_n^{(1,1)}$-NA-Toda model. We further verify (using the $VA_n^{(1,1)}$ PB's algebra only) that these W_p^A indeed does close the W_{n+1} algebra. Thus $h(V^+)$ maps the $VA_n^{(1,1)} = V_{n+1}^{(1,1)}$ algebra into the W_{n+1}-one. This shows that W_{n+1}, which leaves in the *universal enveloping* of $V_{n+1}^{(1,1)}$ appears as an *algebra of symmetries* of the $A_n^{(1,1)}$-NA-Toda theories as well.

The gauge transformation between different set of constraints imposed on G_n-WZW currents play an important role in the description of the symmetries of a larger class of

$G_n^{(j,1)}$-NA-Toda models ($J_{-\alpha_j} = V^+_{(j)}$, $J_{\lambda_j H} = 0$, $\quad j = 1,\cdots,n$ arbitrary fixed). Again as in the $j = 1$ case we find a transformation $h(V_j^+)$ (as solution of eqn. (1.13))which maps them into G_n-abelian Toda theory. The new phenomena occurs when we consider the transformation $H(j_1,j_2) = h(V_{j_1}^+)h(V_{j_2}^+)^{-1}$ between $G_n^{(j_1,1)}$ and $G_n^{(j_2,1)}$-NA Toda models ($j_1 \neq j_2$). Both contain equal number of independent fields, both are W_{n+1}-invariant and the transformation $H(j_1,j_2)$ realizes the map

$$V_{n+1}^{(j_1,1)} \longrightarrow W_{n+1} \longrightarrow V_{n+1}^{(j_2,1)} \tag{1.14}$$

that mixes their algebras of symmetries. However their lagrangeans, their symmetry algebras $V_{n+1}^{(j_1,i)}$ and $V_{n+1}^{(j_2,1)}$ as well as the spins of their conserved currents are quite different. Nevertheless, as we claim in Sect. 7, they are classically equivalent models related by complicated nonlocal change of field variables: $g(j_1) = \bar{H}(j_1,j_2)g(j_2)H(j_1,j_2)$. This is the reason why we are mainly considering the $j = 1$ model, all the rest $j \neq 1$ $G_n^{(j,1)}$-NA-Toda models being equivalent to it.

The $G_n^{(j,1)}$-NA-Toda models we are studying in this paper are the *nearest neighbours* of the abelian G_n Toda theories. They are defined by the set of constraints and gauge fixings conditions (1.11) and (1.12). The only difference with the abelian G_n-Toda is that the constraint $J^{(A)}_{-\alpha_j}$ is removed ($J^{(NA)}_{-\alpha_j} = V_j^+(z)$) and one new PF-type constraint has to be imposed $J_{\lambda_j H} = 0$. These modifications of the abelian Toda constraints reflects on the properties of the remaining currents: W_{n+1} splits in two nonlocal currents V^+ and V^-, the new nonlocal algebra $VG_n^{(1,1)}$ replaces the WG_n algebra and finally the chiral and antichiral nonlocal charges $Q^\pm$ and $\bar{Q}^\mp$ together with the topological charge H_0 generate the $SL(2,R)_q$. One could wonder *how general* is this way of describing the NA-Toda models. Say, abandoning more abelian Toda constraints $J_{-\alpha_i} = 1 \quad i = 1,\cdots,l \quad l \leq n$, i.e. $J_{-\alpha_i} = V_i^+$ and requiring $J_{\lambda_i H} = 0 \quad i = 1,\cdots,l$ are we getting *new NA-Toda models*? The answer to this question is indeed positive: This set of constraints and gauge fixing conditions

$$J^{NA(l)} = \sum_{i=1}^{l} V_i^+ E_{-\alpha_i} + \sum_{i=l+1}^{n} E_{-\alpha_i} + \sum_{i=1}^{l} V_i^- E_{\alpha_i+\cdots+\alpha_n} + \sum_{i=l+1}^{n} W_{n-i+2} E_{\alpha_i+\cdots+\alpha_n} \tag{1.15}$$

(for $G_n = A_n$) defines a family of conformal invariant $G_n^{([j]_l,l)}$-NA-Toda models ($[j]_l = [j_1, j_2, \cdots, j_l]$, labels the positions of the PF-constraints). Their properties are quite similar to the simplest $G_n^{(1,1)}$-NA-Toda model. They have $2l$ chiral nonlocal currents $V_i^\pm$, ($i = 1,\cdots,l$) and (n-l) chiral local ones W_{n-i+2}, $\quad i = l+1,\cdots,n$). The complete discussion of the symmetries of $A_n^{([j]_l,l)}$-NA-Toda models, their general solutions, their relation with the abelian A_n-Toda, etc will be presented in our forthcoming paper [21].

Although the $VG_n^{([j]_l,l)}$-algebras do not exhaust all the parafermionic extensions of the WG_n-algebras, they make the picture of the extended Virasoro algebras more complete. The application of the quantum VG_n-algebras and their minimal conformal models is not restricted to the problem of classification of the universality classes in two dimensions only. As it is well known [40] certain 2-d NA-Toda theories naturally appears in the construction of cylindrically symmetric instantons solutions of self dual Yang-Mills theories in four dimensions. This is a strong indication that the quantum A_n-NA-Toda models (and their integrable off-critical pertubations) provide powerful tools for the nonperturbative quantization of instantons and merons.

2 NA-Toda's as gauged WZW models

The $G_n^{(1,1)}$-NA-Toda theories we are going to study are originally defined as an integrable system of field equations given by the zero curvature condition:

$$[\partial - \mathcal{A}, \bar{\partial} - \bar{\mathcal{A}}] = 0, \tag{2.1}$$

for the specific Lax connections [1]:

$$\begin{aligned}
\mathcal{A} &= \frac{\psi\partial\chi}{2\mathcal{K}_{11}\Delta} e^{k_{12}\phi_1}\lambda_1\cdot H + \sum_{j=1}^{n-1}\partial\phi_j\frac{\alpha_{j+1}\cdot H}{\alpha_{j+1}^2} + \frac{\partial\chi}{\Delta}e^{k_{12}\frac{1}{2}\phi_1}E_{\alpha_1} + (\frac{2}{k})\sum_{i=1}^{n-1}e^{-\frac{1}{2}\tilde{k}_{ij}\phi_j}E_{\alpha_{i+1}}, \\
\bar{\mathcal{A}} &= -\frac{\chi\bar{\partial}\psi}{2\mathcal{K}_{11}\Delta} e^{k_{12}\phi}\lambda_1\cdot H - \sum_{j=1}^{n-1}\bar{\partial}\phi_j\frac{\alpha_{j+1}\cdot H}{\alpha_{j+1}^2} - \frac{\bar{\partial}\psi}{\Delta}e^{k_{12}\frac{1}{2}\phi_1}E_{-\alpha_1} - (\frac{2}{k})\sum_{i=1}^{n-1}e^{-\frac{1}{2}\tilde{k}_{ij}\phi_j}E_{-\alpha_{i+1}},
\end{aligned} \tag{2.2}$$

where H_i, $E_{\pm\alpha}$ denote the generators of the G_n-algebra; $\pm\alpha_i$ are its simple roots, α-an arbitrary root; k_{ij} and $\mathcal{K}_{ij}$-the Cartan matrix and its inverse, respectively, $\tilde{k}_{ij}$,$\tilde{\mathcal{K}}_{ij}$ and $\tilde{\alpha}_j$ are the corresponding matrices and roots for G_{n-1}, $\lambda_i = \mathcal{K}_{ij}\alpha_j$ is the i^{th} fundamental weight.

The problem we address here is *to find an action which reproduces the equations of motion for the fields $\psi(z,\bar{z})$, $\chi(z,\bar{z})$, $\phi_i(z,\bar{z})$ $(i = 1, 2, ..., n-1)$* encoded on eqns. (2.1),

$$\partial\bar{\partial}\phi_l = (\frac{2}{k})^2 e^{-\tilde{k}_{ls}\phi_s} - \tilde{\mathcal{K}}_{1l}(\frac{\tilde{\alpha}_l^2}{2})\frac{\partial\chi\bar{\partial}\psi}{\Delta^2}e^{k_{12}\phi_1},$$

$$\bar{\partial}(\frac{\partial\chi}{\Delta}e^{k_{12}\phi_1}) = -\frac{\bar{\partial}\psi\partial\chi}{2\mathcal{K}_{11}\Delta^2}\chi e^{2k_{12}\phi_1}, \qquad \partial(\frac{\bar{\partial}\psi}{\Delta}e^{k_{12}\phi_1}) = -\frac{\bar{\partial}\psi\partial\chi}{2\mathcal{K}_{11}\Delta^2}\psi e^{2k_{12}\phi_1} \tag{2.3}$$

where we have used $\tilde{\mathcal{K}}_{1i}\mathcal{K}_{11} = \mathcal{K}_{1\,i+1}$. As is well known, the G_n-abelian Toda [9] and the A_1-NA-Toda theories [23] are equivalent to specific gauged WZW models: $G_n/N_L \otimes N_R$ and $SL(2,R)/U(1)$ respectively. This fact suggests to look for a subgroup $H \subset G_n$ such that the corresponding G_n/H-WZW model provides an action for the $G_n^{(1,1)}$-NA-Toda theories. According to the Hamiltonian reduction recipe [8, 9, 5] the first step consists in imposing a specific set of constraints on the WZW-currents:

$$\begin{aligned}
J_{-\alpha_i} &= \bar{J}_{\alpha_i} = 1, \quad i = 2, \cdots, n \\
J_{-[\alpha]} &= \bar{J}_{[\alpha]} = 0, \quad \alpha \ \text{non} - \ \text{simple root} \\
J_{\lambda_1\cdot H} &= \bar{J}_{\lambda_1\cdot H} = 0
\end{aligned} \tag{2.4}$$

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[1] all algebraic notations and definitions used here are collected in App. A

on the WZW currents

$$\begin{aligned} J &= (\frac{k}{2})g^{-1}\partial g = \sum_{all\ roots} J_{\{\alpha\}}E_{\{\alpha\}} + \sum_{i=2}^{n} J_i \frac{\alpha_i \cdot H}{\alpha_i^2} + J_{\lambda_1 H}\lambda_1 \cdot H, \\ \bar{J} &= -(\frac{k}{2})\bar{\partial} g g^{-1} = \sum_{all\ roots} \bar{J}_{\{\alpha\}}E_{\{\alpha\}} + \sum_{i=2}^{n} \bar{J}_i \frac{\alpha_i \cdot H}{\alpha_i^2} + \bar{J}_{\lambda_1 H}\lambda_1 \cdot H, \end{aligned}$$

An important characteristic of the constraints (2.4) is the group of *residual gauge transformations* $H_+^L \otimes H_-^R \in G_n^L \otimes G_n^R$ that leaves them unchanged:

$$\begin{aligned} H_+^L &= \{g_+^L(z) \in G_n^L : g_+^L(z) = exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H \omega_0(z)]exp[\sum_{[\alpha]_1} \omega_{[\alpha]_1}(z)E_{[\alpha]_1}]\} \\ H_-^R &= \{g_-^R(\bar{z}) \in G_n^R : g_-^R(\bar{z}) = exp[\sum_{-[\alpha]_1} \bar{\omega}_{-[\alpha]_1}(\bar{z})E_{-[\alpha]_1}]exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H \bar{\omega}_0(\bar{z})]\} \end{aligned}$$

where $[\alpha]_1$ denotes all positive roots except α_1. This symmetry allows us to remove the remaining irrelevant (fields) degrees of freedom by choosing an appropriate gauge. Note that eqs. (2.4) can be also considered as specific gauge fixing conditions for the " constraints" subgroup $H_-^c \otimes H_+^c \in G_n^L \otimes G_n^R$

$$\begin{aligned} H_+^c &= \{g_+^c(\bar{z}) \in G_n^R : g_+^c(\bar{z}) = exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H \bar{\omega}_0(\bar{z})]exp[\sum_{[\alpha]_1} \bar{\omega}_{[\alpha]_1}(\bar{z})E_{[\alpha]_1}]\} \\ H_-^c &= \{g_-^c(z) \in G_n^L : g_-^c(z) = exp\sum_{-[\alpha]_1} \omega_{-[\alpha]_1}(z)E_{-[\alpha]_1}exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H \omega_0(z)]\} \end{aligned}$$

We can the further combine H_+^L, H_-^R and $H_\pm^c$ in two *noncommuting* and *nonchiral* subgroups $H_\pm \in G_n$:

$$\begin{aligned} &H_+ = \{g_+ = g_+^L(z)g_+^c(\bar{z}) \in G_n : g_+(z,\bar{z}) = exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H R]exp[\sum_{[\alpha]_1} \psi_{[\alpha]_1}E_{[\alpha]_1}]\} \\ &H_- = \{g_- = g_-^c(z)g_-^R(\bar{z}) \in G_n : g_-(z,\bar{z}) = exp[\sum_{[\alpha]_1} \chi_{[\alpha]_1}E_{-[\alpha]_1}]exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H R]\} \qquad (2.5) \end{aligned}$$

where $\chi_{[\alpha_1]}(z,\bar{z})$, $\psi_{[\alpha_1]}(z,\bar{z})$ and $R(z,\bar{z})$ are arbitrary functions of $z = \tau + \sigma$ and $\bar{z} = \tau - \sigma$. In this way, all the information concerning the "nonphysical" (gauge) degrees of freedom (that should be removed by Hamiltonian reduction procedure) is encoded in the subgroups $H_\pm \in G_n$ and in the following two constant matrices $\epsilon^\pm = \sum_{i=2}^n (\frac{\alpha_i^2}{k})E_{\pm\alpha_i}$ (which indicate the currents that are constrained to one). The remaining "physical" degrees of freedom belong to the factor group $H_0^f = H_0/H_0^0$, ($H_0^0 = \{g_0^0 \in G_n : g_0^0 = exp[\frac{1}{2\mathcal{K}_{11}}\lambda_1 H R], [g_0^0, \epsilon^\pm] = 0\}$):

$$H_0^f = \{g_0^f \in G_n : g_0^f(z,\bar{z}) = exp[\chi E_{-\alpha_1}]exp[\sum_{i=1}^{n-1} \frac{2\alpha_{i+1}H}{\alpha_{i+1}^2}\phi_i]exp[\psi E_{\alpha_1}] \qquad (2.6)$$

This decomposition of G_n into $H_\pm$ and H_0^f provides us with a specific parametrization for each $g(z,\bar z) \in G_n : g(z,\bar z) = g_- g_0^f g_+$. The latter is the *crucial ingredient* in the derivation of the NA-Toda eqns. (2.3) from the original G_n-WZW equations $\bar\partial J = \partial \bar J = 0$ by imposing the constraints (2.4) and the corresponding gauge fixing conditions. The proof is quite similar to the one of the abelian Toda case [9](see also [22]). We leave the details to appendix A.

The natural splitting of the original G_n valued WZW fields $g(z,\bar z)$into *irrelevant* $g_\pm \in H_\pm$ and *physical* ones $g_0^f \in H_0^f$ induced by the constraints (2.4) is an indication that the $G_n^{(1,1)}$-NA Toda models (2.3) can be described as gauged $H_- \setminus G_n/H_+$ ($\equiv G_n/H$)-WZW models. Given the subgroups $H_\pm \in G_n$ and the constant matrices $\epsilon^\pm \in \mathcal{H}_\pm$. The standard procedure [9], [37] to construct the corresponding action consists in the introduction of auxiliary gauge fields $A(z,\bar z) \in \mathcal{H}_-$ and $\bar A(z,\bar z) \in \mathcal{H}_+$:

$$A = h_-^{-1}\partial h_-, \qquad \bar A = \bar\partial h_+ h_+^{-1}, \quad h_\pm \in H_\pm \tag{2.7}$$

interacting in an $H_\pm$ invariant way with the WZW field $g \in G_n$:

$$S(g,A,\bar A) = S(g) - \frac{k}{2\pi}\int dz d\bar z \,\mathrm{Tr}\left\{A(\bar\partial g g^{-1} - \epsilon_+) + \bar A(g^{-1}\partial g - \epsilon_-) + A g \bar A g^{-1} + A_0 \bar A_0\right\} \tag{2.8}$$

where $S(g)$ is the G_n-WZW action:

$$S(g) = -\frac{k}{4\pi}\int dz d\bar z \,\mathrm{Tr}(g^{-1}\partial g g^{-1}\bar\partial g) - \frac{k}{12\pi}\int_B d^3x \epsilon_{ijk}\,\mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g)$$

and $A_0 = h_0^{-1}\partial h_0$ ($\bar A_0 = \bar\partial h_0 h_0^{-1}$) ($h_0 \in H_0^0$) is the diagonal part of A and $\bar A$. Since $H_\pm$ (2.5) are by construction semidirect products of nilpotent $N_\pm^{(1)}$ and diagonal H_0^0 subgroups, the gauge fields $A \in \mathcal{H}_-$ and $\bar A \in \mathcal{H}_+$ covariantly split into nilpotent $A_-(\bar A_+)$ and diagonal $A_0(\bar A_0)$ parts $A = A_- + A_0, \quad \bar A = \bar A_0 + \bar A_+$.

The structure of the $A(\bar A)$ dependent terms in (2.8) represents a mixture of the familiar (vector) $U(1) \equiv H_0^0$ gauged WZW model(of PF-type) and the nilpotent $N_\pm^{(1)}$ gauged WZW, similar to the one that gives the abelian Toda model [9]. The specific combination of terms that appears in (2.8)is fixed by the requirement of $H_\pm$-invariance of the action $S(g,A,\bar A)$, i.e. under the following transformations

$$\begin{array}{rcl} g' &=& \alpha_- g \alpha_+, \quad A_0' = A_0 + \alpha_0^{-1}\partial\alpha_0, \quad \bar A_0' = \bar A_0 + \bar\partial\alpha_0\alpha_0^{-1} \\ A' &=& \alpha_-^{-1}A\alpha_- + \alpha_-^{-1}\partial\alpha_-, \qquad \bar A' = \alpha_+\bar A\alpha_+^{-1} + \bar\partial\alpha_+\alpha_+^{-1} \end{array} \tag{2.9}$$

where $\alpha_\pm(z,\bar z) \in H_\pm$ and $\alpha_0 \in H_0^0$.

What remains to be shown is that the action (2.8) indeed describes the $G_n^{(1,1)}$-NA-Toda theories. The way we are going to prove it consists in deriving an effective action (1.1) of the NA-Toda model by integrating out the auxiliary fields A, $\bar A$ in (2.8). In order to perform the (matrix) functional integral of Gauss type in A and $\bar A$, we first simplify (2.8) by gauge fixing the $H_\pm$-symmetries. We choose $\alpha_\pm = g_\pm^{-1}$, therefore $g' = \alpha_- g\alpha_+ = \alpha_- g_- g_0^f g_+ \alpha_+ = g_0^f$, i.e., $S(g,A,\bar A) = S(g_0^f, A', \bar A')$. Equally well, one can consider this as a change of the variables $A, \bar A \to A', \bar A'$ in the functional integral. Taking into account the specific form of $g_0^f \in H_0^f$

(see eqn. (2.6)) and that $A' \in \mathcal{H}_-$, $\bar{A}' \in \mathcal{H}_+$ we verify that the following trace identities hold:

$$\mathrm{Tr} A' \bar{\partial} g_0^f (g_0^f)^{-1} = \mathrm{Tr} A'_0 \bar{\partial} g_0^f (g_0^f)^{-1}, \qquad \mathrm{Tr} \bar{A}' (g_0^f)^{-1} \partial g_0^f = \mathrm{Tr} \bar{A}'_0 (g_0^f)^{-1} \partial g_0^f$$
$$\mathrm{Tr} A' g_0^f \bar{A}' (g_0^f)^{-1} = \mathrm{Tr} A'_0 g_0^f \bar{A}' (g_0^f)^{-1} + \mathrm{Tr} A'_- g_0^f \bar{A}'_+ (g_0^f)^{-1}.$$

As a consequence, the functional integral over A, $\bar{A}$ splits in a product of two integrals: the first one lying in the diagonal subalgebra $\mathcal{H}_0$ of $\mathcal{H}_0^f$:

$$Z_0 = \int DA'_0 D\bar{A}'_0 \exp\left\{-G_0(A'_0, \bar{A}'_0)\right\},$$
$$G_0 = -\frac{k}{2\pi} \mathrm{Tr} \int dz d\bar{z} [A'_0 g_0^f \bar{A}'_0 (g_0^f)^{-1} + \bar{A}'_0((g_0^f)^{-1} \partial g_0^f) + A'_0(\bar{\partial} g_0^f (g_0^f)^{-1}) + A'_0 \bar{A}'_0], \qquad (2.10)$$

and the second one, lying in the nilpotent subalgebras $\mathcal{N}_\pm^{(1)}$:

$$Z_{+-} = \int DA' D\bar{A}'_+ \exp\left\{-G_{+-}(A'_-, \bar{A}'_+)\right\},$$
$$G_{+-} = -\frac{k}{2\pi} \mathrm{Tr} \int dz d\bar{z} [A'_- g_0^f \bar{A}'_+ (g_0^f)^{-1} - \bar{A}'_+ \epsilon_- - A'_- \epsilon_+]. \qquad (2.11)$$

According to the definition of $\mathcal{H}_0$, an arbitrary $A'_0 \in \mathcal{H}_0^0$ can be parametrized by only one function $a'_0(z, \bar{z})$:

$$A'_0 = \frac{1}{2\mathcal{K}_{11}} \lambda_1 \cdot H a'_0, \qquad \bar{A}'_0 = \frac{1}{2\mathcal{K}_{11}} \lambda_1 \cdot H \bar{a}'_0. \qquad (2.12)$$

We first simplify $G_0(a'_0, \bar{a}'_0)$ taking into account the explicit form (2.6), (2.12) of g_0^f, and $A'_0, \bar{A}'_0$ and the basic trace formulas for the G_n-generators

$$\mathrm{Tr}(H_i H_j) = \delta_{ij}; \quad \mathrm{Tr}(H_i E_\alpha) = 0; \quad \mathrm{Tr}(E_\alpha E_\beta) = \frac{2}{\alpha^2} \delta_{\alpha+\beta,0}$$

The result is

$$G_0(a'_0, \bar{a}'_0) = -\frac{k}{2\pi} \int \frac{dz d\bar{z}}{2\mathcal{K}_{11}} \left\{ a'_0 \bar{a}'_0 \Delta - e^{k_{12}\phi_1} (a'_0 \chi \bar{\partial} \psi + \bar{a}'_0 \psi \partial \chi) \right\},$$

where $\Delta = 1 + \frac{1}{2\mathcal{K}_{11}} e^{k_{12}\phi_1} \psi \chi$. Thus the matrix integral (2.10) reduces to simple functional Gauss integral for the scalars $a'_0, \bar{a}'_0$:

$$Z_0 = \frac{2\mathcal{K}_{11}}{k\Delta} e^{-S_0^{eff}}, \quad S_0^{eff} = \frac{k}{2\pi} \int dz d\bar{z} e^{2k_{12}\phi_1} \frac{\psi \chi \bar{\partial} \psi \partial \chi}{2\mathcal{K}_{11} \Delta} \qquad (2.13)$$

The calculation of $S(g_0^f)$ can be easely performed applying the Polyakov-Wiegman decomposition formula for each of the multipliers in g_0^f (see (2.6)):

$$S(g_0^f) = -\frac{k}{4\pi} \int dz d\bar{z} \left\{ \tilde{\eta}_{ik} \partial \phi_i \bar{\partial} \phi_k + 2 \exp\left\{k_{12}\phi_1\right\} \partial \chi \bar{\partial} \psi \right\} \qquad (2.14)$$

where $\tilde{\eta}_{ik} = 4\frac{\alpha_i \alpha_j}{\alpha_i^2 \alpha_j^2}$ is the Killing-Cartan form for G_{n-1} (obtained by deleting the first point of the Dynkin diagram of G_n). In order to take the integral (2.11) we have to rewrite G_{+-} separating the exact square term

$$G_{+-} = -\frac{k}{2\pi}\int dz d\bar{z} tr \left\{ (A'_- - g_0^f \epsilon_- (g_0^f)^{-1}) g_0^f (\bar{A}'_+ - (g_0^f)^{-1} \epsilon_+ g_0^f)(g_0^f)^{-1} - \epsilon_+ g_0^f \epsilon_- (g_0^f)^{-1} \right\} \tag{2.15}$$

Its contribution to the effective action yields

$$S_{+-}^{eff} = \frac{k}{2\pi}\int dz d\bar{z} tr[\epsilon_+ g_0^f \epsilon_- g_0^{f^{-1}}] = \frac{k}{2\pi}\int dz d\bar{z} (\frac{2}{k})^2 \sum_{i=1}^{n-1} \frac{2}{\alpha_{i+1}^2} \exp\left\{-\tilde{k}_{ij}\phi_j\right\} \tag{2.16}$$

Combining together S_0^{eff}, $S(g_0^f)$ and S_{+-}^{eff} we find that the effective classical action (all the determinant factors from the Gauss integration and changes of variables neglected) for the $H_\pm$ gaugeed G_n-WZW model has the form

$$S_{G/H}^{eff} = -\frac{k}{2\pi}\int dz d\bar{z} \left\{ \frac{1}{2}\tilde{\eta}_{ik}\partial\phi_i \bar{\partial}\phi_k + e^{k_{12}\phi_1}\frac{\bar{\partial}\psi\partial\chi}{\Delta} - (\frac{2}{k})^2 \sum_{i=1}^{n-1} \frac{2}{\alpha_{i+1}^2} e^{-\tilde{k}_{ij}\phi_j} \right\}. \tag{2.17}$$

Finally, comparing the equations of motion derived from (2.17) with the NA-Toda ones (2.3), we realize that they do coincide. Therefore, the $S_{G/H}^{\mathbf{eff}}$ is the action for the $G_n^{(1,1)}$-NA-Toda theories we were looking for. This completes our proof that the class of NA-Toda models we are considering are equivalent to the gauged $H_- \setminus G_n/H_+$-WZW models.

Note that the second term in (2.17) contains both symmetric and antisymmetric parts:

$$\frac{e^{k_{12}\phi_1}}{\Delta}\bar{\partial}\psi\partial\chi = \frac{e^{k_{12}\phi_1}}{\Delta}(g^{\mu\nu}\partial_\mu\psi\partial_\nu\chi + \epsilon_{\mu\nu}\partial_\mu\psi\partial_\nu\chi),$$

where $g_{\mu\nu}$ is the 2-D metric of signature $g_{\mu\nu} = diag(1,-1)$. For $n = 1$ ($G_n \equiv A_1$, ϕ_1 is zero) the antisymmetric term is a total derivative:

$$\epsilon_{\mu\nu}\frac{\partial_\mu\psi\partial_\nu\chi}{1+\psi\chi} = \frac{1}{2}\epsilon_{\mu\nu}\partial_\mu\left(\ln\{1+\psi\chi\}\,\partial_\nu \ln\frac{\chi}{\psi}\right),$$

and it can be neglected. This A_1-NA-Toda model is known to describe the 2-D black hole solution for (2-D) string theory [23]. The $G_n^{(1,1)}$-NA-Toda model ($G_n = B_n$ or D_n) can be used in the description of specific (n+1)-dimensional black string theories [10], with
n-1-flat and 2-non flat directions ($g^{\mu\nu}G_{ab}(X)\partial_\mu X^a \partial_\nu X^b$, $X^a = (\psi, \chi, \phi_i)$), containing axions ($\epsilon_{\mu\nu}B_{ab}(X)\partial_\mu X^a \partial_\nu X^b$) and tachions ($\exp\{-k_{ij}\phi_j\}$), as well. For this geometric interpretation of the NA-Toda models, it is convenient to rewrite the antisymmetric term in (2.17) as:

$$\frac{e^{k_{12}\phi_1}}{\Delta}\epsilon_{\mu\nu}\partial_\mu\psi\partial_\nu\chi = -\frac{1}{2}\epsilon_{\mu\nu}k_{12}\partial_\mu\phi_1(\psi\partial_\nu\chi - \chi\partial_\nu\psi)\frac{\exp\{k_{12}\phi_1\}}{\Delta} + \mathcal{K}_{11}\epsilon_{\mu\nu}\partial_\mu\left(\ln\{\Delta\}\,\partial_\nu \ln\frac{\chi}{\psi}\right).$$

Discarding the total derivative term, the action (2.17) takes its final form:

$$\begin{aligned} S^{eff}_{G/H} &= S^{NA}_{G_n^{(1,1)}} = -\frac{k}{2\pi}\int d^2z\{\tilde{\eta}_{ik}g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_k + \frac{e^{k_{12}\phi_1}}{\Delta}g^{\mu\nu}\partial_\mu\psi\partial_\nu\chi \\ &- (\frac{2}{k})^2\sum_{i=1}^{n-1}\frac{2}{\alpha_{i+1}^2}e^{-\tilde{k}_{ij}\phi_i} - \frac{1}{2}\epsilon_{\mu\nu}k_{12}\partial_\mu\phi_1(\psi\partial_\nu\chi - \chi\partial_\nu\psi)\frac{e^{k_{12}\phi_1}}{\Delta}\}. \end{aligned} \tag{2.18}$$

Our definition of the $G_n^{(1,1)}$-NA Toda models (2.17) is based on an appropriate set of constraints (2.4) on the G_n-WZW currents and their residual gauge symmetries. As we have shown this data can be transformed into a (complete) set of subgroups $H_\pm$, the factor group H_0^f of G_n and the matrices $\epsilon_\pm$. This allows us to represent these NA-Toda models as $H_- \setminus G_n/H_+$-gauged WZW models. One might wonder whether they can be derived following the original Leznov-Saveliev (LS) approach [40] to the NA-Toda theories and vice-versa, i.e. given a model from the LS-scheme, could one write it as gauged WZW model? As it is shown in appendix A starting from our NA-Toda data, $H_\pm, H_0^f$ (and H_0), $\epsilon_\pm$ one can construct an unique grading operator $Q = \sum_{i=2}^n \frac{2\lambda_i}{\alpha_i^2}H$ such that $[Q, \mathcal{H}_\pm] = \pm\eta\mathcal{H}_\pm$, $[Q, H_0] = 0$, $[\mathcal{H}_0^0, \epsilon_\pm] = 0$, $H_0^f = H_0/H_0^0$ and finally to derive eqns (2.3) following the LS-approach. There exists however a *difference* between the LS-NA-Toda models and our $G_n^{(1,1)}$-models. It consists in the *following*: given a grading operator (from the Kac classification table) $Q = \sum_{i=1}^n \frac{2s_i\lambda_i H}{\alpha_i^2}$. To construct a model of the LS-type one has to find an appropriate (linear) combinations $\epsilon_\pm$ of the G_n step operators of Q-grade ± 1, such that Q and $\epsilon_\pm$ closes an $SL(2,R)$ algebra. The $\epsilon_\pm$ of the $G_n^{(1,1)}$- models closes an $SL(2,R)$ algebra not with Q but with $Q' = Q + \mathcal{H}_0^0$, ($\mathcal{H}_0^0 = \beta\lambda_1 H, [\epsilon_\pm, \mathcal{H}_0^0] = 0$), i.e. $[\epsilon_+, \epsilon_-] == Q + \mathcal{H}_0^0$. The equivalence of the $G_n^{(1,1)}$-NA Toda models with certain $H_- \setminus G_n/H_+(\epsilon_\pm)$-WZW coset models raises the question: What is the gauged WZW model describing the NA-Toda theories of the LS-type? One has to repeat literally the construction presented in this section using the corresponding (LS) $\epsilon_\pm$'s and as $H_\pm$ – the positive (negative) Q-grade subgroups of G_n.

The main advantage of the description of the NA-Toda models as G/H-WZW models is that it provides a powerful tool for the construction of conserved currents, for the derivation of their V_n-algebras (see Sect 3) as well as for their quantization (see Sect 9).

3 Conserved Currents and $V_{n+1}^{(1,1)}$-algebras

We start our analysis of the symmetries of the $G_n^{(1,1)}$-NA-Toda theories (2.18) with the construction of the improved (classical) stress-tensor $T_{\mu\nu}$ and the global $U(1)$ current J_μ. Since the action (2.18) is manifestly translation, Lorentz and dilation invariant, the corresponding $T_{\mu\nu}$ is conserved, symmetric and traceless and its two nonvanishing (chiral) components $T(z)$ and $\bar{T}(\bar{z})$ are given by

$$T(z) = \frac{1}{2}\eta_{ik}\partial\phi_i\partial\phi_k + \sum\frac{2}{\alpha_i^2}\partial^2\phi_i + \frac{\partial\chi\partial\psi}{\Delta}e^{k_{12}\phi_1} + \gamma\partial(\frac{\psi\partial\chi}{\Delta}e^{k_{12}\phi_1}) \tag{3.1}$$

where $\gamma = \sum_{i=1}^{n-1}\tilde{\mathcal{K}}_{1,i}$ (i.e. $\gamma_{A_n} = \frac{n-1}{2}, \gamma_{B_n} = n-1, \gamma_{C_n} = n - \frac{3}{2}$). $\bar{T}$ have the same form with $\partial, \psi, \chi \longrightarrow \bar{\partial}, \chi, \psi$. Thus our NA-Toda models (2.18) are indeed conformally invariant.

Another evident symmetry of the action (2.18) is under global $U(1)$ gauge transformations: $\phi'_i = \phi_i$, $\psi' = \psi e^{-\alpha}$ and $\chi' = \chi e^{\alpha}$ (α is a constant). The corresponding (*nonchiral*) $U(1)$ current derived from (2.18) is of the form

$$J_\mu = -(\frac{k}{4\pi})\frac{e^{k_{12}\phi_1}}{\Delta}(\psi\partial_\mu\chi - \chi\partial_\mu\psi - k_{12}\psi\chi\partial_\mu\phi_1) \tag{3.2}$$

and its conservation reads $\partial\bar{J} + \bar{\partial}J = 0$, where $J = \frac{1}{2}(J_0 - J_1)$ and $\bar{J} = \frac{1}{2}(J_0 + J_1)$.

Similarly to the abelian Toda case [20] one might expect that the NA-Toda theories obey a larger set of symmetries generated by certain higher spin conserved currents. This is indeed the case, however it is rather difficult to derive them from the effective action (2.18). A powerful and systematic method of exhausting all the symmetries of the models (2.18) is based on their equivalence with specific H-reduced G_n-WZW models as we have shown in Sect.2. Since the equations of motion of WZW theory ($\bar{\partial}J_\alpha = \partial\bar{J}_\alpha = 0$) reduced by the constraints (2.4) do coincide with the NA-Toda field equations (2.3), the remaining unconstrained WZW currents appears as the conserved currents for the $G_n^{(1,1)}$-NA-Toda theories as well. The recipe for deriving the explicit form of these currents in terms of fields χ, ψ and ϕ_i consist in the following; i) choose an independent set of remaining currents by gauge fixing the residual gauge symmetries of (2.4); ii) realize all WZW currents in the g-parametrization, $g = g_- g_0^f g_+$ (see Sect. 2) and next solve the constraints (and gauge fixings conditions) against the physical fields χ, ψ and ϕ_i; iii) substitute these solutions in the remaining currents.

Let us first find an independent set of remaining currents and calculate their (improved) spins. As we have shown in Sect.2 the constraints (2.4) remains invariant under the group $H_+^L \otimes H_-^R$. This allows us to choose a specific DS-type gauge [9] such that the only independent nonvanishing remaining current are:

$$J_{-\alpha_1} = V^+, \quad J_{\alpha_1+\alpha_2+\cdots\alpha_n} = V^- \quad J_{\alpha_n+\alpha_{n-1}+\cdots\alpha_k} = W^A_{n-k+2} \tag{3.3}$$

for A_n and

$$\begin{array}{rcl} J_{-\alpha_1} & = & V^+, \qquad J_{\alpha_1+2\alpha_2+\cdots 2\alpha_n} = V^- \\ J_{\alpha_k+2\alpha_{k+1}+\cdots 2\alpha_n} & = & W^B_{2(n-k+1)}, \quad k = 2, 3, \cdots, n-1, \quad J_{\alpha_n} = W_2 \end{array} \tag{3.4}$$

for B_n and C_n.

For each chiral sector, the gauge fixing condition for the simple root constraints $J_{-\alpha_i} = 1$, $(i \neq 1)$ requires $J_{\frac{2\lambda_i H}{\alpha_i^2}} = 0$. In order to make these constraints consistent with the conformal invariance (including Lorentz), we have to improve the WZW-stress tensor T_{WZW} by

$$\tilde{T} = T_{WZW} + \sum_{i=2}^{n} \partial J_{\frac{2\lambda_i H}{\alpha_i^2}} \tag{3.5}$$

such that the spin of $J_{-\alpha_i}$ with respect to $\tilde{T}$ is zero. Since $J_{-\alpha_1}$ is unconstrained we can add a term proportional to $\partial J_{\lambda_1 H}$ i.e.

$$T = \tilde{T} + X\partial J_{\lambda_1 H} \tag{3.6}$$

The condition that fixes X comes from the consistency of the conformal transformation of $J_{\lambda_1 H}$ generated by T,

$$\{T(\sigma), J_{\lambda_1 H}(\sigma')\} = J_{\lambda_1 H}(\sigma')\delta'(\sigma-\sigma') + \partial_{\sigma'} J_{\lambda_1 H}(\sigma')\delta(\sigma-\sigma') + (X\mathcal{K}_{11} + \sum_{i=2}^{n}\mathcal{K}_{1i})\delta''(\sigma-\sigma')$$

with the constraint $J_{\lambda_1 H} = 0$, i.e.

$$X = -\frac{1}{\mathcal{K}_{11}}\sum_{i=2}^{n}\mathcal{K}_{1i} \tag{3.7}$$

Then the improved spins of the remaining currents J_α (α being one of the roots appearing in (3.3), (3.4),) is given by,

$$s(\alpha) = 1 + X\lambda_1\alpha + \sum_{i=2}^{n} 2\frac{\lambda_i\alpha}{\alpha_i^2} \tag{3.8}$$

For the $A_n^{(1,1)}$-NA-Toda models we have $X = -\frac{(n-1)}{2}$ and eqns. (3.6) gives

$$s^- = s(-\alpha) = \frac{n+1}{2} \qquad s^+ = s(\alpha_1+\alpha_2+\cdots\alpha_n) = \frac{n+1}{2} \tag{3.9}$$

In the B_n case we find $X = 1-n$ and rescaling that $\lambda_1 = \alpha_1+\alpha_2+\cdots+\alpha_n$ we obtain

$$\begin{aligned} s^- &= s(-\alpha_1) = n \qquad s^+ = s(\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_n) = n \\ s_2 &= s(\alpha_n) = 2, \qquad s_k = s(\alpha_k+2\alpha_{k+1}+\cdots+2\alpha_n) = 2(n-k+1) \end{aligned}$$

The same is true for C_n-NA-Toda theories.

We have to note that the PF-type constraint $J_{\lambda_1 H} = \bar{J}_{\lambda_1 H} = 0$ results in a system of differential equations for the field R,

$$\partial R = \frac{e^{k_{12}\phi_1}\psi\partial\chi}{\Delta}, \qquad \bar{\partial} R = \frac{e^{k_{12}\phi_1}\chi\bar{\partial}\psi}{\Delta} \tag{3.10}$$

Therefore the elimination of R by solving (3.10) introduce certain nonlocal terms (of the type $e^{\alpha R}$) in the part of the remaining currents. To construct the conserved currents for $G_n^{(1,1)}$-NA-Toda for generic n is rather cumbersome task. There exist however few exceptions when we can easely perform all the calculations for arbitrary G_n. These are the two (simple root) nonlocal currents $V^+ = J_{-\alpha_1}$ and $\bar{V}^- = \bar{J}_{\alpha_1}$,

$$V^+(z) = \frac{k}{2}\frac{e^{k_{12}\phi_1+\frac{1}{2\mathcal{K}_{11}}R}\partial\chi}{\Delta}, \qquad \bar{V}^-(\bar{z}) = \frac{k}{2}\frac{e^{k_{12}\phi_1+\frac{1}{2\mathcal{K}_{11}}R}\bar{\partial}\psi}{\Delta} \tag{3.11}$$

and the chiral components of the stress tensor $T(z) = J_{\alpha_n}$, and $\bar{T}(\bar{z}) = \bar{J}_{-\alpha_n}$ which indeed coincides with the improved stress tensor (3.1), derived directly from (2.18). As in the case of the abelian Toda theories [20], [9] the higher spin currents ($s \geq 3$) have quite complicated form and the knowledge of their explicit form is not necessary in the derivation of the complete algebra of symmetries. Nevertheless we present here few examples for the $A_n^{(1,1)}$-NA-Toda ($n = 1, 2$) mainly concerning the rest of the nonlocal currents $V^-(z)$, $\bar{V}^+(\bar{z})$. Their explicit form happens to be important for the derivation of the $SL(2,R)_q$ algebra of symmetries in Sect.5. For $n = 1$ the remaining nonlocal currents of spin 1 have a form even simpler than (3.11),

$$V^-_{n=1} = (\frac{k}{2})e^{-R}\partial\psi, \qquad \bar{V}^+_{n=1} = (\frac{k}{2})e^{-R}\bar{\partial}\chi \tag{3.12}$$

The full set of conserved currents in the A_2 case contains together with (3.1), (3.2) and (3.11) two extra nonlocal currents of spin 3/2,

$$V^-_{n=2} = (\frac{k}{2})^2 e^{-\frac{3}{4}R}\left(\partial^2\psi + \frac{1}{16}\psi(\partial R)^2 - \psi(\partial\phi_1)^2 - \psi\partial^2\phi_1 - \frac{1}{4}\psi\partial^2 R - \frac{1}{2}\partial\psi\partial R\right) \tag{3.13}$$

and $\bar{V}^+_{n=2} = V^-_{n=2}(\psi \to \chi, \partial \to \bar{\partial})$.

We now come to the *main problem* addressed in this paper: *to derive the complete algebra of the symmetries of the $G_n^{(1,1)}$-NA-Toda theories given by the action (2.18).* As we have shown above this algebra is generated by the $n+1$ chiral $V^\pm, T, W_p^{(G)}$ and $n+1$ antichiral $\bar{V}^\pm, \bar{T}, \bar{W}_p^{(G)}$ conserved currents (3.3), (3.4), etc together with the nonchiral $U(1)$ current $(J(z,\bar{z}), \bar{J}(z,\bar{z}))$. Given the explicit form of the conserved currents, the standard method for deriving their algebra consists in realizing them in terms of the fields χ, ψ and ϕ_i, their conjugate momenta (obtained from (2.18)) and their space derivatives. From the canonical PB's one can, in *principle*, calculate the algebra we seek. This method of calculating is known to be difficult and cumbersome, even for the simple cases $n = 1, 2, 3$ where all necessary ingredients are at hand. Fortunately a short cut exists and is given by a simple procedure proposed by Polyakov [7] for deriving the W_3 algebra from the constraint $SL(3,R)$ current algebra transformations. This method does not require any knowledge of the explicit form of the currents. We are going to demonstrate now how it works in our case of parafermionic H-reduction of the G_n-WZW model described in Sect.2. The starting point are the G_n current algebra infinitesimal transformations

$$\delta J = [\epsilon, J] - \frac{k}{2}\partial\epsilon \tag{3.14}$$

with $J = \sum_{all\ roots} J_{\{\alpha\}}E_{\{\alpha\}} + \sum J_i H_i$ and the same decomposition for the chiral parameters $\epsilon(z) = \sum_{all\ roots} \epsilon_{\{\alpha\}}E_{\{\alpha\}} + \sum \epsilon_i H_i$. We next substitute the constraints (2.4) and the gauge fixing condition (in the DS gauge) in (3.14) and further require that the remaining gauge transformation (generated by $V^\pm$, T and $W_p^{(G)}$)to leave (2.4) invariant. As a result, (3.14) gives rise to a system of $n+1$ linear algebraic equations and $n^2 - 2$ first order differential equations for the redundant ϵ's we have to solve in terms of the independent parameters $\epsilon^\pm$, ϵ, η_p and the currents $V^\pm$, T, $W_p^{(G)}$. The remaining $n+1$ equations of (3.14) represent the effective transformation laws of $V^\pm$, T, $W_p^{(G)}$ we are looking for. It becomes evident from the explicit form of eqs (3.14) that our system of differential equations, is diagonal on the derivatives and that the first equation (the coefficient of H_1) splits from the others. Its integration is straightforward for generic G_n,

$$\epsilon_1 = \frac{1}{2k^2}\int \epsilon(\sigma - \sigma')(\epsilon^+ V^- - \epsilon^- V^+)(\sigma')d\sigma' \tag{3.15}$$

It gives rise to the only nonlocal quadratic terms in the $VG_n^{(1,1)}$ algebra. The integration of the rest is reasonably simple but the explicit form of the recursive relations to be solved depends heavely on the G_n algebra in consideration. This is why we choose the simplest case of PF reduction of the A_n– current algebra in order to demonstrate our method in solving the system of algebraic and differential equations (eqn. (3.14) reduced by the constraints (2.4) in the DS gauge). It is convenient to write eqn. (3.14) in the Cartan- Weyl basis and realize the A_n generators E_α , H_i in terms of $(n+1)\times(n+1)$ matrices $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$: $H_i = E_{ii}$, $E_\alpha = E_{ij}$, $(i<j)$, $E_{-\alpha} = E_{ij}$, $(i>j)$, $i,j = 1,\cdots n+1$ and $\sum_{i=1}^{n+1} H_i = 0$. The constraints (2.4) takes the form

$$J_{i,i-1} = 1, \quad i = 3,\cdots n+1 \qquad J_{ij} = 0, \quad i > j \neq i-1, \qquad J_{11} = 0 \tag{3.16}$$

the remaining currents are then given as

$$J_{2,1} = V^+, \qquad J_{1,n+1} = V^-, \qquad J_{p,n+1} = W_{n-p+2}, \qquad p = 2,\cdots,n \tag{3.17}$$

and all other elements J_{ii} ,$i = 2,\cdots,n+1$, J_{kl}, $(k < l \neq n+1)$ are zero in the DS–gauge. Apart from the equations for the transformations of currents (3.17) the rest of the system (3.14) can be written in the following matrix form

$$-\frac{k}{2}\partial\epsilon_{ik} = J_{ij}\epsilon_{jk} - \epsilon_{ij}J_{jk} \tag{3.18}$$

(the equation for $(ik) = \{(21),(p,n+1),(1,n+1)\}$ excluded). We next choose the independent parameters to be $\epsilon^- = \epsilon_{12}$, $\epsilon^+ = \epsilon_{n+1,1}$ for the transformations generated by the nonlocal currents $V^\pm$, $\epsilon = \epsilon_{n+1,n}$ for conformal transformation $(T = W_2)$ and $\eta_{n-p+2} = \epsilon_{n+1,p}$ the transformation generated by the highest spin currents W_{n-p+2}, $p = 2,\cdots n-1$. The problem is to solve eqs. (3.18) for all others ϵ_{ik} in terms of $\epsilon^\pm$, ϵ and η_p and $V^\pm$, T and W_p. We first derive the *conformal transformations* $\delta_\epsilon V^\pm|_{\eta_p=\epsilon^\pm=0}$, etc , setting $\eta_p = \epsilon^\pm = 0$ in eqn. (3.18). In this case all ϵ_{ik}'s for $i > k$ satisfy simple algebraic equations and their solution is

$$\epsilon_{i,i-1} = \epsilon \quad i = 3,\cdots,n+1, \qquad \epsilon_{21} = \epsilon V^+, \qquad \epsilon_{ik} = 0 \;\; i > k \neq i-1 \tag{3.19}$$

The diagonal elements ϵ_{ii}, $i = 1,\cdots n+1$ with $\sum_{i=1}^{n+1}\epsilon_{ii} = 0$ satisfy the following system of differential recursive relations

$$\epsilon_{ii} - \epsilon_{i-1,i-1} = \frac{k}{2}\partial\epsilon, \quad i = 3,\cdots,n$$
$$-\epsilon_{n,n} - \sum_{l=1}^{n}\epsilon_{l,l} = \frac{k}{2}\partial\epsilon, \quad \epsilon_{11} = 0$$

The solution is

$$\epsilon_{ss} = \frac{(2s-n-3)}{2}(\frac{k}{2})\partial\epsilon, \quad s = 2,\cdots n \quad \epsilon_{11} = 0 \tag{3.20}$$

We next consider the equations for the upper triangular part of ϵ_{ik} and find that all elements of the first row, $\epsilon_{1,s}$ vanish except the last one,

$$\epsilon_{1,s} = 0, \quad s = 2,\cdots,n; \quad \epsilon_{1,n+1} = \epsilon V^- \tag{3.21}$$

The recursive relations for $\epsilon_{l,l+1}$, $l = 1, \cdots, n$ are of the form

$$\epsilon_{l,l+1} - \epsilon_{l-1,l} = \frac{k}{2}\partial\epsilon_{ll} + \epsilon T\delta_{l,n}$$

and can be easely solved by taking

$$\epsilon_{l,l+1} = \frac{k}{2}\sum_{s=2}^{l}\partial\epsilon_{ss} + \epsilon T\delta_{l,n} = \frac{(l-1)}{2}(l-1-n)(\frac{k}{2}\partial)^2\epsilon + \epsilon T\delta_{l,n} \qquad (3.22)$$

Similarly for $\epsilon_{l,l+2}$ we get for $l = 2, \cdots, n-1$

$$\epsilon_{l,l+2} = \frac{k}{2}\sum_{s=2}^{l}\partial\epsilon_{s,s+1} + \epsilon W_3\delta_{l,n-1} = (\frac{k}{2})^2\sum_{s_1=2}^{l}\sum_{s_2=2}^{s_1}\partial^2\epsilon_{s_2,s_2} + \epsilon W_3\delta_{l,n-1}$$

and for generic $\epsilon_{l,l+m}$, $\;l = 2, \cdots n-m+1;\;\; m = 1, \cdots, n-1$,

$$\epsilon_{l,l+m} = \frac{k}{2}\sum_{s=2}^{l}\partial\epsilon_{s,s+m-1} + \epsilon W_{m+1}\delta_{l,n-m+1} \qquad (3.23)$$

Taking into account (3.20) and the well known multiple sum formula,

$$\sum_{s_1=1}^{l-1}\sum_{s_2=1}^{s_1}\cdots\sum_{s_{m-1}=1}^{s_{m-2}} s_{m-1} = \frac{(l+m-2)}{m!(l-2)!} = \left(\begin{array}{c} l+m-2 \\ m \end{array}\right) \qquad (3.24)$$

we derive the explicit form for $\epsilon_{l,l+m}$,

$$\epsilon_{l,l+m} = \epsilon W_{m+1}\delta_{l,n-m+1} + \left(\begin{array}{c} l+m-2 \\ m \end{array}\right)\{\frac{l+m-1}{m+1} - \frac{n+1}{2}\}(\frac{k}{2}\partial)^{m+1}\epsilon \qquad (3.25)$$

$l = 2, \cdots n-m+1, \;\; m = 1, \cdots n-1$. The eqs. (3.19), (3.20), (3.22) and (3.25) give the general solution for eqn. (3.18) for the case $\epsilon^{\pm} = \eta_p = 0$, $p = 3, \cdots n$. The remaining part of eqns. (3.14) represent the effective infinitesimal transformation for $V^{\pm}$, $T = W_2$ and W_p we are looking for, i.e.

$$\begin{aligned}
\delta_\epsilon V^{+} &= \epsilon_{22}V^{+} - \frac{k}{2}\partial\epsilon_{21}, \qquad \delta_\epsilon V^{-} = V^{-}\sum_{s=1}^{n}\epsilon_{ss} - \frac{k}{2}\partial\epsilon_{1,n+1}, \\
\delta_\epsilon W_s &= \epsilon W_{s+1} - (s-1)W_s(\frac{k}{2}\partial\epsilon) + \sum_{p=1}^{s-2}\epsilon_{n-s+2,n-s+p+2}W_{s-p} - \epsilon_{n-s+1,n+1} \\
&- \frac{k}{2}\partial\epsilon_{n-s+2,n+1} \qquad (3.26)
\end{aligned}$$

Substituting the explicit form of the redundant ϵ'_{ik} in (3.26) and rescaling the conformal parameter $\epsilon = -\frac{2}{k}\tilde{\epsilon}$ we obtain the desired conformal transformations

$$\delta V^{\pm} = \frac{n+1}{2}V^{\pm}\partial\tilde{\epsilon} + \tilde{\epsilon}\partial V^{\pm}$$

$$\begin{aligned}\delta W_s &= sW_s\partial\tilde{\epsilon}+\tilde{\epsilon}\partial W_s-(-\frac{k}{2})^s\begin{pmatrix} n-1 \\ s-1\end{pmatrix}\frac{n(n+1)(s-1)}{2s(s+1)}\partial^{s+1}\tilde{\epsilon} \\ &- \sum_{p=1}^{s-2}\begin{pmatrix} n-s+p \\ p\end{pmatrix}[\frac{n-s+p+1}{p+1}-\frac{n+1}{2}]W_{s-p}(\frac{k}{2})^p\partial^{p+1}\tilde{\epsilon}, \end{aligned}\qquad (3.27)$$

$s=2,3,\cdots n$, $W_2=T$. Note that the nonhomogeneous conformal transformations of W_s, $s=3,\cdots n$, (the last two terms in δW_s) reflect the fact that we are working in the DS gauge, where the W_s are not primary fields with respect to $T=W_2$. One can find an appropriate "gauge transformation" that maps the DS gauge in a gauge where all $\tilde{W}_s$ are primary fields (see sect. 4 of ref. ([25]) for the A_3-case). For example $\tilde{W}_3=W_3-\frac{n-2}{2}\partial T$ is primary field. The construction of primary $\tilde{W}_s$ for $s>4$ requires further investigation.

In order to find the transformations generated by the nonlocal currents $V^{\pm}$ we have to solve eqns. (3.18) for the particular case where $\epsilon=\eta_p=0$, leaving this time, $\epsilon^{\pm}$ unconstrained. Following the same strategy we first consider the equations for the lower triangular part of ϵ_{ik}, $i>k$. Starting with the n^{th} row we find

$$\epsilon_{n-k,s}=0, \quad s=2,\cdots n-k-1; \quad k=0,1,\cdots n-3 \qquad (3.28)$$

and the following recurrence relations for $\epsilon_{n-s,1}$

$$\epsilon_{n-s,1}+\epsilon^+W_{s+1}+\frac{k}{2}\partial\epsilon_{n-s+1,1}=0, \quad \epsilon_{n,1}=-\frac{k}{2}\partial\epsilon^+ \qquad (3.29)$$

$s=1,2,\cdots n-2$. The solution of (3.29) is given by

$$\epsilon_{l,1}=\sum_{s=0}^{n-l-1}(-1)^s(\frac{k}{2}\partial)^s(\epsilon^+W_{n-l-s+1})+(-\frac{k}{2}\partial)^{n-l+1}\epsilon^+, \quad l=2,\cdots n \qquad (3.30)$$

For the diagonal elements $\epsilon_{i,i}$ we get

$$\begin{aligned}\sum_{i=1}^{n}\epsilon_{i,i} &+ \epsilon_{n,n}=0, \quad \epsilon_{2,2}=\cdots=\epsilon_{3,3}=\epsilon_{n,n} \\ \frac{k}{2}\partial\epsilon_{1,1} &+ \epsilon^+V^--\epsilon^-V^+=0\end{aligned}$$

and therefore ($\partial=2\partial_\sigma$, $\bar{\partial}\epsilon_{1,1}=0$)

$$\begin{aligned}\epsilon_{11}(\sigma) &= \frac{1}{2k}\int\epsilon(\sigma-\sigma')[\epsilon^-(\sigma')V^+(\sigma')-\epsilon^+(\sigma')V^-(\sigma')]d\sigma' \\ \epsilon_{s,s} &= -\frac{1}{n}\epsilon_{1,1}, \quad s=2,\cdots n\end{aligned}\qquad (3.31)$$

From the upper triangular part of eqn. (3.18) we derive

$$\begin{aligned}\epsilon_{1,l} &= (\frac{k}{2}\partial)^{l-2}\epsilon^-, \quad l=2,\cdots n+1 \\ \epsilon_{l,l+1} &= \frac{n-l+1}{n}\epsilon^-V^++\frac{l-1}{n}\epsilon^+V^-, \quad l=2,\cdots,n \\ \epsilon_{l,l+2} &= \frac{k}{2}(\partial\epsilon^-)V^++\frac{k}{2}(l-1)\partial(\epsilon^-V^+)+\frac{l(l-1)}{2n}\frac{k}{2}\partial(\epsilon^+V^--\epsilon^-V^+)\end{aligned}\qquad (3.32)$$

$l = 2, \cdots, n-1$, and for the generic $\epsilon_{l,l+m}$ the recursive relation is

$$\epsilon_{l,l+m} = \epsilon_{1,m+1} V^+ + \sum_{s=1}^{l-1} (\frac{k}{2}\partial) \epsilon_{s+1,s+m}, \quad m = 2, \cdots, n+1-l \tag{3.33}$$

The solution of eqn. (3.33) can be written in the following compact form

$$\begin{aligned}\epsilon_{l,l+m} &= \frac{1}{n} \begin{pmatrix} l+m-3 \\ m-1 \end{pmatrix} (\frac{k}{2}\partial)^{m-1} [(n - \frac{l+m-2}{m}) \epsilon^- V^+ + \frac{l+m-2}{m} \epsilon^+ V^-] \\ &+ \sum_{s=0}^{m-2} \begin{pmatrix} l+s-2 \\ s \end{pmatrix} (\frac{k}{2}\partial)^s (V^+ (\frac{k}{2}\partial)^{m-s-1} \epsilon^-), \quad m = 1, \cdots, n+1-l \end{aligned} \tag{3.34}$$

The $\epsilon^\pm$-transformation laws for $V^\pm$, T and W_s derived from (3.15) are

$$\begin{aligned}\delta V^+ &= \epsilon_{2,2} V^+ - \epsilon_{1,1} V^+ - \epsilon^+ W_n - \frac{k}{2} \partial \epsilon_{2,1} \\ \delta V^- &= (\epsilon_{1,1} + \sum_{i=1}^{n} \epsilon_{i,i}) V^- + \sum_{s=2}^{n} \epsilon_{1,s} W_{n-s+2} - \frac{k}{2} \partial \epsilon_{1,n+1} \\ \delta W_p &= \epsilon_{n-p+2,1} V^- + \sum_{s=n-p+3}^{n} \epsilon_{n-p+2,s} W_{n-s+2} - \epsilon_{n-p+1,n+1} - \frac{k}{2} \partial \epsilon_{n-p+2,n+1} \end{aligned} \tag{3.35}$$

$p = 2, \cdots, n$. Taking into account the explicit form (3.30), (3.31), (3.32) and (3.34) of the ϵ's that contribute to (3.35) we find the transformations generated by the nonlocal currents $V^\pm$ to be in the form ($\tilde{\epsilon}^\pm = -\frac{k}{2}\epsilon^\pm$):

$$\begin{aligned}\delta_{\epsilon^\pm} V^+ &= -\frac{(n+1)}{nk^2} \int \epsilon(\sigma - \sigma') [\tilde{\epsilon}^+(\sigma') V^-(\sigma') - \tilde{\epsilon}^-(\sigma') V^+(\sigma')] V^+(\sigma) d\sigma' \\ &+ \sum_{s=0}^{n-2} (\frac{k}{2})^{s-1} (-\partial)^s (\tilde{\epsilon}^+ W_{n-s}) - (\frac{k}{2})^{n-1} (-\partial)^n \tilde{\epsilon}^+ \\ \delta_{\epsilon^\pm} V^- &= \frac{(n+1)}{nk^2} \int \epsilon(\sigma - \sigma') [\tilde{\epsilon}^+(\sigma') V^-(\sigma') - \tilde{\epsilon}^-(\sigma') V^+(\sigma')] V^-(\sigma) d\sigma' \\ &- \sum_{s=0}^{n-2} (\frac{k}{2})^{s-1} W_{n-s} (\partial)^s \tilde{\epsilon}^- + (\frac{k}{2})^{n-1} (\partial)^n \tilde{\epsilon}^- \\ \delta_{\epsilon^\pm} T &= \frac{n+1}{2} V^- \partial \tilde{\epsilon}^+ + \frac{n-1}{2} \tilde{\epsilon}^+ \partial V^- + \frac{n+1}{2} V^+ \partial \tilde{\epsilon}^- + \frac{n-1}{2} \tilde{\epsilon}^- \partial V^+ \\ \delta_{\epsilon^+} W_p &= V^- [\sum_{s=0}^{p-3} (-1)^s (\frac{k}{2})^{s-1} \partial^s (\tilde{\epsilon}^+ W_{p-s-1}) + (-\frac{k}{2})^{p-2} \partial^{p-1} \tilde{\epsilon}^+] \\ &+ \frac{n-1}{np} \begin{pmatrix} n-2 \\ p-1 \end{pmatrix} (\frac{k}{2})^{p-2} \partial^{p-1} (\tilde{\epsilon}^+ V^-) + \frac{n-1}{n(p-1)} \begin{pmatrix} n-2 \\ p-2 \end{pmatrix} (\frac{k}{2})^{p-2} \partial^{p-1} (\tilde{\epsilon}^+ V^-) \\ &- \sum_{s=n-p+3}^{n} W_{n-s+2} \begin{pmatrix} s-3 \\ s+p-n-3 \end{pmatrix} \frac{s-2}{n(s+p-2-n)} (\frac{k}{2})^{s+p-n-4} \partial^{s+p-n-3} (\tilde{\epsilon}^+ V^-) \end{aligned}$$

$$\begin{aligned}
\delta_{\epsilon^-} W_p &= \sum_{s=0}^{p-2} \begin{pmatrix} n-p+s-1 \\ s \end{pmatrix} \frac{2}{k}(\frac{k}{2}\partial)^s [V^+ (\frac{k}{2}\partial)^{p-s-1} \tilde{\epsilon}^-] \\
&+ \frac{2}{nk}[(n - \frac{n-1}{p}) \begin{pmatrix} n-2 \\ p-1 \end{pmatrix} + (n - \frac{n-1}{p-1}) \begin{pmatrix} n-2 \\ p-2 \end{pmatrix}](\frac{k}{2}\partial)^{p-1}(\tilde{\epsilon}^- V^+) \\
&+ \frac{2}{k}\sum_{s=0}^{p-3} \begin{pmatrix} n-p+s \\ s \end{pmatrix} (\frac{k}{2}\partial)^{s+1}[V^+(\frac{k}{2}\partial)^{p-s-2}(\tilde{\epsilon}^- V^+)] \\
&- \frac{2}{k}\sum_{s=n-p+3}^{n} W_{n-s+2}\{ \sum_{l=0}^{s+p-n-4} \begin{pmatrix} n-p+l \\ l \end{pmatrix} (\frac{k}{2}\partial)^l [V^+(\frac{k}{2}\partial)^{s+p-n-l-3}\tilde{\epsilon}^-] \\
&+ \begin{pmatrix} s-3 \\ s+p-n-3 \end{pmatrix} (1 - \frac{s-2}{n(s+p-n-2)})(\frac{k}{2}\partial)^{s+p-n-3}(\tilde{\epsilon}^- V^+)\} \qquad (3.36)
\end{aligned}$$

We next consider the transformations generated by W_3 taking $\epsilon^+ = \epsilon = 0$, $\eta_p = 0$ for $p = 4, \cdots, n$ leaving $\eta_3 \equiv \eta$ as a free parameter in eqns. (3.18). Solving eqn. (3.18) for $i \geq k$ we find

$$\begin{aligned}
\epsilon_{n-k,s} &= 0, \quad k = 0, \cdots, n-4; \quad s = 1, \cdots, n-k-3 \\
\epsilon_{l,l-2} &= \eta, \quad l = 4, \cdots, n; \quad \epsilon_{3,1} = \eta V^+ \\
\epsilon_{l,l-1} &= (l-n-1)\frac{k}{2}\partial\eta, \quad l = 3, \cdots n; \quad \epsilon_{2,1} = -(n-1)V^+\frac{k}{2}\partial\eta - \frac{k}{2}\eta\partial V^+ \\
\epsilon_{1,1} &= 0, \quad \epsilon_{l,l} = -\frac{2}{n}\eta T + \{\frac{(n-1)(n-2)}{3} - \frac{(l-2)(2n-l-1)}{2}\}(\frac{k}{2}\partial)^2\eta, \quad l = 2, \cdots, n-1 \\
\epsilon_{n,n} &= \frac{n-2}{n}\eta T - \frac{(n-2)(n-1)}{6}(\frac{k}{2}\partial)^2\eta \qquad (3.37)
\end{aligned}$$

The solution of eqn. (3.18) for $i < k$ is given by

$$\begin{aligned}
\epsilon_{1,l} &= 0 \quad l = 2, \cdots, n-1; \quad \epsilon_{1,n} = \eta V^-, \quad \epsilon_{1,n+1} = \frac{k}{2}\partial(\eta V^-) \\
\epsilon_{l,l+m} &= \eta W_{m+2}\delta_{n,l+m} + (\eta W_{m+2} + (\frac{k}{2}\partial)(\eta W_{m+1}))\delta_{n+1,l+m} - \frac{2}{n}\begin{pmatrix} l+m-2 \\ m \end{pmatrix}(\frac{k}{2}\partial)^m(\eta T) \\
&+ \{\begin{pmatrix} l+m \\ m+2 \end{pmatrix} + \frac{(n^2-1)}{3}\begin{pmatrix} l+m-2 \\ m \end{pmatrix} - n\begin{pmatrix} l+m-1 \\ m+1 \end{pmatrix}\}(\frac{k}{2}\partial)^{m+2}\eta \\
& m = 1, \cdots, n-1, \quad l = 2, \cdots, n-m+1 \qquad (3.38)
\end{aligned}$$

where we denote $V^+V^- = W_{n+1}$ in order to include the case $\epsilon_{2,n+1}$ in the general formula for $\epsilon_{l,l+m}$. The corresponding W_3-*transformation* of the currents $V^\pm$, $T = W_2$, W_p calculated by substituting (3.37) and (3.38) in (3.14) has the form

$$\delta_\eta V^+ = -\frac{2}{n}\eta T V^+ + \frac{(n^2-1)}{3}V^+(\frac{k}{2}\partial)^2\eta + \eta(\frac{k}{2}\partial)^2 V^+ + n(\frac{k}{2}\partial\eta)(\frac{k}{2}\partial V^+)$$

$$\delta_\eta V^- = \frac{2}{n}\eta T V^- + \frac{(n+1)(n-4)}{6}V^-(\frac{k}{2}\partial)^2\eta - \eta(\frac{k}{2}\partial)^2 V^- - 2(\frac{k}{2}\partial\eta)(\frac{k}{2}\partial V^-)$$

$$
\begin{aligned}
\delta_\eta W_p ={}& -(p+1)(\frac{k}{2}\partial\eta)W_{p+1} - 2\eta(\frac{k}{2}\partial)W_{p+1} - (\frac{k}{2}\partial)^2(\eta W_p) + \frac{(p-1)(p-2)}{2}W_p(\frac{k}{2}\partial)^2\eta \\
&+\frac{2}{n}\binom{n}{p}(\frac{k}{2}\partial)^p(\eta T) - \{\binom{n+2}{p+2} + \frac{(n^2-1)}{3}\binom{n}{p} - n\binom{n+1}{p+1}\}(\frac{k}{2}\partial)^{p+2}\eta \\
&+\sum_{s=1}^{p-2} W_{p-s}\{-\frac{2}{n}\binom{s+n-p}{s}(\frac{k}{2}\partial)^s(\eta T) + [\binom{s+n-p+2}{s+2} + \frac{n^2-1}{3}\binom{s+n-p}{s} \\
&-n\binom{s+n-p+1}{s+1}](\frac{k}{2}\partial)^{s+2}\eta\} + \delta_{n,p}(\frac{k}{2}\partial\eta)W_{p+1}
\end{aligned}
\tag{3.39}
$$

Following the same procedure we find the transformations generated by W_n,

$$
\begin{aligned}
\delta_{\eta_n}V^+ &= -\frac{1}{n}V^+[(-\frac{k}{2}\partial)^{n-1}\eta_n - \sum_{s=1}^{n-2}(-\frac{k}{2}\partial)^{s-1}(\eta_n W_{n-s}] + (-\frac{k}{2}\partial)^{n-1}(\eta_n V^+) \\
&+ \sum_{l=1}^{n-1}(-\frac{k}{2}\partial)^{n-l-1}\{V^+[(-\frac{k}{2}\partial)^l\eta_n - \sum_{s=1}^{l-1}(-\frac{k}{2}\partial)^{s-1}(\eta_n W_{l-s+1})]\} \\
\delta_{\eta_n}V^- &= \frac{1}{n}V^-\{(-\frac{k}{2}\partial)^{n-1}\eta_n\} - (-\frac{k}{2}\partial)^{n-1}(\eta_n V^-) \\
&+ \sum_{s=1}^{n-2} W_{n-s}[(\frac{k}{2}\partial)^{s-1}\eta_n V^-] - \frac{1}{n}V^-\sum_{s=1}^{n-2}(-\frac{k}{2}\partial)^{s-1}(\eta_n W_{n-s}) \\
\delta_{\eta_n}W_p &= \epsilon_{n-p+2,1}V^- + \epsilon_{n-p+2,2}W_n + \sum_{s=1}^{p-2}\epsilon_{n-p+2,s+n-p+2}W_{p-s} - \epsilon_{n-p+1,n+1} \\
&- \frac{k}{2}\partial\epsilon_{n-p+2,n+1}
\end{aligned}
\tag{3.40}
$$

where

$$
\begin{aligned}
\epsilon_{n-p+2,1} &= (-\frac{k}{2}\partial)^{p-2}(\eta_n V^+) + \\
&+ \sum_{l=1}^{p-2}(-\frac{k}{2}\partial)^{p-l-2}[V^+\left((-\frac{k}{2}\partial)^l\eta_n - \sum_{s=1}^{l-1}(-\frac{k}{2}\partial)^{s-1}(\eta_n W_{l-s+1})\right)] \\
\epsilon_{n-p+2,2} &= (-\frac{k}{2}\partial)^{p-1}\eta_n - \sum_{s=1}^{p-2}(-\frac{k}{2}\partial)^{s-1}(\eta_n W_{p-s+2}) \\
\epsilon_{n-p+2,s+n-p+2} &= \binom{s+n-p-1}{s-1}(\frac{k}{2}\partial)^{s-1}(\eta_n W_n) \\
&+ \sum_{l=1}^{s-1}(\frac{k}{2}\partial)^{s-l-1}[V^+(\frac{k}{2}\partial)^{l-1}(\eta_n V^-)]\binom{s+n-p-l-1}{s-l-1} \\
&+ (-1)^s[\binom{s+n-p-1}{s-1} - \frac{1}{n}\binom{s+n-p}{s}][\sum_{l=1}^{n-2}(-\frac{k}{2}\partial)^{l+s-1}(\eta_n W_{n-l})
\end{aligned}
$$

$$\begin{aligned}
&- \quad (-\frac{k}{2}\partial)^{n+s-1}\eta_n] \\
\epsilon_{n-p+1,n+1} &+ \quad \frac{k}{2}\partial\epsilon_{n-p+2,n+1} = (\frac{k}{2}\partial)^{p-1}(\eta_n W_n)\left(\begin{array}{c} n-1 \\ p-1 \end{array}\right) + V^+[(\frac{k}{2}\partial)^{p-2}(\eta_n V^-)] \\
&+ \quad (-1)^p[n\left(\begin{array}{c} n-1 \\ p-1 \end{array}\right) - \left(\begin{array}{c} n \\ p \end{array}\right)][(-\frac{k}{2}\partial)^{p+n-1}\eta_n - \sum_{s=1}^{n-2}(-\frac{k}{2}\partial)^{p+s-1}(\eta_n W_{n-s})] \\
&+ \quad \sum_{s=1}^{p-2}(\frac{k}{2}\partial)^{p-s-1}[V^+(\frac{k}{2}\partial)^{s-1}(\eta_n V^-)]\left(\begin{array}{c} n-s-1 \\ p-s-1 \end{array}\right).
\end{aligned}$$

The derivation of the remaining W_p-transformations for $3 < p \le n-1$ is more complicated and is presented in the appendix B.

In order to find the explicit form of the classical PB's algebra $V_{n+1}^{(1,1)}$ generated by the $V^\pm$, T, W_p we have to remember the standard relation between the infinitesimal transformations and the currents PB's :

$$\delta_{\epsilon^\pm} J(w) = \int dz \epsilon^\pm(z)\{V^\mp(z), J(w)\} \tag{3.41}$$

where J is any of the currents $V^\pm$, T, W_p, $p = 3, \cdots, n$. Starting from (3.36) we easely derive the algebra of the nonlocal currents

$$\begin{aligned}
\{V^\pm(\sigma), V^\pm(\sigma')\} &= -\frac{n+1}{nk^2}\epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') \\
\{V^+(\sigma), V^-(\sigma')\} &= \frac{n+1}{nk^2}\epsilon(\sigma-\sigma')V^+(\sigma)V^-(\sigma') + (\frac{k}{2})^{n-1}\partial_{\sigma'}^n\delta(\sigma-\sigma') \\
&- \sum_{s=0}^{n-2}(\frac{k}{2})^{s-1}W_{n-s}(\sigma')\partial_{\sigma'}^s\delta(\sigma-\sigma')
\end{aligned} \tag{3.42}$$

In the simplest case, $n = 1$ (A_1-model) the full $V_2^{(1,1)}$ algebra is spanned by $V^\pm$ (of spin $\frac{n+1}{2} = 1$) only. It turns out that this nonlocal V_2 algebra coincides with the semiclassical limit of the Fattev-Zamolodchikov PF- algebra [3] studied in ref. [38] (see also our eqns. (1.4), (1.5)). The algebra $V_3^{(1,1)}$ of the symmetries of A_2 NA Toda model is related to the semi-classical limit of the Polyakov-Bershadsky $W_3^{(2)}$ algebra [26], but with the local $U(1)$ current gauged away, i.e. one additional constraint $J_{\lambda_1 H} = 0$ is imposed in the corresponding reduction of the A_2 WZW model. The $V_3^{(1,1)}$ algebra is a PF–type extension of the Virasoro algebra

$$\{T(\sigma), T(\sigma')\} = 2[\partial_{\sigma'}\delta(\sigma-\sigma')]T(\sigma') + \delta(\sigma-\sigma')\partial_{\sigma'}T(\sigma') - \frac{k^2}{2}\partial_{\sigma'}^3\delta(\sigma-\sigma') \tag{3.43}$$

with two spins $s = 3/2$ (nonlocal) currents

$$\{T(\sigma), V^\pm(\sigma')\} = s[\partial_{\sigma'}\delta(\sigma-\sigma')]V^\pm(\sigma') + \delta(\sigma-\sigma')\partial_{\sigma'}V^\pm(\sigma') \tag{3.44}$$

The PB's of the $V^\pm$ in this case are given by (3.42) with $n = 2$. This algebra is quite similar to the semi–classical limit of the $N = 2$ superconformal algebra.

The $V_4^{(1,1)}$ algebra of symmetries of A_3-NA-Toda theory provides an interesting example of new type of mixed parafermionic-W_3-algebra. It represents *a nonlocal and nonlinear (non–Lie)* extension of the Virasoro algebra (3.43) with two spins $s = 2$ nonlocal currents $V^{\pm}$ and one local spin $s = 3$, $\omega_3 = W_3 - \partial_\sigma T$. Together with (3.43) (with central charge $-2k^2$) and (3.42), it contains two new PB's,

$$\begin{aligned}\{\omega_3(\sigma), V^{\pm}(\sigma')\} &= \mp\frac{5k}{3}V^{\pm}(\sigma')\partial^2_{\sigma'}\delta(\sigma-\sigma') \mp \frac{5k}{2}[\partial_{\sigma'}\delta(\sigma-\sigma')]\partial_{\sigma'}V^{\pm}(\sigma') \\ &\pm \delta(\sigma-\sigma')\left(\frac{2}{3k}TV^{\pm} - k\partial^2_{\sigma'}V^{\pm}(\sigma')\right) \end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\{\omega_3(\sigma), \omega_3(\sigma')\} = 4\left(V^+V^- + \frac{1}{6}T^2\right)(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + 2\delta(\sigma-\sigma')\partial_{\sigma'}(V^+V^- + \frac{1}{6}T^2) \\ -\frac{k^2}{6}\delta(\sigma-\sigma')\partial^3_{\sigma'}T - \frac{3k^2}{4}[\partial_{\sigma'}\delta(\sigma-\sigma')]\partial^2_{\sigma'}T - \frac{5k^2}{4}[\partial^2_{\sigma'}\delta(\sigma-\sigma')]\partial_{\sigma'}T - \\ -\frac{5k^2}{6}T(\sigma')\partial^3_{\sigma'}\delta(\sigma-\sigma') + \frac{k^4}{6}\partial^5_{\sigma'}\delta(\sigma-\sigma') \end{aligned} \tag{3.46}$$

The eqns. (3.44), (3.45) and (3.46) are derived from the infinitesimal transformations (3.39) taking into account that we have introduced the primary field ω_3 instead of W_3 (from the D-S gauge). It is straightforward to write the PB's $\{W_{p_1}, W_{p_2}\}$ and $\{W_{p_1}, V^{\pm}\}$ for the arbitrary $V_{n+1}^{(1,1)}$ algebra. As we have mentioned they are encoded on the corresponding infinitesimal transformations (B.17),(B.18) and and (3.36). In the nonprimary basis (DS-gauge) for W_p's the algebra looks rather complicated.

The method we have used in the derivation of the $V_{n+1}^{(1,1)}$ works equally well for arbitrary G_n. The explicit construction of the solutions of eqns.(3.14) far say, B_n, C_n or D_n however requires a bit more work.

In order to demonstrate that the $V_n^{(1,1)}$ algebra is in fact the algebra of symmetries of the $G_n^{(1,1)}$-NA-Toda model (2.18) we need to know the transformations for the fields ψ, χ and ϕ_i generated by the currents $V^{\pm}$, T and W_p. We apply once more the Polyakov method [7], this time imposing all constraints (3.16) in the chiral A_n-gauge transformations of the WZW field g: $\delta g_{ik} = -g_{il}(z,\bar{z})\epsilon_{lk}$. We have already calculated the redundant gauge parameters. What is still missing is the reduced form of g_{ik}. Using the explicit solutions of the constraint equations (3.18) we find

$$\begin{aligned} g_{11} &= e^R; \quad g_{22} = e^{\phi_1 - \frac{1}{n}R}(1 + \chi\psi e^{-\phi_1}); \qquad g_{2l} = (\frac{k}{2}\partial)^{l-2}g_{22}; \quad g_{l2} = (\frac{k}{2}\bar{\partial})^{l-2}g_{22}, \cdots \\ g_{1l} &= (\frac{k}{2}\partial)^{l-2}(e^{\frac{n-1}{2n}R}\psi); \quad g_{l1} = (\frac{k}{2}\bar{\partial})^{l-2}(e^{\frac{n-1}{2n}R}\chi), \cdots, etc \end{aligned} \tag{3.47}$$

We are now able to write the field transformations generated by $V^{\pm}$

$$\begin{aligned} \delta_{\epsilon^{\pm}} g_{11} = -g_{11}\epsilon_{11} - \sum_{l=2}^{n+1} g_{1l}\epsilon_{l1}; \qquad \delta_{\epsilon^{\pm}} g_{12} = \frac{1}{k}g_{11}\epsilon^- - g_{12}\epsilon_{22}; \\ \delta_{\epsilon^{\pm}} g_{22} = \frac{1}{k}g_{21}\epsilon^- - g_{22}\epsilon_{22}; \qquad \delta_{\epsilon^{\pm}} g_{21} = -g_{21}\epsilon_{11} - \sum_{l=2}^{n+1} g_{2l}\epsilon_{l1}, \end{aligned} \tag{3.48}$$

etc. All ϵ's in (3.48) are given by eqns. (3.30), (3.31) and (3.34) and are indeed linear functions of $\epsilon^{\pm}$. The corresponding conformal transformations take the following simple form

$$\begin{aligned} \delta_\epsilon \psi &= \frac{1-n}{2}\psi\partial\epsilon + \epsilon\partial\psi, \quad \delta_\epsilon\chi = \epsilon\partial\chi, \quad \delta_\epsilon R = \epsilon\partial R \\ \delta_{\bar\epsilon}\psi &= \bar\epsilon\bar\partial\psi; \quad \delta_{\bar\epsilon}\chi = \frac{1-n}{2}\chi\bar\partial\bar\epsilon + \bar\epsilon\bar\partial\chi, \quad \delta_{\bar\epsilon}R = \bar\epsilon\bar\partial R \\ \delta_\epsilon\phi_l &= \frac{l(l-n)}{2}\partial\epsilon + \epsilon\partial\phi_l; \quad \delta_{\bar\epsilon}\phi_l = \frac{l(l-n)}{2}\bar\partial\bar\epsilon + \bar\epsilon\bar\partial\phi_l \end{aligned} \tag{3.49}$$

$l = 1, \cdots n-1$.The eqns. (3.49) show that ψ and χ are primary conformal fields of spin $s = \Delta - \bar\Delta$ and dimension $d = \Delta + \bar\Delta$: $(s_\psi, d_\psi) = (\frac{1-n}{2}, \frac{1-n}{2})$ and $(s_\chi, d_\chi) = (\frac{1-n}{2}, \frac{1-n}{2})$. For the vertices e^{ϕ_l} we have $(s_l, d_l) = (0, l(l-n))$. The non-local field e^R is spinless and dimensionless.

One can further calculate the corresponding W_p transformations of ψ, χ and ϕ_l taking into account the explicit form of eqns. (3.37), (3.38),(B.15) and (B.17) of the ϵ_{ik}'s in terms of η_p and $V^{\pm}$, T , W_p. Consider, for example, W_3 transformations ($\eta_3 = \eta$)

$$\begin{aligned} \delta_\eta e^R &= (n-1)\partial\eta\partial e^R - 2\eta[\tilde T - \frac{(n-1)}{2}\partial^2 R + \frac{(n-1)}{2}(\partial R)^2 - \frac{1}{2}\partial^2]e^R, \\ \delta_\eta\psi &= -\frac{(n-1)}{2n}\psi\partial\eta\partial R + (n-2)\partial\eta\partial\psi + \frac{1}{n}\eta[(n+1)T - (n-1)\tilde T - \frac{(n^2-1)}{2}\partial^2 R \\ &- \frac{(n^2-1)}{4n}(\partial R)^2 - (n-1)\partial R\partial - n\partial^2]\psi - \frac{(n-1)(n-2)}{3}\psi\partial^2\eta \\ \delta_\eta\chi &= -\frac{(n-1)}{2n}\chi[(n-1)\partial\eta\partial R - 2\eta(\tilde T - \frac{n}{2}\partial^2 R - \frac{1}{2n}(\partial R)^2] + \\ &+ (n-1)\partial\eta(e^{\phi_1} + \psi\chi)e^{-\frac{(n+1)}{2n}R} - \eta e^{-\frac{(n+1)}{2n}R}\partial(e^{\phi_1 - \frac{1}{n}R} + e^{-\frac{1}{n}R}\psi\chi) \end{aligned} \tag{3.50}$$

etc. , where $\tilde T = T - \sum_{i=1}^{n-1}[\partial\phi_i\partial\phi_i - \frac{1}{2}\partial\phi_i\partial\phi_{i-1} - \frac{1}{2}\partial\phi_i\partial\phi_{i+1} + \partial^2\phi_i]$ and T is given by (3.1).

Although the field equations of the $A_n^{(1,1)}$-NA-Toda model (2.18),

$$\begin{gathered} \partial\bar\partial\phi_i = (\frac{2}{k})^2 e^{\phi_{i+1}+\phi_{i-1}-2\phi_i} - \frac{(n-i)}{n}e^{-\phi_i}\frac{\partial\chi\bar\partial\psi}{\Delta^2} \\ \bar\partial\left(\frac{\partial\chi e^{-\phi_1}}{\Delta}\right) = -\frac{(n+1)}{2n}\frac{\chi\partial\chi\bar\partial\psi e^{-2\phi_1}}{\Delta^2}, \quad \partial\left(\frac{\bar\partial\psi e^{-\phi_1}}{\Delta}\right) = -\frac{(n+1)}{2n}\frac{\psi\partial\chi\bar\partial\psi e^{-2\phi_1}}{\Delta^2} \end{gathered} \tag{3.51}$$

are by construction invariant under all the $V_n^{(1,1)}$-transformations, the proof of the invariance of the action (2.18) is rather complicated. There are however, few exceptions. The reparametrization (conformal) invariance of (2.18) is straightforward, due to the simple form of the conformal transformations (3.49). We next verify the invariance of (2.18) under non-local transformations (3.48), generated by V^+ (and $\bar V^-$),

$$\begin{aligned} \delta_{\epsilon^-}\psi &= \frac{1}{k}e^{\frac{(n+1)}{2n}R}\epsilon^- + \frac{1}{2k}\frac{(n+1)}{2n}\psi\int\epsilon(\sigma-\sigma')\epsilon^-(\sigma')V^+(\sigma')d\sigma' \\ \delta_{\epsilon^-}\chi &= -\frac{1}{2k}\frac{(n+1)}{2n}\chi\int\epsilon(\sigma-\sigma')\epsilon^-(\sigma')V^+(\sigma')d\sigma' \\ \delta_{\epsilon^-}\phi_i &= 0 \end{aligned} \tag{3.52}$$

Using the definition (3.10) of the nonlocal field R and the fact that $\tilde{\epsilon}_{11} = \epsilon_{11}|_{\epsilon^+=0}$:

$$\frac{k}{2}\partial\tilde{\epsilon}_{11} = \epsilon^- V^+, \quad \bar{\partial}\epsilon_{11} = 0$$

(see eqn. (3.31)) we find

$$\delta_{\epsilon^-} S^{NA}_{A_n^{(1,1)}} = \frac{1}{2\pi}\frac{(n+1)}{2n}\int d^2z\bar{\partial}(V^+\epsilon^- R) = 0$$

Similarly for $\bar{V}^-$-transformation we obtain

$$\delta_{\bar{\epsilon}^+} S^{NA}_{A_n^{(1,1)}} = -\frac{1}{2\pi}\frac{(n+1)}{2n}\int d^2z\partial(\bar{V}^-\bar{\epsilon}^+ R) = 0$$

The remaining nonlocal transformations (generated by V^- and $\bar{V}^+$) are not as simple as (3.52) and the check of the invariance of (2.18): $\delta_{\epsilon^+} S^{NA}_{A_n^{(1,1)}} = \delta_{\bar{\epsilon}^-} S^{NA}_{A_n^{(1,1)}} = 0$ is more complicated. The explicit proof of the W_p invariance of $S^{NA}_{A_n^{(1,1)}}$ is still lacking, except in the simplest case $n = 3$, i.e. the A_3-NA-Toda model.

4 $SL(2,R)_q$ Symmetries

The appearence of *nonlocal currents* in the theory is always an indication of the existence of some underlying *quantum group structure* (see ref. [27], [29]). We shall demonstrate that the charges of the *chiral* nonlocal currents $Q^+ = \int V^+ d\sigma$ and $Q^+ = \int \sigma^{n-1}V^- d\sigma$ have nonvanishing PB's with the *antichiral* nonlocal charges $\bar{Q}^- = \int V^- d\sigma$ and $\bar{Q}^+ = \int \sigma^{n-1}\bar{V}^+ d\sigma$ and together with the nonchiral $U(1)$ charge $Q_0 = \int J_0 d\sigma$ (see eqn. (3.2)) they generate a *q-deformed affine* $SL(2,R)$ *PB's algebra.* The PB's of the charges of *chiral local* currents (T, W_p) with the charges of the *antichiral* ($\bar{T}$, $\bar{W}_p$) do indeed *vanish.* The presence of the $\hat{S}L(2,R)_q$ *Poisson bracket algebra* as Noether symmetry of the classical $G_n^{(1,1)}$-NA-Toda theory is one of the basic features of these models.

We begin with the PB's algebra of $V^+(z)$ and $\bar{V}^-(\bar{z})$. Using the explicit form of the conjugate momenta Π_ψ and Π_χ (derived from ((2.18), $\Pi_\rho = \frac{\delta\mathcal{L}}{\delta\partial_0\rho}$, $\rho = \psi, \chi$), we eliminate the time derivatives from (3.11) and (3.10)

$$\begin{aligned} V^+ &= (\frac{k}{2})\left(\chi' + \frac{1}{2}k_{12}\chi\phi_1' - (\frac{2\pi}{k})\Pi_\psi e^{k_{12}\phi_1}\Delta\right)\frac{e^{k_{12}\phi_1 - \frac{1}{2\mathcal{K}_{11}}(\frac{2\pi}{k})R_0}}{\Delta^{\frac{1}{2}}} \\ V^- &= (\frac{k}{2})\left(-\psi' - \frac{1}{2}k_{12}\psi\phi_1' - (\frac{2\pi}{k})\Pi_\chi e^{k_{12}\phi_1}\Delta\right)\frac{e^{k_{12}\phi_1 - \frac{1}{2\mathcal{K}_{11}}(\frac{2\pi}{k})R_0}}{\Delta^{\frac{1}{2}}}. \end{aligned} \tag{4.1}$$

For further convenience we have splited the nonlocal field R (defined by eqn. (3.10) in two parts

$$R = -\frac{2\pi}{k}R_0 + \mathcal{K}_{11}ln\Delta, \quad R_0' = \frac{1}{2}(\psi\Pi_\psi - \chi\Pi_\chi)$$

By simple manipulations involving the canonical equal time PB's,

$$\{\Pi_\psi(\sigma),\psi(\sigma')\}=-\delta(\sigma-\sigma'),\ \{\Pi_\chi(\sigma),\chi(\sigma')\}=-\delta(\sigma-\sigma'),\ \{\Pi_{\phi_i}(\sigma),\phi_j(\sigma')\}=-\delta_{ij}\delta(\sigma-\sigma')$$

(all other PB vanish) and their space derivatives we find

$$\begin{aligned}\{V^+(\sigma),\bar V^-(\sigma')\} &= -\frac{k\pi}{2}e^{k_{12}\phi_1(\sigma)+k_{12}\phi_1(\sigma')-\frac{1}{2\mathcal{K}_{11}}(\frac{2\pi}{k})(R_0(\sigma)+R_0(\sigma'))}[e^{-k_{12}\phi_1(\sigma')}(\frac{\Delta(\sigma')}{\Delta(\sigma)})^{\frac{1}{2}}\partial_\sigma\delta(\sigma-\sigma') \\ &+ e^{-k_{12}\phi_1(\sigma)}(\frac{\Delta(\sigma)}{\Delta(\sigma')})^{\frac{1}{2}}\partial_{\sigma'}\delta(\sigma-\sigma')-\frac{\partial_\sigma(e^{-k_{12}\phi_1}\Delta)}{\Delta}\delta(\sigma-\sigma')]\end{aligned}\tag{4.2}$$

Integrating (4.2) we get the PB's for the charges Q^+ and $\bar Q^-$

$$\{Q^+,\bar Q^-\}=\frac{k\pi}{2}\int_{-\infty}^{\infty}d\sigma\partial_\sigma e^{\frac{1}{\mathcal{K}_{11}}R+k_{12}\phi_1-ln\Delta}\tag{4.3}$$

One can simplify the r.h.s. of (4.3) taking into account the relation of the field in the exponent

$$\varphi=R+\mathcal{K}_{11}(k_{12}\phi_1-ln\Delta)\tag{4.4}$$

with the $U(1)$ current (3.2), namely

$$J_\mu=\frac{k}{2\pi}\epsilon_{\mu\nu}\partial^\nu(\varphi+k_{12}\mathcal{K}_{11}\phi_1)$$

Note that $I_\mu=-\frac{k}{2\pi}\mathcal{K}_{11}\epsilon_{\mu\nu}\partial^\nu k_{12}\phi_1$ is automatically conserved topologically current and its charge $\int I_0 d\sigma$ have vanishing PB's with either V^+ and $\bar V^-$. This fact suggests the following redefinition of the $U(1)$ charge

$$\begin{aligned}&H_1=Q_0-\int I_0 d\sigma=-\frac{k}{2\pi}(\varphi(\infty)-\varphi(-\infty))\\ &\{H_1,Q^+\}=Q^+,\quad \{H_1,\bar Q^-\}=-\bar Q^-\end{aligned}\tag{4.5}$$

and the nonlocal charges Q^+ and $\bar Q^-$ as well

$$E_1=\sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1+\hat\kappa}{2}}}{(q^2-1)^{\frac{1}{2}}}Q^+,\qquad F_1=\sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1+\hat\kappa}{2}}}{(q^2-1)^{\frac{1}{2}}}\bar Q^-\tag{4.6}$$

where $q_{(G_n)}=e^{-(\frac{2\pi}{k})\frac{1}{2\mathcal{K}_{11}}}$ and $\hat\kappa=-\frac{k}{2\pi}(\varphi(\infty)+\varphi(-\infty))$. As a consequence of the PB's of φ with V^+ and $\bar V^-$,

$$\{\varphi(\sigma),V^+(\sigma')\}=\frac{\pi}{k}V^+(\sigma')\epsilon(\sigma-\sigma');\quad \{\varphi(\sigma),\bar V^-(\sigma')\}=-\frac{\pi}{k}\bar V^-(\sigma')\epsilon(\sigma-\sigma')$$

we realize that $\hat\kappa$ has vanishing PB's with Q^+ and $\bar Q^-$.

The result of all this rearangements of the variables is that the algebra (4.3), (4.5) takes the standard form of the $q_{(G_n)}$-deformed $SL(2,R)$ algebra (for an arbitrary G_n):

$$\{E_1,F_1\}=\frac{q^{H_1}-q^{-H_1}}{q-q^{-1}},\qquad \{H_1,E_1\}=E_1,\qquad \{H_1,F_1\}=-F_1.\tag{4.7}$$

We should mention that the $U(1)$ charge Q_0 (or H_1) appears as a *topological* charge for the lagrangean derived from (2.18) by the familiar change of variables:

$$\psi = \sqrt{2\mathcal{K}_{11}}e^{-\frac{1}{2}k_{12}\phi_1-\theta}sh(r), \qquad \chi = \sqrt{2\mathcal{K}_{11}}e^{-\frac{1}{2}k_{12}\phi_1+\theta}sh(r)$$

The computation of the PB's algebra of the remaining nonlocal charges Q^- and $\bar{Q}^+$ (as well as the mixed PB's $\{Q^\pm, \bar{Q}^\pm\}$) is rather difficult problem even in the few cases ($n \leq 3$) we know their explicit form. The complications arise from the fact that the currents $V^-_{(n)}$ and $\bar{V}^+_{(n)}$ contain n^{th} order time derivatives and their elimination by using the field equations (2.3) is a cumbersome task even for $n = 3$. The simplest case $n = 1$ is an exception. The currents $V^-_{(1)}$ and $\bar{V}^+_{(1)}$ given by (3.12) and $V^+_{(1)}$ and $\bar{V}^-_{(1)}$ have a very similar form. The calculation of the PB's of the corresponding charges $Q^+, \bar{Q}^-$ is straightforward and yields

$$\{E_0, F_0\} = \frac{q^{H_0} - q^{-H_0}}{q - q^{-1}}, \qquad q = q_{(A_1)} = e^{\frac{2\pi}{k}} \tag{4.8}$$

where

$$E_0 = \sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1-\hat{\kappa}}{2}}}{(1-q^2)^{\frac{1}{2}}}Q^-, \qquad F_0 = \sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1-\hat{\kappa}}{2}}}{(1-q^2)^{\frac{1}{2}}}\bar{Q}^+, \qquad H_o = -H_1$$

The two remaining PB's vanish, i.e. $\{Q^\pm, \bar{Q}^\pm\} = 0$. One can write the PB's (4.7 and 4.8) in the following compact form

$$\begin{aligned} &\{H_i, E_j\} = \kappa_{ij}E_j \qquad \{H_i, F_j\} = -\kappa_{ij}F_j, \qquad i,j = 0,1 \\ &\{E_i, F_j\} = \delta_{ij}\frac{q_n^{H_i} - q_n^{-H_i}}{q_n - q_n^{-1}}, \qquad \kappa_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned} \tag{4.9}$$

which is known to be the centerless affine $SL(2,R)_q$ algebra in the principal gradation [27] (the Serre relations are omited). The question in order *is whether the algebra* (4.8) (and therefore the larger $\hat{S}L(2,R)_q$ (4.9)) *takes place for all* $G_n^{(1,1)}$*-NA- Toda models.* Starting from the explicit form of the currents $V^-_{(2)}$ and $\bar{V}^+_{(2)}$ (see eqn. (3.13)) and eliminating the 2^{nd} time derivative we verify that this is indeed the case for $A_2^{(1,1)}$-NA-Toda theory. This leads us to the following *conjecture*: *the affine* $SL(2,R)_q$ *PB algebra (4.9) appears to be an algebra of (Noether) symmetries of the classical* $G_n^{(1,1)}$*-NA-Toda models.*

At this stage one might wonder about the relation of the $\hat{S}L(2,R)_{q_{(G_n)}}$ PB's algebra with the Poisson-Lie group $G_n(r)$ (familiar from the G_n-WZW and abelian Toda models [8, 15]). One expect it to take place as a symmetry of the Poisson structure of $G_n^{(1,1)}$-NA-Toda theories as well. It is well known that the symplectic form of the G_n-WZW model is invariant with respect to (a) Loop group $\hat{G}_n$ generated by G_n current algebra and (b) The Poisson– Lie group $G_n(r)$ of the monodromy matrices $M \in G_n$ satisfying apart from the G_n-multiplication laws, the Sklyanin PB's algebra [28]

$$\{M(\sigma) \overset{\otimes}{,} M(\sigma')\} = -\frac{2\pi}{k}[r, M(\sigma) \otimes M(\sigma')] \tag{4.10}$$

as well. The G_n- albelian Toda theory realized as $H = \mathcal{N}_L \otimes \mathcal{N}_R$-reduced G_n-WZW model manifests similar properties. Its symplectic structure is invariant under the action of $G_n(r)$ and WG_n- algebras [13], [15], [18]. As we have shown in Sect 3. the Poisson structure of the $G_n^{(1,1)}$-NA-Toda (generated by the Hamiltonian derived from (2.18)) is invariant under the nonlocal algebra $VG_n^{(1,1)}$ and with respect to $\hat{S}L(2,R)_{q(G_n)}$ as well. In order to find the Poisson-Lie group of the $G_n^{(1,1)}$-NA-Toda models we have to construct the monodromy matrix, to calculate the corresponding classical r-matrix and to verify that eqn. (4.10) holds. As usual the monodromy matrix is defined as a solution of the linear problem

$$(\partial - \mathcal{A})M(\sigma,t) = (\bar{\partial} - \bar{\mathcal{A}})M(\sigma,t) = 0, \qquad \mathcal{A}_x = \frac{1}{2}(\mathcal{A} - \bar{\mathcal{A}}) \tag{4.11}$$

with $\mathcal{A}$ and $\bar{\mathcal{A}}$ given by eqn. (2.2). The solution of (4.11) normalized by the condition $M(-\infty, t_0) = 1$ can be written as the P-ordered exponential:

$$M(\sigma) = Pe^{\int_{-\infty}^{\sigma} A_x(\sigma')d\sigma'}. \tag{4.12}$$

We next realize A_x in terms of the fields ψ, χ, ϕ_i and their momenta $\Pi_\psi, \Pi_\chi, \Pi_\phi$ (derived from eq(2.18)),

$$\begin{aligned} A_x &= \frac{\pi}{k}\Big[\frac{1}{2\mathcal{K}_{11}}(\psi\Pi_\psi + \chi\Pi_\chi)\lambda_1 \cdot H + \frac{1}{2}\sum_{m,l=1}^{n-1} \tilde{\mathcal{K}}_{lm}\Pi_{\phi_l}\alpha_{m+1} \cdot H + \Pi_\chi e^{-\frac{1}{2}k_{12}\phi_1} E_{-\alpha_1} \\ &+ \Pi_\psi e^{-\frac{1}{2}k_{12}\phi_1} E_{\alpha_1} + \frac{k}{2\pi}\sum_{i=1}^{n-1} e^{-\frac{1}{2}\tilde{k}_{ij}\phi_j}(E_{\alpha_{i+1}} + E_{-\alpha_{i+1}})\Big] \end{aligned}$$

By straightforward calculation (similar to the abelian Toda case), we derive the so called Fundamental Poisson Brackets [16]:

$$\{A_x(\sigma) \overset{\otimes}{,} A_x(\sigma')\} = -\frac{2\pi}{k}[r, A_x(\sigma) \otimes I + I \otimes A_x(\sigma')]\delta(\sigma - \sigma') \tag{4.13}$$

where $r \in G_n \otimes G_n$ denote one of the solutions

$$r^+ = \frac{1}{4}(\sum_i H_{\alpha_i} \otimes H_{\alpha_i} + 2\sum_{\alpha>0} E_\alpha \otimes E_{-\alpha}), \quad r^- = -\frac{1}{4}(\sum_i H_{\alpha_i} \otimes H_{\alpha_i} + 2\sum_{\alpha>0} E_{-\alpha} \otimes E_\alpha)$$

of the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

Finally the Sklyanin relation (4.10) is a simple consequence of (4.12) and (4.13). The *conclusion* is that the Poisson-Lie group $G(r)_n$ generated by the monodromy matrices (4.12) of the $G_n^{(1,1)}$-NA-Toda model coincides with the one that appears in the G_n-WZW and the G_n-abelian Toda models. With the monodromy matrix at hand one can further define the group of (nonlocal) dressing transformations preserving the form of the Lax connection (2.2) and mapping one solution of the G_n-NA-Toda (with charge Q_0) into another (of charge $Q_0' \neq Q_0$)

(see [29] for the abelian Toda case). The *nonlocal transformations* (**??**) generated by currents $V^+, \bar{V}^-$ (and $\bar{V}^+, V^-$) share many of the properties of the *dressing transformations*. They leave invariant G_n-NA-Toda equations (2.3), preserve the form (2.2) of $\mathcal{A}$ and $\bar{\mathcal{A}}$ and transform solutions with Q_0 into solutions of charge $Q_0 \pm 1$. This is a hint that the $\hat{S}L(2,R)_q$ *PB's algebra* of the nonlocal (Noether) charges $Q^{\pm}, \bar{Q}^{\pm}, Q_0$ might appear as a *subalgebra* of the *dressing* Poisson algebra. However we have no proof of such statement. It requires the explicit construction of the dressing PB'S algebra for the $G_n^{(1,1)}$-NA-Toda theory.

In our derivation of the $\hat{S}L(2,R)_q$ algebra we left *unanswered the important question whether the equations (2.3) admit solutions such that* $\varphi(\infty, t_0) \neq \varphi(-\infty, t_0)$, i.e. $H_1 \neq 0$. In the next section we construct the general (exact) solutions of eqns. (2.3) and study their asymptotics.

5 General Solutions

The solutions of eqn. (2.3) can be found by direct application of the Leznov-Saveliev method [31]. As it is well known their explicit form contains multiple integrals and is not appropriate for the analysis of the asymptotics of a specific combination of these solutions such as φ defined by eqn. (4.4). For this purpose we need the NA-Toda analog of the Gervais-Bilal's solution [20] for the abelian Toda equations. It turns out that the NA-Toda fields $\psi, \chi, \phi_i, i = 1, \cdots, n-1$ can be realized in terms of the corresponding abelian Toda fields, $\varphi_A, A = 1, \cdots, n$ together with the *chiral* currents $V^+(z)$ and $\bar{V}^-(\bar{z})$ considered as *independent variables*. The exact *statement* is as follows: Let $\psi, \chi, \phi_i, i = 1, \cdots, n-1$ satisfy eqn (2.3), R is defined by eqn (3.10) and $V^+(z)$ and $\bar{V}^-(\bar{z})$ are given by (3.11). Then the fields

$$\varphi_i = \phi_{i-1} + \tilde{\mathcal{K}}_{1,i-1}\frac{\tilde{\alpha}_{i-1}^2}{2}R - \mathcal{K}_{1,i}\frac{\alpha_i^2}{2}lnV^+\bar{V}^-, \phi_0 = 0, \quad i = 1, \cdots n \tag{5.1}$$

with $\alpha_0^2 = 2$ and $\tilde{\mathcal{K}}_{1,0} = 1$, satisfy the G_n abelian Toda equations

$$\partial\bar{\partial}\varphi_i = (\frac{2}{k})^2 e^{-k_{ij}\varphi_j} \tag{5.2}$$

The proof is based on the following suggestive form of the first eqn. (2.3)

$$\partial\bar{\partial}\phi_i = (\frac{2}{k})^2 e^{-\tilde{k}_{il}\phi_l} - (\frac{2}{k})^2\tilde{\mathcal{K}}_{1,l}(\frac{\tilde{\alpha}_l^2}{2})e^{-k_{12}\phi_1 - \frac{1}{\mathcal{K}_{11}}R + lnV^+\bar{V}^-},$$

of the eqn (3.11) for the nonlocal field R :

$$\partial\bar{\partial}R = (\frac{2}{k})^2 e^{-k_{12}\phi_1 - \frac{1}{\mathcal{K}_{11}}R + lnV^+\bar{V}^-}$$

and on the following algebraic identities for the algebras A_n, B_n, C_n and D_n,

$$\frac{1}{\mathcal{K}_{11}} = k_{11} + \tilde{\mathcal{K}}_{11}k_{12}, \qquad (\frac{\alpha_j^2}{2})k_{ij}\mathcal{K}_{1j} = \delta_{1i},$$

$$k_{ij}(\frac{\tilde{\alpha}_{j-1}^2}{2})\tilde{\mathcal{K}}_{1,j-1} = 0, \qquad i = 2, \cdots n$$

Finally we observe that as a consequence of (3.10) and (3.11) the fields ψ and χ can be realized in terms of R, $V^+(z)$ and $\bar{V}^-(\bar{z})$ only:

$$\psi V^+ = (\frac{2}{k})e^{\frac{1}{2\mathcal{K}_{11}}R}\partial R, \quad \chi \bar{V}^- = (\frac{2}{k})e^{\frac{1}{2\mathcal{K}_{11}}R}\bar{\partial} R \tag{5.3}$$

The conclusion is that eqns. (5.1) and (5.2) allows us to write the solutions of (2.3) in terms of the solutions of eqns. (5.2).

We next take the general solutions of the G_n-abelian Toda eqns. (5.2) in the form proposed by Gervais and Bilal [20]

$$e^{\varphi_1} = (\frac{k}{2})^{-n}F_i\bar{F}_i, \cdots \quad e^{-k_{l-1,l}\varphi_l} = (\frac{k}{2})^{l(l-n-1)}\frac{1}{l!}F_{i_1,i_2,\cdots i_l}(z)\bar{F}_{i_1,\cdots i_l}(\bar{z}) \tag{5.4}$$

$l = 2, \cdots n; i_s = 1, \cdots, n+1$, where $F_{i_1,i_2,\cdots i_l}$ are rank l antisymmetric tensors (for example, $F_{i_1,i_2} = F_{i_1}F'_{i_2} - F'_{i_2}F_{i_1}$). The $(n+1)$-chiral functions F_i ($\bar{F}_i$) are not independent. The condition they have to satisfy comes from the last equation ($i = n$) of the system (5.2). For the A_n case it has the following form:

$$F_{i_1,\cdots i_{n-1}}\bar{F}_{i_1,\cdots i_{n-1}} = (F'_{i_1,\cdots i_n}\bar{F}'_{i_1,\cdots i_n})(F_{j_1,\cdots j_n}\bar{F}_{j_1,\cdots j_n}) - (F'_{i_1,\cdots i_n}\bar{F}_{i_1,\cdots i_n})(F_{j_1,\cdots j_n}\bar{F}'_{j_1,\cdots j_n})$$

which turns out to be equivalent to the Wronskian condition

$$\epsilon_{i_1,\cdots,i_{n+1}}F_{i_1}F^{(\prime)}_{i_2}\cdots F^{(n)}_{i_{n+1}} = \epsilon_{i_1,\cdots,i_{n+1}}\bar{F}_{i_1}\bar{F}^{(\prime)}_{i_2}\cdots \bar{F}^{(n)}_{i_{n+1}} = 1 \tag{5.5}$$

The correspondent requirement for B_n is

$$\begin{aligned} 2(F_{i_1,\cdots i_{n-1}}\bar{F}_{i_1,\cdots i_{n-1}})(F_{j_1,\cdots j_n}\bar{F}_{j_1,\cdots j_n}) &= (F'_{i_1,\cdots i_n}\bar{F}'_{i_1,\cdots i_n})(F_{j_1,\cdots j_n}\bar{F}_{j_1,\cdots j_n}) \\ &- (F'_{i_1,\cdots i_n}\bar{F}_{i_1,\cdots i_n})(F_{j_1,\cdots j_n}\bar{F}'_{j_1,\cdots j_n}) \end{aligned} \tag{5.6}$$

This way one can write the solutions of the $G_n^{(1,1)}$-NA Toda theory ϕ_i, ψ, χ in terms of the $n+1$-dependent functions F_i, ($\bar{F}_i$) and the chiral currents $V^+, \bar{V}^-$. It is convenient to introduce a new set of $n+1$ *independent* functions $f_i(z)$ and $\bar{f}_i(\bar{z})$

$$f_i = (V^+)^{\mathcal{K}_{11}}F_i, \quad \bar{f}_i = (\bar{V}^-)^{\mathcal{K}_{11}}\bar{F}_i, \quad f_{ij} = (V^+)^{2\mathcal{K}_{11}}F_{ij}, \quad \bar{f}_{ij} = (\bar{V}^-)^{2\mathcal{K}_{11}}\bar{F}_{ij}, \text{ etc.} \tag{5.7}$$

This change of variables is based on the following *observation*: let us shift the Toda fields φ_l as follows

$$\varphi'_l = \varphi_l + l\mathcal{K}_{11}lnV^+\bar{V}^-$$

where $l = 1, 2, \cdots, n$ for the A_n case and $l = 1, \cdots n-1$ for B_n. The last field φ'_n in B_n is given by

$$\varphi_n{}' = \varphi_n + \frac{n}{2}\mathcal{K}_{11}lnV^+\bar{V}^-$$

($\mathcal{K}_{11} = 1$ for B_n). Then the first $n-1$ of the new fields φ'_j, ($j = 1, \cdots n-1$) satisfy the same abelian Toda eqns. (5.2), while the last one becomes

$$\begin{aligned} \partial\bar{\partial}\varphi'_n &= (V^+\bar{V}^-)^n e^{-2\varphi'_n+\varphi'_{n-1}}, \text{for } A_n \\ \partial\bar{\partial}\varphi'_n &= (V^+\bar{V}^-)e^{-2\varphi'_n+\varphi'_{n-1}}, \text{for } B_n \end{aligned}$$

The general solutions are given again by eqn (5.4) (but now with $f_i, \bar{f}_i$, etc). The fact that the last equation of the system (5.2)has been modified leads to the evident changes in eqns. (5.5) and (5.6)

$$(V^+)^n = \epsilon_{i_1,\cdots,i_{n+1}} f_{i_1} \cdots f^{(n)}_{i_{n+1}}, \qquad (\bar{V}^-)^n = \epsilon_{i_1,\cdots,i_{n+1}} \bar{f}_{i_1} \cdots \bar{f}^{(n)}_{i_{n+1}} \tag{5.8}$$

for A_n and

$$2(V^+\bar{V}^-)(f_{i_1,\cdots i_n}\bar{f}_{i_1,\cdots i_n}) = (\epsilon_{i_1,\cdots,i_{n+1}} f_{i_1} \cdots f^{(n)}_{i_{n+1}})(\epsilon_{j_1,\cdots,j_{n+1}} \bar{f}_{j_1} \cdots \bar{f}^{(n)}_{j_{n+1}}) \tag{5.9}$$

for B_n.

Substituting (5.4) in (5.1) and (5.3) we obtain the general solutions of eqns. (2.3) in the following form[2]

$$e^R = (\frac{k}{2})^{-n} f_i \bar{f}_i, \qquad e^{-k_{12}\phi_1} = (\frac{k}{2})^{1-n} \frac{1}{2} f_{ij}\bar{f}_{ij}(f_i\bar{f}_i)^{\frac{1}{\mathcal{K}_{11}}-2}(V^+\bar{V}^-)^{-1},$$

$$\psi = (\frac{k}{2})^{\frac{1-n}{2}}(f_i\bar{f}_i)^{\frac{1}{2\mathcal{K}_{11}}-1}(f_i'\bar{f}_i)(V^+)^{-1}, \quad \chi = (\frac{k}{2})^{\frac{1-n}{2}}(f_i\bar{f}_i)^{\frac{1}{2\mathcal{K}_{11}}-1}(f_i\bar{f}_i')(\bar{V}^-)^{-1}, \text{ etc.} \tag{5.10}$$

We are now able to write the explicit solution for the field φ given in eqn. (4.4)

$$\varphi = -\mathcal{K}_{11} \ln\{(\frac{k}{2})\frac{(f_i'\bar{f}_i')(f_j\bar{f}_j) + (\frac{1}{2\mathcal{K}_{11}} - 1)(f_i'\bar{f}_i)(f_j\bar{f}_j')}{(f_i\bar{f}_i)^2 V^+\bar{V}^-}\} \equiv -\mathcal{K}_{11} lnG \tag{5.11}$$

whose asymptotics are under investigation. The scaling properties of the function G suggest to look for a class of solutions such that $V^+(z)\bar{V}^-(\bar{z}) = A(t)$ i.e. its fixed time limit for $\sigma \longrightarrow \pm\infty$ are constants, $A_+ = A_- = A(t=0)$. This leads to the following ansatz:

$$f_i = \alpha_i e^{a_i(t+\sigma)}, \quad \bar{f}_i = \bar{\alpha}_i e^{a_i(t-\sigma)} \tag{5.12}$$

Taking into account the explicit form (5.8),(5.12) of $V^+\bar{V}^-$ we conclude that the requirement $V^+(z)\bar{V}^-(\bar{z}) = A(t)$ is satisfied only if

$$\sum_{i=1}^{n+1}(a_i - \bar{a}_i) = 0 \qquad for \;\; A_n \tag{5.13}$$

and

$$a_i = \bar{a}_i \quad , i = 1, 2, \cdots, n+1 \qquad for \;\; B_n \tag{5.14}$$

For the A_n case it is convenient to parametrize the solutions of (5.13) as follows

$$a_1 - \bar{a}_1 = b_1 + b_2 + \cdots + b_n, \qquad a_l - \bar{a}_l = -b_{l-1}, \;\; l = 2, 3, \cdots n+1$$

and for simplicity we specify $b_n > b_{n-1} > \cdots > b_1 > 0$. Then taking the limits $\sigma \longrightarrow \pm\infty$ at $\tau = 0$ in eqn. (5.11) we find

$$\varphi(\infty, 0) = -\mathcal{K}_{11} ln(\frac{1}{2\mathcal{K}_{11}} a_1\bar{a}_1 A(\frac{k}{2})), \quad \varphi(-\infty, 0) = -\mathcal{K}_{11} ln(\frac{1}{2\mathcal{K}_{11}} a_{n+1}\bar{a}_{n+1} A(\frac{k}{2}))$$

---

[2] We are sistematicaly omiting the explicit solutions for ϕ_1, $i = 2, \cdots n-1$ which contains rank $i-1$ antisymmetric tensors since we do not need them in the calculations of the asymptotics of φ.

Since $a_1\bar{a}_1 \neq a_{n+1}\bar{a}_{n+1}$, therefore $\varphi(\infty,0) \neq \varphi(-\infty,0)$, (i.e. $H_1 \neq 0$) for the class of solutions (5.12),(5.13) of $A_n^{(1,1)}$-model we have chosen. This makes complete our statement that $\hat{S}L(2,R)_q$ PB algebra appears as a symmetry of the $A_n^{(1,1)}$-NA Toda models.

The condition (5.14) for B_n-case gives $\varphi(\infty,0) = \varphi(-\infty,0)$ and therefore $H_1 = 0$ for the class of solutions (5.12). Whether there exist $B_n^{(1,1)}$-solutions such that $H_1 \neq 0$ is still an open question.

6 W_{n+1}-structures in NA-Toda models

The way we have constructed the solutions of the $G_n^{(1,1)}$-NA-Toda models, address the *question about the origin of the relation between the abelian and NA-Toda theories.* As we have mentioned in the introduction, the explanation of this phenomena can be found by realizing both as gauged G_n/H_i-WZW models, $(N^+ \setminus G_n/N^-)$ and $H^+ \setminus G_n/H^-$ and looking for G_n-gauge transformation $h = h(V^+) \bigotimes \bar{h}(\bar{V}^-) \in G_n \bigotimes G_n$ mapping H^{NA} in H^A. This transformation can be considered as a map between the constraints, gauge fixing conditions and the remaining currents of NA-Toda (1.11) into the corresponding ones of the abelian Toda (1.12). Therefore, it should satisfy the eqns. (1.13):

$$-J_{ij}^{(A)}H_{jk} + H_{ij}J_{jk}^{(NA)} = \frac{k}{2}\partial H_{ik}, \quad \bar{J}_{ij}^{(NA)}\bar{H}_{jk} - \bar{H}_{ij}\bar{J}_{jk}^{(A)} = \frac{k}{2}\bar{\partial}\bar{H}_{ik}, \tag{6.1}$$

where $H_{ik} = (h^{-1})_{ik}$, $\bar{H}_{ik} = (\bar{h}^{-1})_{ik}$ and, for simplicity, we are considering the A_n-case only (in the convenient Weyl basis –see eqns. (3.16), (3.17)).

In order to derive the solutions of eqn. (6.1), we apply once more the method we have used in Sect. 3. Due to the specific forms of J_{ij}^A and J_{ij}^{NA}, the eqn.(6.1) for $i > k$ $(i,k = 1,2,...,n+1)$ imply that

$$H_{ik} = 0, \quad i > k \tag{6.2}$$

i.e., $H = h^{-1}$ is an upper triangular matrix. For the diagonal elements H_{ii} we find

$$H_{ss} = H_{11}(V^+)^{-1}, \quad s = 2,3,...,n+1 \tag{6.3}$$

Imposing the condition $\det H = 1$:

$$\prod_{i=1}^{n+1} H_{ii} = 1 \tag{6.4}$$

(we have used (6.2) in deriving (6.4)), we find that

$$H_{ss} = e^{-\frac{1}{n+1}\Phi}, \qquad H_{11} = e^{\frac{n}{n+1}\Phi}, \qquad \Phi = \ln V^+ \tag{6.5}$$

$s = 2,3,...,n+1$, is the solution of eqns. (6.3) and (6.4). We next analize the equations for the elements of the first row H_{1k}:

$$H_{1k} = \frac{k}{2}\partial H_{1,k-1}, \qquad k = 3,4,...,n+1 \qquad H_{12}V^+ = \frac{k}{2}\partial H_{11}$$

The solutions of these recursive relations is given by

$$H_{1s} = -n\left(\frac{k}{2}\partial\right)^{s-1} e^{-\frac{1}{n+1}\Phi}, \qquad s = 2,3,...,n+1 \tag{6.6}$$

The elements $H_{l,l+1}$ satisfy more complicated recursive relations:

$$H_{l,l+1} = H_{l-1,l} + \frac{k}{2}\partial H_{ll}, \quad l = 2,3,...,n, \qquad H_{12} = \frac{k}{2}\partial H_{11} = -n\left(\frac{k}{2}\partial\right)H_{22},$$

which can be solved by taking

$$H_{l,l+1} = \sum_{s=1}^{l} H_{ss} = (l-1-n)\frac{k}{2}\partial\left(e^{-\frac{1}{n+1}\Phi}\right), \quad l = 2,3,\cdots,n. \tag{6.7}$$

Similarly, for generic $H_{l,l+m}$, we find

$$H_{l,l+m} = \frac{k}{2}\sum_{s=1}^{l}\partial H_{s,s+m-1}, \quad l = 1,2,...,n-m+1, \quad m = 1,2,...,n-1 \tag{6.8}$$

Taking into account (6.7) and (3.24), we obtain the solution of (6.8), in the form:

$$H_{l,l+m} = \left[\begin{pmatrix} l+m-1 \\ m \end{pmatrix} - (n+1)\begin{pmatrix} l+m-2 \\ m-1 \end{pmatrix}\right]\left(\frac{k}{2}\partial\right)^{m} e^{-\frac{1}{n+1}\Phi}. \tag{6.9}$$

The remaining part of the eqns. (6.1):

$$W^{(A)}_{n-l+2}H_{n+1,n+1} + H_{l-1,n+1} - \sum_{s=l}^{n} H_{ls}W^{(NA)}_{n-s+2} = -\frac{k}{2}\partial H_{l,n+1} \tag{6.10}$$

provides the explicit realization of the conserved currents of the *abelian* Toda theory $W^{(A)}_{n-l+2}$, in terms of the ones of the *NA-Toda* $W^{(NA)}_{n-s+2}$ and $V^{\pm}$. By means of (6.5) and (6.9), we can write (6.10) in the following compact form:

$$\begin{aligned} W^{A}_{n+1} &= V^{+}V^{-} + ne^{\frac{1}{n+1}\Phi}\left(\frac{k}{2}\partial\right)^{n+1} e^{-\frac{1}{n+1}\Phi} - n\sum_{s=2}^{n} W^{NA}_{n-s+2}e^{\frac{1}{n+1}\Phi}\left(\frac{k}{2}\partial\right)^{s-1} e^{-\frac{1}{n+1}\Phi}, \\ W^{A}_{n-l+2} &= W^{NA}_{n-l+2} - (\Gamma_{l-1,n-l+2} + \Gamma_{l,n-l+1})e^{\frac{1}{n+1}\Phi}\left(\frac{k}{2}\partial\right)^{n-l+2} e^{-\frac{1}{n+1}\Phi} \\ &+ \sum_{s=l+1}^{n} \Gamma_{l,s-l}W^{NA}_{n-s+2}e^{\frac{1}{n+1}\Phi}\left(\frac{k}{2}\partial\right)^{s-l} e^{-\frac{1}{n+1}\Phi}, \end{aligned} \tag{6.11}$$

for $l = 2,3,...,n$, where the coefficients $\Gamma_{l,m}$ are given by

$$\Gamma_{l,m} = \begin{pmatrix} l+m-1 \\ m \end{pmatrix} - (n+1)\begin{pmatrix} l+m-2 \\ m-1 \end{pmatrix}.$$

Let us give two explicit examples of the relations (6.11):

$$\begin{aligned} T^{(A)} &= T^{(NA)} - \left(\frac{k}{2}\right)^2 \frac{n}{2}\left(\partial^2\Phi - \frac{1}{n+1}(\partial\Phi)^2\right), \\ W_3^{(A)} &= W_3^{(NA)} + \frac{2}{n+1}T^{(NA)}\left(\frac{k}{2}\partial\right)\Phi + (3n-4)e^{\frac{1}{n+1}\Phi}\left(\frac{k}{2}\partial\right)^3 e^{-\frac{1}{n+1}\Phi}. \end{aligned}$$

(for $n = 2, W_3^{NA} = V^+V^-$). Comparing the form of the eqns. (6.1) for the chiral $H_{jk}(z)$ with the one for the antichiral $\bar{H}_{ik}(\bar{z})$, we conclude that

$$\bar{H}_{ik} = H_{ki}(\partial \to \bar{\partial}, \Phi = \ln V^+ \to \bar{\Phi} = ln\bar{V}^-).$$

The corresponding relations between the antichiral currents $\bar{W}_p^A$ and $\bar{W}_p^{NA}$, $\bar{V}^{\pm}$ have the same form as (6.11), with $\partial \to \bar{\partial}$ and $\Phi \to \bar{\Phi}$.

Our initial motivation of studying the solutions of eqns. (6.1) was to find an explanation of the change of variables (5.1), (5.2), that transforms part of the NA-Toda into the abelian Toda equations. Denoting by g_{ik}^{NA} the A_n-WZW field $g_{ik} \in A_n$, constrained by eqns. (3.16), and by g_{ik}^A, the reduced form of g_{ik} by the abelian Toda constraints (1.12), we realize that

$$g^{NA} = \bar{H}g^A H. \tag{6.12}$$

The explicit form of g_{ik}^{NA} is given by eqns. (3.47), while g_{ik}^A are known to be (see Sect. 2 of ref. [9])

$$\begin{aligned} &g_{11}^A = e^{\varphi_1^A}, \;\; g_{22}^A = e^{\varphi_2^A - \varphi_1^A} - e^{\varphi_1^A}(\frac{k}{2}\partial\varphi_1^A)(\frac{k}{2}\bar{\partial}\varphi_1^A), \\ &g_{1l}^A = \left(\frac{k}{2}\partial\right)^{l-1} e^{\varphi_1^A}, \qquad g_{l1}^A = \left(\frac{k}{2}\bar{\partial}\right)^{l-1} e^{\varphi_1^A}, \quad \ldots \quad \text{etc.} \end{aligned} \tag{6.13}$$

More generally $\phi_i^A = \ln D_i$, where D_i are certain subdeterminants fo the matrix g_{ik}^A [9]. With the explicit form of H and $\bar{H}$ at hand, we verify that (6.12) indeed reproduces eqns. (5.1).

The most *important consequence* of the H and $\bar{H}$-transformations (6.1) is the explicit realization (6.11) of the *abelian Toda conserved currents* W_p^A $(p = 2, 3, ..., n+1)$ in terms of the *NA-Toda currents* $V^{\pm}$, W_p^{NA} $(p = 2, 3, ..., n)$. As it is well known, the $W_p^{(A)}$'s generate the W_{n+1}-algebra [20]. On the other hand, we have shown in Sect. 3 that $V^{\pm}$ and W_p^{NA} are the generators of the *non-local (non-Lie) algebra* $V_{n+1}^{(1,1)}$, which is the algebra of the symmetries of the A_n-NA Toda theory. The eqns. (6.11) suggest that the *W_{n+1}-algebra [20] lies in the universal enveloping of the $V_{n+1}^{(1,1)}$-algebra*, i.e., the W_{n+1}-generators are specific combinations of certain products of the $V_{n+1}^{(1,1)}$-generators. Using the $V_{n+1}^{(1,1)}$-PB's only, we verify that for $n = 1, 2, 3$ the W_{n+1}-generators, constructed by the V_{n+1}-generators, according to the rule (6.11), indeed satisfy the standard W_{n+1}-PB's relations. The shortest way to prove this for arbitrary n is to derive the W_{n+1}-infinitesimal transformations $\delta_{\eta_p}W_{n-l+2}^{(A)}$ from the V_{n+1}-transformations $\delta_{\epsilon^\pm}W_p^{(NA)}$, $\delta_{\epsilon^\pm}V^{\pm}$, $\delta_{\eta_p}W_{n-l+2}^{(NA)}$, $\delta_{\epsilon^\pm}H_{ik}$ etc, solving explicitly the eqn. (6.10), written this time for the infinitesimal transformations. It is not difficult to verify that

$$\delta_{\epsilon^-}W_{n-l+2}^{(A)} = 0, \quad i.e. \quad \left\{V^+, W_{n-l+2}^{(A)}\right\} = 0,$$

and that $\delta_{\epsilon^+} W^{(A)}_{n-l+2} = \delta_{\eta^{(A)}} W^{(A)}_{n-l+2}$. However, the complete proof is still missing.

One might wonder whether these W_{n+1}-transformations that appear in the NA-Toda theories are in fact symmetries, i. e., whether the action (2.18) is invariant under the W_{n+1}-transformations of the NA-Toda fields ψ, χ, Φ_l. It is indeed the case, but again our proof is restricted to the particular cases $n = 1, 2, 3$.

7 $G_n^{(j,1)}$-NA-Toda Models

The $G_n^{(j,1)}$-NA-Toda models are straightforward generalization of the $G_n^{(1,1)}$-NA-Toda ones (2.3), (2.18). They are defined as the Hamiltonian reduction of the G_n-WZW model by the constraints

$$\begin{aligned} J_{-\alpha_i} &= \bar{J}_{\alpha_i} = 1, \quad i = 1, \cdots n \quad i \neq j \\ J_{-[\alpha]} &= \bar{J}_{[\alpha]} = 0, \quad \alpha \text{ non simple root} \\ J_{\lambda_j \cdot H} &= \bar{J}_{\lambda_j \cdot H} = 0 \end{aligned} \tag{7.1}$$

i.e. the current $J_{-\alpha_j}$ ($\bar{J}_{\alpha_j}$), (j is arbitrary fixed) is now left unconstrained. Similarly to the $j = 1$ case the $G_n^{(j,1)}$ models can be realized as gauged $G_n/H^{(j)}$-WZW models. The subgroups $H_\pm^{(j)}$ with $H_+^{(j)} = N_+^{(j)} \otimes H_0^{(j)^0}$ and $H_-^{(j)} = N_-^{(j)} \otimes H_0^{(j)^0}$ are introduced by means of the grading operator $Q_j = \sum_{k\neq j}^n \lambda_k \cdot H$. The nilpotent subgroup $N_\pm^{(j)}$ are generated by Q_j-positive (negative) step operators (i.e. all except $E_{\pm\alpha_j}$ since $[Q_j, E_{\pm\alpha_j}] = 0$). The $U(1)$-subgroup $H_0^{(j)^0}$ is generated by $\frac{2\lambda_j \cdot H}{\alpha_j^2}$. This Q_j-gradation of G_n reflects the algebraic structure of the constraints (7.1) and suggests the following "*nonabelian Gauss decomposition*" for each $g \in G_n$ (valid for the connected part of G_n):

$$g = g_- g_0^f g_+, \quad g_\pm \in H_\pm^{(j)}, \quad g_0^f \in H_0^{f\,(j)}$$

$$\begin{aligned} g_- &= \exp\left\{ \sum_{[\alpha]\neq\alpha_j} \chi_{[\alpha]} E_{-[\alpha]} \right\} \exp\left\{ \frac{1}{\mathcal{K}_{jj}} \frac{\lambda_j \cdot H \phi_j}{\alpha_j^2} \right\} \\ g_+ &= \exp\left\{ \frac{1}{\mathcal{K}_{jj}} \frac{\lambda_j \cdot H}{\alpha_j^2} \phi_j \right\} \exp\left\{ \sum_{[\alpha]\neq\alpha_1} \psi_{[\alpha]} E_{[\alpha]} \right\} \\ g_0^f &= \exp\left\{ \chi E_{-\alpha_j} \right\} \exp\left\{ \sum_{i\neq j}^n \phi_i \frac{2\alpha_i \cdot H}{\alpha_i^2} \right\} \exp\left\{ \psi E_{\alpha_j} \right\} \end{aligned} \tag{7.2}$$

The action of the $G_n/H^{(j)}$-WZW model that describes the $G_n^{(j,1)}$-NA-Toda theory is given again by eqn(2.10) where now,

$$\begin{aligned} A^j &= h_-^{-1} \partial h_-, \quad \bar{A}^j = \bar{\partial} h_+ h_+^{-1}, \quad h_\pm(z,\bar{z}) \in H_\pm^j \\ A^j &= A_0 + A_-, \quad \bar{A}^j = \bar{A}_0 + \bar{A}_-, \quad A_- \in N_-^j, \quad \bar{A}_+ \in N_+^j \\ A_0^j &= \frac{2}{\mathcal{K}_{jj}\alpha_j^2} a_0 \lambda_j \cdot H, \quad \bar{A}_0^j = \frac{2}{\mathcal{K}_{jj}\alpha_j^2} \bar{a}_0 \lambda_j \cdot H, \quad \epsilon_\pm = \sum_{i\neq j} (\frac{{\alpha_i}^2}{2}) E_{\pm\alpha_i} \end{aligned}$$

Following the recipe developed in Sect 2, we integrate out the auxiliary gauge fields A^j and $\bar{A}^j$ in order to obtain the corresponding action for the $G_n^{(j,1)}$-NA Toda models

$$\begin{aligned} S_n^{(j)} &= -\frac{k}{2\pi}\int dzd\bar{z}\Big(\frac{1}{2}\eta_{ab}^{(1)}\partial\rho_a^{(1)}\bar{\partial}\rho_b^{(1)} + \frac{1}{2}\eta_{a'b'}^{(2)}\partial\rho_{a'}^{(2)}\bar{\partial}\rho_{b'}^{(2)} \\ &- (\frac{2}{k})^2\sum_{a=1}^{j-1}\frac{2}{\alpha_a^2}e^{-k_{ab}^{(1)}\rho_b^{(1)}} - (\frac{2}{k})^2\sum_{a'=1}^{n-j}\frac{2}{\alpha_{a'}^2}e^{-k_{a'b'}^{(2)}\rho_{b'}^{(2)}} + \frac{2}{\alpha_j^2}e^{k_{j,j-1}\rho_{j-1}^{(1)}+k_{j,j+1}\rho_1^{(2)}}\frac{\partial\chi\bar{\partial}\psi}{\Delta_j}\Big) \end{aligned} \tag{7.3}$$

where $\rho_a^{(1)} = \phi_a$, $(a = 1, \cdots, j-1)$, $\rho_a^{(2)} = \phi_{a'+j}$, $(a' = 1, \cdots, n-j)$ and

$$\Delta_j = 1 + \frac{1}{2\mathcal{K}_{jj}}\psi\chi e^{k_{j,j-1}\rho_{j-1}^{(1)}+k_{j,j+1}\rho_1^{(2)}}$$

We have assumed for simplicity that deleting the j^{th} vertex of the G_n Dynkin diagram the resulting G'_{n-1} algebra is a direct product of two subalgebras G_1 and G_2 of rank $j-1$ and $n-j$ respectively, i.e. $G'_{n-1} = G_1 \otimes G_2$. The exception arises when a specific vertex of D_n, E_6, E_7 or E_8 is deleted. In such cases,$G'_{n-1} = G_1 \otimes G_2 \otimes G_3$ and the generalization of (7.3) is evident.

As in the $j = 1$ case (see Sect. 3), the symmetries of the action (7.3) are generated by the $n+1$-chiral "remaining currents" $W_{s(\alpha)}$, $V^{\pm}$ (and $\bar{W}_{s(\alpha)}$, $\bar{V}^{\pm}$) and the global (nonchiral) $U(1)$ current

$$J_\mu^j = -\frac{k}{4\pi}\frac{e^{k_{ji}\phi_i}}{\Delta_j}\left(\psi\partial_\mu\chi - \chi\partial_\mu\psi - \psi\chi\partial_\mu k_{ji}\phi_i\right) \tag{7.4}$$

Their conformal spin (dimension) are given by the following j-analog of eqn. (3.8):

$$s(\alpha) = 1 + X_j\frac{2\lambda_j\cdot\alpha}{\alpha_j^2} + \sum_{i\neq j}\frac{2\lambda_i\cdot\alpha}{\alpha_i^2}, \quad X_j = -\frac{1}{\mathcal{K}_{jj}}\sum_{i\neq j}\mathcal{K}_{ji} \tag{7.5}$$

Choosing a specific Drinfeld-Sokolov (DS) type gauge we find the remaining currents for the $A_n^{(j,1)}$ case to be

$$\begin{aligned} W_2 &= J_{\alpha_n} \quad W_3 = J_{\alpha_n+\alpha_{n-1}}, \quad \cdots, W_{n-j+1} = J_{\alpha_n+\cdots+\alpha_{j+1}} \\ \tilde{W}_2 &= J_{\alpha_1} \quad \tilde{W}_3 = J_{\alpha_1+\alpha_2}, \quad \cdots, \tilde{W}_j = J_{\alpha_1+\cdots+\alpha_{j-1}} \\ V_j^+ &= J_{-\alpha_j}, \quad V_j^- = J_{\alpha_1+\cdots+\alpha_n} \end{aligned} \tag{7.6}$$

where the index $s(\alpha)$ in W_s and in $\tilde{W}_s$ denote their spin ($X_j = -\frac{n-1}{2}$ for A_n). The nonlocal currents $V_j^{\pm}$ are both of spin $s^{\pm} = \frac{n+1}{2}$. For $B_n^{(j,1)}$-models we have $X_j = \frac{(1-j)}{2}$ and

$$\begin{aligned} W_2 &= J_{\alpha_n} \quad W_4 = J_{2\alpha_n+\alpha_{n-1}}, \quad W_6 = J_{2\alpha_n+2\alpha_{n-1}+\alpha_{n-2}}, \quad \cdots, W_{2(n-i+1)} = J_{2\alpha_n+2\alpha_{n-1}+\cdots+2\alpha_{i+1}+\alpha_i} \\ \tilde{W}_2 &= J_{\alpha_1} \quad \tilde{W}_3 = J_{\alpha_1+\alpha_2}, \quad \cdots, \tilde{W}_j = J_{\alpha_1+\cdots+\alpha_{j-1}} \\ V_j^+ &= J_{-\alpha_j}, \quad V_j^- = J_{\alpha_1+\cdots+\alpha_n} \end{aligned} \tag{7.7}$$

The spin of $V_j^{\pm}$ is now $s_j^{\pm} = n - \frac{1}{2}(j-1)$.

The structure of the constraints (7.1) allows to choose one of the nonlocal currents to be $V_j^+ = J_{-\alpha_j}$, ($\bar{V}_j^+ = \bar{J}_{\alpha_j}$) which have the explicit form (c.f. eqn. (3.11) for $j = 1$)

$$V_j^+(z) = \frac{k}{2}\frac{\partial\chi}{\Delta_j}e^{k_{ji}\phi_i + \frac{1}{2\mathcal{K}_{jj}}R_j}, \quad \bar{V}_j^-(z) = \frac{k}{2}\frac{\bar{\partial}\psi}{\Delta_j}e^{k_{ji}\phi_i + \frac{1}{2\mathcal{K}_{jj}}R_j} \tag{7.8}$$

Applying the method we have used in Sect 3. in the derivation of the $V_n^{(1,1)}$-algebra one can find the algebra of the symmetries $V_n^{(j,1)}$($\bar{V}_n^{(j,1)}$) of the $A_n^{(1,1)}$-NA-Toda model. The corresponding recursive (differential) relations and their solutions are quite similar to those of $j = 1$ obtained in Sect 3. We present here the explicit form of the *simplest nontrivial example* of such $j \neq 1$ type of algebra namely, $V_4^{(2,1)}$-algebra. According to eqn. (7.6) it consist of four spin two currents $V^{\pm}$, $T = \tilde{W}_2 + W_2$ and $V^0 = \tilde{W}_2 - W_2$ satisfying,

$$\begin{aligned}
\{T(\sigma), T(\sigma')\} &= 2T(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \delta(\sigma-\sigma')\partial_{\sigma'}T(\sigma') - 4\partial_{\sigma'}^3\delta(\sigma-\sigma') \\
\{T(\sigma), V^{\alpha}(\sigma')\} &= 2V^{\alpha}(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \delta(\sigma-\sigma')\partial_{\sigma'}V^{\alpha}(\sigma'), \quad \alpha = 0, \pm \\
\{V^{\pm}(\sigma), V^{\pm}(\sigma')\} &= \frac{1}{8}\epsilon(\sigma-\sigma')V^{\pm}(\sigma)V^{\pm}(\sigma'), \quad \{V^0(\sigma), V^{\pm}(\sigma')\} = \frac{1}{8}\epsilon(\sigma-\sigma')V^0(\sigma)V^{\pm}(\sigma') \\
\{V^0(\sigma), V^0(\sigma')\} &= -\frac{1}{4}\epsilon(\sigma-\sigma')[V^+(\sigma)V^-(\sigma') + V^-(\sigma)V^+(\sigma')] \\
&\quad +2T(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \delta(\sigma-\sigma')\partial_{\sigma'}T(\sigma') - 4\partial_{\sigma'}^3\delta(\sigma-\sigma') \\
\{V^-(\sigma), V^+(\sigma')\} &= -\frac{1}{8}\epsilon(\sigma-\sigma')[V^0(\sigma)V^0(\sigma') + V^-(\sigma)V^+(\sigma')] \\
&\quad +T(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \frac{1}{2}\delta(\sigma-\sigma')\partial_{\sigma'}T(\sigma') - 2\partial_{\sigma'}^3\delta(\sigma-\sigma')
\end{aligned} \tag{7.9}$$

(k is fixed to 2 in eqn. (3.14)). It turns out that $V_4^{(2,1)}$ has the same structure as the $V_{2,2}$-algebra of ref. [36] (see eqn. (2.37) of ref. [36]). In our case the $V_4^{(2,1)}$ algebra (7.9) appears as the algebra of symmetries of the $A_3^{(2,1)}$-NA-Toda model

$$\begin{aligned}
\mathcal{L}_{(3)}^{(2,1)} &= \partial A\bar{\partial}A + \partial B\bar{\partial}B - e^{2A} - e^{2B} + \frac{1}{2}e^{A+B}\frac{(\bar{\partial}\psi\partial\chi + \bar{\partial}\chi\partial\psi)}{1 + \frac{1}{2}e^{A+B}\psi\chi} \\
&+ \frac{1}{4}\frac{(\bar{\partial}(A+B)(\chi\partial\psi - \psi\partial\chi) - \partial(A+B)(\chi\bar{\partial}\psi - \psi\bar{\partial}\chi))}{1 + \frac{1}{2}e^{A+B}\psi\chi}
\end{aligned} \tag{7.10}$$

The $\mathcal{L}_{(3)}^{(2,1)}$ differs from the one derived from (7.3) by an appropriate total derivative term similar to the one introduced in the $j = 1$ action (2.18).

As one might expect the charges $Q_{(j)}^+$, $\bar{Q}_{(j)}^-$, of the nontrivial nonlocal currents $V_{(j)}^+$, $\bar{V}_{(j)}^-$ have nonvanishing PB's and together with the $U(1)$ charge, $Q_0 = \int J_0^{(j)}$ close $SL(2,R)_{q_{(j)}}$ PB's algebra ($q_{(j)} = e^{-\frac{2\pi}{k}(\frac{1}{2\mathcal{K}_{jj}})}$). The calculation is identical to the case $j = 1$ case considered in Sect 4. The final result is

$$\{Q_{(j)}^+, \bar{Q}_{(j)}^-\} = \frac{k\pi}{2}\int_{-\infty}^{\infty} d\sigma\partial_\sigma e^{\frac{1}{\mathcal{K}_{jj}}\varphi^{(j)}}, \quad \varphi^{(j)} = R^{(j)} + \mathcal{K}_{jj}(k_{ji}\phi_i - ln\Delta_j) \tag{7.11}$$

The derivation of the PB's of the remaining nonlocal charges $\{Q^-_{(j)}, \bar{Q}^+_{(j)}\}$ is an open problem. It is important to note that as in the $j = 1$ case we have used the $\mathcal{L}^{(j)}_n$ modified by a specific total derivative term in the calculation of the conjugate momenta Π_{ψ,χ,ρ_i}.

We next consider the problem of mapping the $G^{(j,1)}_n$-models into the G_n abelian Toda models. Our starting point is again the observation that there exists a transformation of variables

$$\begin{aligned}
\varphi_i &= \rho^{(1)}_i - \frac{2}{\alpha^2_j}(k_{ja}\eta^{(1)^{-1}}_{ia})R_j - \mathcal{K}_{ij} lnV^+_j\bar{V}^-_j, \quad i = 1, \cdots, j-1 \\
\varphi_j &= R_j - \mathcal{K}_{jj} lnV^+_j\bar{V}^-_j, \\
\varphi_{j+l} &= \rho^{(2)}_l - \frac{2}{\alpha^2_j}(k_{ja}\eta^{(2)^{-1}}_{la})R_j - \mathcal{K}_{l+j,j} lnV^+_j\bar{V}^-_j, \quad l = 1, \cdots n-j
\end{aligned} \tag{7.12}$$

which transforms part of the equations of motion of (7.3) into the abelian G_n Toda equations (5.2) for the new fields φ_l, $(l = 1, \cdots n)$, where the identity

$$\frac{1}{\mathcal{K}_{jj}} = \sum_{i=1}^{j-1} \frac{2k_{ja}}{\alpha^2_j} k_{j,i}\eta^{(1)^{-1}}_{i,a} + \sum_{b=1}^{n-j} \frac{2k_{j,j+a}}{\alpha^2_j} k_{j,j+b}\eta^{(2)^{-1}}_{b,a} - k_{jj}$$

was verified for A_n, B_n, C_n and D_n. The nonlocal field R_j satisfy now

$$\bar{\partial}\partial R_j = \frac{\bar{\partial}\psi\partial\chi}{\Delta^2_j} e^{k_{jb}\phi_b}$$

For example in the $A^{(2,1)}_3$ case (7.10) we have

$$\varphi_1 = \frac{1}{2}R_{(2)} + A - \frac{1}{2}lnV^+\bar{V}^-, \qquad \varphi_2 = R_{(2)} - lnV^+\bar{V}^- \qquad \varphi_3 = \frac{1}{2}R_{(2)} + B - \frac{1}{2}lnV^+\bar{V}^-$$

Following the same line of argument presented in Sect.6, we seek a G_n gauge transformation mapping the constraints (7.1) and the remaining currents (7.6) into the constraints and remaining currents (1.12) leading to the abelian Toda theory:

$$\bar{H}\bar{J}^{(A)} = \bar{J}^{(NA)}_{(j)}\bar{H} - \frac{k}{2}\bar{\partial}\bar{H} \tag{7.13}$$

For the A_n case, $\bar{J}^{(A)}$ and $\bar{J}^{(NA)}_{(j)}$ have the following matrix form:

$$\begin{aligned}
(\bar{J}^{(A)})_{il} &= \delta_{i,l-1} + \bar{W}_{n-l+2}\delta_{i,n+1}, \quad \bar{W}_1 = 0 \\
(\bar{J}^{(NA)}_{(j)})_{il} &= (\bar{V}^-)^{\delta_{ij}\delta_{l,j+1}}\delta_{i,l-1} + \delta_{l,1}\sum_{s=2}^{j} \tilde{\bar{W}}_s\delta_{is} + \delta_{i,n+1}\delta_{l,1}\bar{V}^+ \\
&+ \sum_{p=j+1}^{n} \bar{W}_{n-p+2}\delta_{p,l}\delta_{i,n+1}
\end{aligned} \tag{7.14}$$

Substituting (7.14) in (7.13) and requiring $det\ \bar{H} = 1$ and $\bar{H}_{i,l} = 0, \quad i < l$ we obtain:

$$\bar{H}_{ii} = \begin{cases} (\bar{V}^-)^{\frac{n-j+1}{n+1}}, & i = 1, \cdots j \\ (\bar{V}^-)^{\frac{-j}{n+1}}, & i = j+1, \cdots n+1 \end{cases}$$

We next consider the equations for $\bar{H}_{i,1}$:

$$(\bar{V}^-)^{\delta_{ij}} \bar{H}_{i+1,1} = -\sum_{p=2}^{j} \bar{\tilde{W}}_p \bar{H}_{11} \delta_{ip} + \frac{k}{2} \partial \bar{H}_{i1}$$

Their solutions are given by

$$\begin{aligned} (\bar{V}^-)^{\delta_{ij}} \bar{H}_{i+1,1} &= -\sum_{l=0}^{j-2} (\frac{k}{2}\partial)^l (\bar{\tilde{W}}_{j-l} \bar{H}_{11}) + (\frac{k}{2}\partial)^i \bar{H}_{11}, \quad i = 1, \cdots j \\ \bar{H}_{j+r,1} &= (\frac{k}{2}\partial)^{r-1} \bar{H}_{j+1,1}, \qquad r = 2, \cdots, n-j+1 \end{aligned} \tag{7.15}$$

The general solution of (7.13) can be written in terms of H_{i1} as follows

$$(\bar{V}^-)^{\delta_{sj}} \bar{H}_{s+1,k} = \sum_{p=0}^{s-k+1} \frac{(p+k-2)!}{(k-2)!p!} (\frac{k}{2}\partial)^p \bar{H}_{s-p+k+2,1}, \quad s = k-1, \cdots j$$

$$\bar{H}_{j+r,k} = \sum_{p=1}^{k} \frac{(r-1)!}{(k-p)!(r+p-k-1)!} (\frac{k}{2}\partial)^{r-k-1+p} \bar{H}_{j+1,p}, r = 2, \cdots, n-j+1, \quad k = 1, \cdots, n+1$$

Note that $\bar{H}_{il} = \bar{H}_{il}(\bar{V}^-, \bar{\tilde{W}}_p)$ contrary to the $j = 1$ case are now functionals of the currents $\bar{\tilde{W}}_p$ as well. This reflects a specific (mixed type) of gauge (7.6) we have chosen. In the DS gauge where all $W_{n-s+2} = J_{\alpha_s + \alpha_{s+1} \cdots + \alpha_n}$, $V^- = J_{\alpha_1 + \cdots + \alpha_n}$ lie on the last column (all $\bar{W}_s$, $\bar{V}^+$ on the last row) the corresponding H_{il} $(\bar{H}_{il})$ indeed depend on V^+ $(\bar{V}^-)$ only. The spins of W_{n-s+2} in the DS gauge is $n-s+2$ for $s \geq j+1$ and $\frac{n+3}{2} - s$ for $s \leq j$.

The main advantage of this current dependent H-transformation is that it provides an explicit realization of the A_n- NA-Toda currents $(W^{(A)}_{n-s+2})$ in terms of the $A_n^{(j,1)}$- NA-Toda currents W_p, $\tilde{W}_p$, $V^{\pm}$ (see eqn. (6.11) for $j = 1$ case) and vice-versa. For our $A_3^{(2,1)}$ example (7.10) we have $h_{il} = (H^{-1})_{il}$, $H_{il} = \bar{H}_{li}(\partial \rightarrow \bar{\partial}, \bar{V}^-, \bar{\tilde{W}}_2 \rightarrow V^+, \tilde{W}_2)$ (k is taken to be 2 in (7.13)):

$$\begin{aligned} h_{11} &= h_{22} = h_{33}^{-1} = h_{44}^{-1} = e^{-\frac{1}{2} ln V^+} \\ h_{12} &= \frac{1}{2} h_{23} = \partial h_{11}, \quad h_{34} = -\partial h_{33}, \quad h_{13} = \partial^2 h_{11} + \tilde{W}_2 h_{11} \\ h_{14} &= \partial^3 h_{11} + h_{11} \partial \tilde{W}_2 + 3 W_2 \partial h_{11}, \quad h_{24} = 3\partial^2 h_{11} + \tilde{W}_2 h_{11} \end{aligned}$$

The corresponding abelian Toda currents T^A, W_3^A, W_4^A are expressed in terms of $V^{\pm}$, W_2, $\tilde{W}_2$ as follows:

$$T^A = T^{NA} - 2\partial^2 \ln V^+ + \frac{1}{2}(\partial \ln V^+)^2, \qquad T^{NA} = W_2 + \tilde{W}_2$$

$$
\begin{aligned}
W_3 &= (W_2 - \tilde{W}_2)\ln V^+ + 2\partial\tilde{W}_2 - 2\partial^3\ln V^+ + (\partial^2\ln V^+)\partial(\ln V^+) \\
W_4 &= V^+V^- - W_2\tilde{W}_2 + \frac{1}{2}(W_2 - \tilde{W}_2)\partial^2(\ln V^+) + \frac{1}{4}(W_2 + \tilde{W}_2)(\ln V^+)^2 + \partial^2\tilde{W}_2 - \frac{1}{2}\partial^4\ln V^+ - \\
&\quad -(\partial\tilde{W}_2)(\partial\ln V^+) - \frac{1}{8}(\partial\ln V^+)^4 - \frac{1}{4}(\partial^2\ln V^+)^2 + \frac{1}{2}(\partial^2\ln V^+)(\partial\ln V^+)^2 \qquad (7.16)
\end{aligned}
$$

One can verify by direct calculation that if $V^{\pm}$, W_2, $\tilde{W}_2$ satisfy *the* $V_4^{(2,1)}$ *algebra* (7.9), then T^A, W_3^A and W_4^A given by eqns. (7.16) indeed close the (classical) W_4 algebra [14] . Therefore the $A_3^{(2,1)}$-NA-Toda model has together with the $V_4^{(2,1)}$ algebra, also the W_4^A *as its algebra of symmetries.*

We now address the question about the relation between $A_n^{(j_1,1)}$ and the $A_n^{(j_2,1)}$-NA-Toda models ($j_1 \neq j_2$). In terms of transformations $H(j_1)$ and $H(j_2)$ mapping them into the A_n abelian Toda theory we compose the new transformation $H(j_1,j_2) = H(j_1)H^{-1}(j_2)$. By construction $H(j_1,j_2)$ transforms the constraints and remaining currents of the $A_n^{(j_1,1)}$into the corresponding ones of the $A_n^{(j_2,1)}$ model:

$$
J_{j_2}^{NA} = H^{-1}(j_1,j_2)J_{j_1}^{NA}H(j_1,j_2) + \frac{k}{2}H^{-1}(j_1,j_2)\partial H(j_1,j_2) \qquad (7.17)
$$

As a byproduct $H(j_1,j_2)$ realizes a map of $V_{n+1}^{(j_1,1)}$-algebra into $V_{n+1}^{(j_2,1)}$ and vice-versa. The simplest example is given by $n = 3$, $j_1 = 2$, $j_1 = 1$ i.e. the transformation of the $A_3^{(2,1)}$ into the $A_3^{(1,1)}$-NA-Toda model ($H(2,1) \equiv H$)

$$
\begin{aligned}
H_{11} &= H_{33} = H_{44} = (V^+)H_{22} = e^{\frac{1}{4}lnV^+}, \quad H_{12} = -\frac{1}{3}\partial H_{22} = -\frac{1}{2}H_{23} \\
H_{13} &= \partial H_{12} + H_{22}\tilde{W}_2, \quad H_{24} = -\partial H_{12} + H_{22}\tilde{W}_2 \\
H_{14} &= \partial^3 H_{12} - \frac{5}{4}(\tilde{W}_2 H_{22})\partial lnV^+ + (\partial\tilde{W}_2)H_{22}
\end{aligned}
$$

(we have chosen $V_{(1)}^+ \equiv V_{(2)}^+$). The current transformations take the form:

$$
\begin{aligned}
T^{(1)} &= T^{(2)} - \frac{1}{2}\partial^2 lnV^+ + \frac{1}{8}(\partial lnV^+)^2, \quad T^{(2)} = W_2 + \tilde{W}_2 \\
W_3^{(1)} &= 2\partial\tilde{W}_2 + \frac{1}{2}(\partial lnV^+)(W_2 - 3\tilde{W}_2) - \frac{1}{4}(\partial^2 lnV^+)\partial lnV^+ + \frac{1}{16}(\partial lnV^+)^3 \quad (7.18)
\end{aligned}
$$

and $V_{(1)}^-$ has a rather complicated form in terms of $V_{(2)}^-, V^+, W_2$ and $\tilde{W}_2$.

The models $A_n^{(j_1,1)}$ and $A_n^{(j_2,1)}$ have identical field contents but their lagrangeans represent different interactions between the neutral fields $\rho_a^{(1)}$ and $\rho_{a'}^{(2)}$, (compare for example $A_3^{(1,1)}$ and $A_3^{(2,1)}$ models). The transformation $H(j_1,j_2)$ changes the (j_1)-constraints into the (j_2) ones and according to the hamiltonian reduction procedure it maps the field equations of $A_n^{(j_1,1)}$ to those of $A_n^{(j_2,1)}$. If we denote by $g_{il}(j_1)$ and $g_{il}(j_2)$ the constrained WZW matrix field $g_{il} \in A_n$ (i.e. $g_{il}(j_\alpha)$ depending on the fields $\psi_{j_\alpha}, \chi_{j_\alpha}$, and $\rho_a^{(i)}(j_\alpha)$ only) then $H(j_1,j_2)$ induces the following field transformations:

$$
\phi_1 = -\frac{2}{3}A + \frac{2}{3}R_{(2)} - \frac{2}{3}lnV^+\bar{V}^-, \quad \phi_2 = B - \frac{1}{3}A + \frac{1}{3}R_{(2)} - \frac{1}{3}lnV^+\bar{V}_-
$$

$$R_{(1)} = A + \frac{1}{2}R_{(2)} + \frac{1}{4}lnV^{+}\bar{V}^{-} \tag{7.19}$$

($R_{(1)}$ and $R_{(2)}$ are nonlocal in terms of ψ, χ and A, B respectively). The remaining ψ, χ transformations are quite implicit. Although the $A_n^{(j_1,1)}$ and $A_n^{(j_2,1)}$-NA-Toda models represent different interactions and have different algebras of symmetry (but equal number of generators) the arguments presented above indicate that they are classically equivalent models. The proof of such statement requires however further investigations.

8 Weyl Group Families of $A_n^{(j_1,1)}$-Models

This section is devoted to the *problem* of the relation between the NA-Toda models that have *identical algebras of symmetries*. Our starting point is the *following fact*: The $V_3^{(1,1)}$- algebra (??),(3.43) and (3.44), ($n = 2, s = \frac{3}{2}$) appears as the symmetry algebra of the $A_2^{(1,1)}$-NA-Toda model (2.18), ($n = 2$) as well as of the reduced Bershadsky-Polyakov (BP) $A_2^{(2)}$-model [7], [26]. The latter is defined by the set of constraints imposed on the A_2-WZW currents,

$$J_{-\alpha_2} = \bar{J}_{\alpha_2} = 0, \qquad J_{-\alpha_1-\alpha_2} = \bar{J}_{\alpha_1+\alpha_2} = 1, \tag{8.1}$$

$$J_{(\lambda_1-\lambda_2)\cdot H} = \bar{J}_{(\lambda_1-\lambda_2)\cdot H} = 0 \tag{8.2}$$

It differs from the standard BP model [26] by the additional constraint (8.2). This new constraints is responsible for the reduction of the $W_3^{(2)}$-algebra (symmetry of the BP model defined by (8.1)) to the nonlocal algebra $V_3^{(2)} \equiv V_3^{(1,1)}$. Following the methods of Sect. 2 we first derive the lagrangean of the reduced $A_2^{(2)}$-BP-model:

$$\mathcal{L}_3^{(2)} = -\frac{k}{2\pi}\{\partial\varphi\bar{\partial}\varphi + e^{\varphi}\frac{\bar{\partial}\psi_0\partial\chi_0}{1 + \frac{3}{4}e^{\varphi}\psi_0\chi_0} - e^{-2\varphi}(1 + \psi_0\chi_0 e^{\varphi})\} \tag{8.3}$$

The algebra of symmetries $V_3^{(2)}$ of (8.3) obtained by direct application of the recipe described in Sect. 3 turns out to be identical to $V_3^{(1,1)}$. The lagrangean of the $A_2^{(1,1)}$-NA-Toda model possess however quite a different form,

$$\mathcal{L}_3^{(1,1)} = -\frac{k}{2\pi}\{\partial\phi\bar{\partial}\phi + e^{-\phi}\frac{\bar{\partial}\psi\partial\chi}{1 + \frac{3}{4}e^{-\phi}\psi\chi} - e^{-2\phi}\} \tag{8.4}$$

We shall prove that the models (8.3) and (8.4) are (classically) equivalent since their Lagrangeans are related by the following change of field variables,

$$\psi = \chi_0 e^{\varphi}(1 + e^{\varphi}\psi_0\chi_0)^{-\frac{1}{4}}, \qquad \chi = \psi_0 e^{\varphi}(1 + e^{\varphi}\psi_0\chi_0)^{-\frac{1}{4}}$$
$$\phi = \varphi - \frac{1}{2}\ln(1 + e^{\varphi}\psi_0\chi_0) \tag{8.5}$$

i.e. $\mathcal{L}_3^{(2)} = \mathcal{L}_3^{(1,1)}+$ total derivative. This can be verified by direct calculation. Our derivation of transformation (8.5) is based on the following *observation*: The constraints (8.1) and (8.2) are the *image* of

$$J'_{-\alpha_2} = \bar{J}'_{\alpha_2} = 1, \quad J'_{-\alpha_1-\alpha_2} = \bar{J}'_{\alpha_1+\alpha_2} = 0, \quad J'_{-\lambda_1\cdot H} = \bar{J}'_{-\lambda_1\cdot H} = 0 \tag{8.6}$$

together with the gauge fixing condition $J'_{\alpha_1} = \bar{J}'_{-\alpha_1} = 0$ (defining the model (8.4)) under the action of a particular A_2-*Weyl reflection*

$$\omega_{\alpha_1}(\alpha) = \alpha_1 - (\alpha \cdot \alpha_1)\alpha, \quad \omega^2_{\alpha_1} = 1$$

In fact ω_{α_1} maps all the algebraic (Hamiltonian reduction) data of the model (8.3), constraints, gauge fixing condition and remaining currents into those of model (8.4): $J' = \omega_{\alpha_1}(J)$. The change of variables (8.5) is a consequence of the relation between the reduced A_2-WZW matrix fields $g^{(2)}_{(3)}$ and $g^{(1,1)}_{(3)}$:

$$g^{(2)}_{(3)} = \omega_{\alpha_1}(g^{(1,1)}_{(3)}) \tag{8.7}$$

The explicit form of $g^{(1,1)}_{(3)}$ in terms of fields ψ, χ, and φ is given by eqn. (3.47). Solving the constraints (8.1) and (8.2) we find the matrix elements of $g^{(2)}_{(3)}$:

$$\begin{aligned}
\left(g^{(2)}\right)_{11} &= e^{\varphi-\frac{1}{2}R_0}, \quad \left(g^{(2)}\right)_{13} = \partial e^{\varphi-\frac{1}{2}R_0}, \quad \left(g^{(2)}\right)_{31} = \bar{\partial} e^{\varphi-\frac{1}{2}R_0} \\
\left(g^{(2)}\right)_{22} &= e^{R_0}(1 + e^{\varphi}\psi_0\chi_0), \quad \left(g^{(2)}\right)_{12} = e^{\frac{1}{4}R_0+\varphi}\psi_0, \quad \left(g^{(2)}\right)_{21} = e^{\frac{1}{4}R_0+\varphi}\chi_0, \\
\left(g^{(2)}\right)_{23} &= \partial\left(g^{(2)}\right)_{21}, \quad \left(g^{(2)}\right)_{32} = \bar{\partial}\left(g^{(2)}\right)_{12}, \cdots etc
\end{aligned}$$

We next write eqns. (8.7) in a matrix form

$$\left(g^{(2)}\right)_{ik} = (\omega_{\alpha_1})_{il}\left(g^{(1,1)}\right)_{lm}(\omega_{\alpha_1})_{mk}, \quad i, k = 1, 2, 3 \tag{8.8}$$

where $(\omega_{\alpha_1}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

As a solution of (8.8), (i.e. ψ, χ, ϕ in terms of ψ_0, χ_0, φ) we find the change of variables (8.5) as well as the relation between the nonlocal fields R and R_0:

$$R = R_0 + \ln(1 + e^{\varphi}\psi_0\chi_0).$$

The generalization of such results for a generic $A^{(1,1)}_n$-model (2.18) is straightforward. Reflecting the $A^{(1,1)}_n$ constraints (2.4) by ω_{α_1} we find the set of constraints that define the new model ω_{α_1} $(A^{(1,1)}_n)$:

$$\begin{aligned}
&J_{-\alpha_i} = \bar{J}_{\alpha_i} = 0, \quad i = 3, 4, \cdots, n; \qquad J_{-[\alpha]} = \bar{J}_{[\alpha]} = 0, \quad [\alpha] \text{ all other roots} \\
&J_{-\alpha_1-\alpha_2} = \bar{J}_{\alpha_1+\alpha_2} = 1, \qquad J_{(\lambda_1-\lambda_2)\cdot H} = \bar{J}_{(\lambda_1-\lambda_2)\cdot H} = 0
\end{aligned} \tag{8.9}$$

It is important to note the reflection ω_{α_1} keeps unchanged the gradation operator, $Q_{(1)} = \sum_{i=2}^{n} \lambda_i \cdot H$ and the nilpotent subgroups $N^{\pm}_{(1)} \subset H^{\pm}_{(1)}$, $\omega_{\alpha_1}(Q_{(1)}) = Q_{(1)}$, $\omega_{\alpha_1}(N^{\pm}_{(1)}) = N^{\pm}_{(1)}$ but it does change $H^{(1)^0}_0$ and $\epsilon^{(1)}_{\pm}$

$$\omega_{\alpha_1}(H^{(1)^0}_0) = exp\left(\frac{1}{2\mathcal{K}_{11}}[(\lambda_2 - \lambda_1) \cdot H]\varphi_1\right), \quad \omega_{\alpha_1}(\epsilon^{\pm}_{(1)}) = E_{\pm(\alpha_1+\alpha_2)} + \sum_{i=3}^{n} E_{\pm\alpha_i}$$

With all this data we derive the Lagrangean for the $\omega_{\alpha_1}(A_n^{(1,1)})$ model:

$$\mathcal{L}^{(1,1)}_{n,\omega_{\alpha_1}} = -\frac{k}{2\pi}\{\frac{1}{2}\tilde{k}_{ij}\partial\varphi_i\bar{\partial}\varphi_j + e^{\varphi_1}\frac{\bar{\partial}\psi_0\partial\chi_0}{1+\frac{n+1}{2n}\psi_0\chi_0 e^{\varphi_1}} - \sum_{i=2}^{n-1} e^{-\tilde{k}_{ij}\varphi_j} - (1+e^{\varphi}\psi_0\chi_0)e^{-\tilde{k}_{1i}\varphi_i}\} \quad (8.10)$$

The change of variables that follows from the A_n-analog of eqn. (8.8) is now given by

$$\phi_i = \varphi_i - \tilde{\mathcal{K}}_{1i}\ln(1+e^{\varphi_1}\psi_0\chi_0), \quad \tilde{\mathcal{K}}_{1i} = \frac{n-i}{n}$$
$$\psi = \chi_0 e^{\varphi_1}(1+e^{\varphi_1}\psi_0\chi_0)^{-\frac{1}{2}\tilde{\mathcal{K}}_{11}}, \quad \chi = \psi_0 e^{\varphi_1}(1+e^{\varphi_1}\psi_0\chi_0)^{-\frac{1}{2}\tilde{\mathcal{K}}_{11}} \quad (8.11)$$

Substituting eqn. (8.11) in (2.18) we find (8.10) modulo total derivative. The following identity

$$\frac{e^{\varphi}}{1+e^{\varphi}\psi\chi}\left(\bar{\partial}\chi\partial\psi - \bar{\partial}\psi\partial\chi + \frac{1}{2}(\chi\bar{\partial}\psi - \psi\bar{\partial}\chi)\partial\varphi - \frac{1}{2}(\chi\partial\psi - \psi\partial\chi)\bar{\partial}\varphi\right) =$$
$$= \frac{1}{2}\partial\left(\ln(1+e^{\varphi}\psi\chi)\bar{\partial}\ln\frac{\psi}{\chi}\right) - \frac{1}{2}\bar{\partial}\left(\ln(1+e^{\varphi}\psi\chi)\partial\ln\frac{\psi}{\chi}\right)$$

is crucial in the proof of the above statement. The *conclusion* is that the pair of NA-Toda models $A_n^{(1,1)}$ and $\omega_{\alpha_1}(A_n^{(1,1)})$ *sharing the same algebra of symmetries* $V_{n+1}^{(1,1)}$ are classically *equivalent*. The same is true for the pair of generic $A_n^{(j_1,1)}$ and $\omega_{\alpha_j}(A_n^{(j_1,1)})$-models. The simplest example of such doublet is given by the $A_3^{(2,1)}$ and the $\omega_{\alpha_1}(A_3^{(2,1)})$-NA-Toda models. Their lagrangeans are given by:

$$\mathcal{L}_3^{(2,1)} = -\frac{k}{2\pi}(\partial A\bar{\partial}A + \partial B\bar{\partial}B - e^{2A} - e^{2B} + e^{A+B}\frac{\bar{\partial}\psi\partial\chi}{1+e^{A+B}\psi\chi}) \quad (8.12)$$

and

$$\begin{aligned}\mathcal{L}^{(2,1)}_{3,\omega_{\alpha_1}} &= -\frac{k}{2\pi}\{\partial\mathcal{A}\bar{\partial}\mathcal{A} + \partial\mathcal{B}\bar{\partial}\mathcal{B} - e^{2\mathcal{A}} - e^{2\mathcal{B}} - (1+e^{-\mathcal{A}-\mathcal{B}}\psi_0\chi_0)(e^{2\mathcal{A}}+e^{2\mathcal{B}}) \\ &+ e^{-\mathcal{A}-\mathcal{B}}\frac{\bar{\partial}\psi_0\partial\chi_0}{1+e^{-\mathcal{A}-\mathcal{B}}\psi_0\chi_0}\} \quad (8.13)\end{aligned}$$

The change of variables that makes their equivalence evident has the form:

$$\begin{aligned} A &= \mathcal{A} + \frac{1}{2}\ln(1+e^{-\mathcal{A}-\mathcal{B}}\psi_0\chi_0), \quad B = \mathcal{B} + \frac{1}{2}\ln(1+e^{-\mathcal{A}-\mathcal{B}}\psi_0\chi_0) \\ \psi &= \chi_0 e^{-\mathcal{A}-\mathcal{B}}(1+e^{-\mathcal{A}-\mathcal{B}}\psi_0\chi_0)^{-\frac{1}{2}}, \quad \chi = \psi_0 e^{-\mathcal{A}-\mathcal{B}}(1+e^{-\mathcal{A}-\mathcal{B}}\psi_0\chi_0)^{-\frac{1}{2}}\end{aligned}$$

Our observation that *each pair of* $A_n^{(1,1)}$*- models, whose constraints are related by* ω_{α_j} *are equivalent*, adresses the *question* about the *family of models* obtained from $A_n^{(j,1)}$ by transforming its constraints (7.1) (or (2.4), for $j=1$) under the *whole Weyl group* S_{n+1} of A_n.

We first consider a subset of models $\Pi_i(A_n^{(1,1)})$ from the A_n family, whose constraints are the images of (2.4) under the action of the following (composite) reflections,

$$\Pi_i = \omega_{\alpha_i}\omega_{\alpha_{i-1}}\cdots\omega_{\alpha_1}, \quad i = 1,\cdots,n$$

In order to derive their lagrangeans we need the explicit form of $\Pi_i(H_\pm) = \Pi_i(N_\pm) \otimes \Pi_i(H_0^{(1)^0})$, $\Pi_i(g_0^f)$ and $\Pi_i(\epsilon_\pm^{(1)})$ (see Sect. 2). It is convenient to first calculate the corresponding grading operator $\Pi_i(Q_{(1)}) = Q_{(1)} - \sum_{j=2}^{i}(j-1)\alpha_j$ where $Q_{(1)} = \sum_{l=2}^{n} \lambda_l \cdot H$. Then the nilpotent subgroups $\Pi_i(N_\pm)$ are spanned by elements of grade ± 1 with respect to $\Pi_i(Q_{(1)})$ and $\Pi_i(g_0)$ is the subgroup of zero $\Pi_i(Q_{(1)})$-grade. The diagonal subgroup $\Pi_i(H_0^{(1)^0})$ has the form

$$\Pi_i(H_0^{(1)^0}) = exp\{\frac{1}{2\mathcal{K}_{11}}(\lambda_1 - \sum_{l=1}^{i} \alpha_l) \cdot H\}$$

Following the recipe of Sect. 2 we find that the lagrangeans of $\Pi_i(A_n^{(1,1)})$ models coincide with (8.10) for $i < n$ and with $A_n^{(n,1)}$ (see eqn (7.3) for $j = n$). Note that due to the Z_2 symmetry of the A_n Dynkin diagram

$$\Pi^+(\alpha_j) = \alpha_{n+1-j}, \quad \mathcal{K}_{jj} = \mathcal{K}_{n+1-j,n+1-j} \tag{8.14}$$

the $A_n^{(j,1)}$ and the $A_n^{(n+1-j,1)}$ models are *identical.* In particular $\mathcal{L}_n^{(1,1)} = \mathcal{L}_n^{(n,1)}$, $(\phi_i' = \phi_{n-i})$ and therefore all the $\Pi_i(A_n^{(1,1)})$-models are equivalent to the $A_n^{(1,1)}$.

Another subfamily of the $A_n^{(1,1)}$ models is defined by transforming the constraints (2.4) under the composite Weyl reflections

$$\Pi_i^- = \Pi_{n+1-i}\Pi^-, \quad \Pi^-(\alpha_j) = -\alpha_{n+1-j} \tag{8.15}$$

(Π^- is an element of the Weyl group, contrary to the Π^+ from (8.14)). Repeating once more the procedure, we have used in the construction of $\Pi_i(A_n^{(1,1)})$ models, we realize that all the $\Pi_i^-(A_n^{(1,1)})$ models are equivalent to the $A_n^{(n,1)}$ model (modulo PC-transformations: $C\psi = \chi$, $C^2 = 1$; $P\sigma = -\sigma$). Taking into account (8.14) *the conclusion* is that the *family* of $A_n^{(1,1)}$ models obtained by Π_i and Π_i^-- Weyl reflections $(i = 1, \cdots n)$ of $A_n^{(1,1)}$ constraints (2.4) have as lagrangeans (8.10) or (2.18). They are related by the change of variables (8.11). Hence all this family of models can be represented by the original $A_n^{(1,1)}$ model only.

Note that this (Π_i , Π_i^-) family of equivalent models coincides with the complete Weyl group $A_n^{(1,1)}$ family only for $n = 2$. Whether the other models generated by Weyl reflections different from Π_i and Π_i^- are also equivalent to $A_n^{(1,1)}$ is still an open problem for generic $n \neq 2$.

9 What does the Quantum V-algebras look like

The main obstacle for quantizing the NA-Toda models (2.18) (and their algebra of symmetries $V_{n+1}^{(1,1)}$) is the additional PF-type constraint $J_{\lambda_1 H} = 0$. As we have shown in Sect. 3 the latter is responsible for the nonlocal terms in the $V_{n+1}^{(1,1)}$-algebra. This suggests the following *strategy*: First consider an *intermediate* "local" NA-Toda model defined by the set of constraints (2.4) but with the *current* $J_{\lambda_1 H}$ *left unconstrained.* Its Lagrangean as well as the PB symmetry algebra $W_{n+1}^{(1,1)}$ can be easely derived by the methods of Sect.2 and 3. The $W_{n+1}^{(1,1)}$

is generated by $n+2$ local currents: $n+1$ of them $G^{(\pm)}$, $W_{n-k+2}, k=2,3,\cdots,n$ have the same spins as the $V_{n+1}^{(1,1)}$-currents and the last one is the chiral $U(1)$-current $J_{\lambda_1 H}\equiv J$ of spin one. An *important difference* with respect to the original NA-Toda model and its nonlocal $V_{n+1}^{(1,1)}$-algebra is that $W_{n+1}^{(1,1)}$ is a *local* quadratic (non-Lie) algebra. It has a structure similar to the W_{n+1} and $W_{n+1}^{(l)}$-algebras [26] and its quantization can be realized by the standard methods of ref. [14], [26], [5].

The *problem* we address in this section is the following: *Given the quantum $W_{n+1}^{(1,1)}$ algebra and its irreducible representations, to derive the quantum $V_{n+1}^{(1,1)}=W_{n+1}^{(1,1)}/U(1)$- algebra and its representations by implementing the constraint* $J_{\lambda_1 H}=0$. The method we are going to use is an appropriate generalization of the derivation of the PF-algebra from the affine $SU(2)$ by imposing the constraint $J_3=0$ [3]. The crucial ingredient of this approach is the free field representation of the $W_{n+1}^{(1,1)}$-currents. In the framework of the quantum Hamiltonian reduction [5], [26] this is rather difficult problem even for $n=3$. To start with it requires the explicit bosonization of the $SL(4,R)$ currents. One expects the nonabelian analog of the quantum Miura transformation [14] to be the effective tool for the solution of this problem. The $n=2$ case is an exception. According to the arguments of Sect. 8, the $W_3^{(1,1)}$ algebra coincides with the $W_3^{(2)}$-one. The free field representation of the $W_3^{(2)}$ currents $G^{\pm}, T$ and J is well known [26]:

$$\begin{aligned}
J(z) &= \sqrt{\frac{2k+3}{3}}\partial\tilde{\Phi}, \quad \sqrt{\frac{2k+3}{3}}\tilde{\Phi} = \frac{\alpha_+}{3}(\vec{\alpha_2}-\vec{\alpha_1})\vec{\varphi}+\phi_0, \quad \alpha_+ = \sqrt{2k+6} \\
G^-(z) &= [(k+3)\partial\phi_0 + \alpha_+\vec{\alpha_2}\partial\vec{\varphi} - (k+2)\partial\chi]e^{\phi_0-\chi} \\
G^+(z) &= [-\partial\chi((k+3)\partial\phi_0 - \alpha_+\vec{\alpha_1}\partial\vec{\varphi}) + (k+2)(\partial^2\chi + (\partial\chi)^2]e^{-\phi_0+\chi} \\
T_W(z) &= \frac{2}{3}[(\vec{\alpha_1}\partial\vec{\varphi})^2 + (\vec{\alpha_2}\partial\vec{\varphi})^2 + (\vec{\alpha_1}\partial\vec{\varphi})(\vec{\alpha_2}\partial\vec{\varphi})] \\
&+ \frac{k+1}{\alpha_+}(\vec{\alpha_1}+\vec{\alpha_2})\partial^2\vec{\varphi} + \frac{1}{2}[(\partial\chi)^2 + \partial^2\chi - (\partial\phi_0)^2], \qquad (9.1)
\end{aligned}$$

where $\varphi_i, i=1,2,$ ϕ_0 and χ are free bosonic fields,

$$< \varphi_i(z_1)\varphi_j(z_2) >= \frac{1}{2}\delta_{ij} lnz_{12}, \quad < \phi_0(z_1)\phi_0(z_2) >= -lnz_{12}, \quad < \chi(z_1)\chi(z_2) >= lnz_{12} \qquad (9.2)$$

and $\vec{\alpha_1}, \vec{\alpha_2}$ are the simple roots of A_2. The difference of our eqn. (9.1) from the original Bershadsky's ones (see eqns. (3.6) of ref. [26]) is due to the fact that we have bosonized the pair of *bosonic* ghosts ($\Phi, \Phi^{\dagger}$) of spin $(1/2, 1/2)$:

$$\Phi = e^{\phi_0-\chi}, \quad \Phi^{\dagger} = e^{-\phi_0+\chi}\partial\chi$$

With eqns. (9.1) at hand one can easely derive the explicit form for the $W_3^{(1,1)}$-algebra,

$$\begin{aligned}
J(z_1)J(z_2) &= \frac{2k+3}{3z_{12}^2} + O(z_{12}), \quad G^{\pm}(z_1)G^{\pm}(z_2) = O(z_{12}) \\
J(z_1)G^{\pm}(z_2) &= \pm\frac{1}{z_{12}}G^{\pm}(z_2) + O(z_{12}),
\end{aligned}$$

$$\begin{aligned} G^{+}(z_1)G^{-}(z_2) &= \frac{(k+1)(2k+3)}{z_{12}^3} + 3\frac{(k+1)}{z_{12}^2}J(z_2) \\ &+ \frac{1}{z_{12}}[3J^2(z_2) - (k+3)T(z_2) + 3\frac{(k+1)}{2}\partial J(z_2)] + O(z_{12}) \end{aligned} \tag{9.3}$$

etc. The central charge of the $W_3^{(1,1)}$ algebra is

$$c_W = \frac{8k}{k+3} - 6k - 1 \tag{9.4}$$

According to the definition of the $V_3^{(1,1)}$ algebra $V_3^{(1,1)} = \{W_3^{(1,1)} : J = 0\}$ its generators $V^{\pm}(z), T_V(z)$ have to commute with $J(z)$,

$$J(z_1)V^{\pm}(z_2) = O(z_{12}) = J(z_1)T_V(z_2) \tag{9.5}$$

i.e. they are related to the $\tilde{\Phi}$ independent parts ($J = \sqrt{\frac{2k+3}{3}}\partial\tilde{\Phi}$) of the $W_3^{(1,1)}$-currents. This suggests to seek for a specific change of the field variables $\phi_0, \varphi_i \longrightarrow \tilde{\Phi}, \rho_i$ that makes explicit the J (or $\tilde{\Phi}$) dependence of $G^{\pm}$ and T_W:

$$G^{\pm} = V^{\pm}e^{\pm a\tilde{\Phi}}, \quad T_W = T_V + T_{\tilde{\Phi}} \tag{9.6}$$

An important condition required for the new fields ρ_i to satisfy is the orthogonality to $\tilde{\Phi}$:

$$\tilde{\Phi}(z_1)\rho_i(z_2) = O(z_{12}) \tag{9.7}$$

One solution for such requirement is given by

$$\tilde{\rho}_1 = -(k+3)\phi_0 + \alpha_+\vec{\alpha}_1\vec{\varphi}, \quad \tilde{\rho}_2 = (k+3)\phi_0 + \alpha_+\vec{\alpha}_2\vec{\varphi}$$

If we further impose orthonormality among the ρ_i's,

$$\rho_i(z_1)\rho_j(z_2) = -\delta_{ij} ln z_{12} + O(z_{12}), \qquad i,j = 1,2 \tag{9.8}$$

the unique change of variables satisfying both (9.7) and (9.8) has the form

$$\begin{aligned} \phi_0 &= -\frac{1}{2k+3}\sqrt{\frac{k+3}{k-3}}[(1+\beta_1)g_1\rho_1 - (1+\beta_2)g_2\rho_2] - \sqrt{\frac{3}{2k+3}}\tilde{\Phi} \\ \vec{\alpha}_1\vec{\varphi} &= -\frac{1}{(2k+3)\sqrt{2k-6}}[(k+3-k\beta_1)g_1\rho_1 - (k+3-k\beta_2)g_2\rho_2] - \sqrt{\frac{3(k+3}{2(2k+3)}}\tilde{\Phi} \\ \vec{\alpha}_2\vec{\varphi} &= -\frac{1}{(2k+3)\sqrt{2k-6}}[(k-(k+3)\beta_1)g_1\rho_1 - (k-(k+3)\beta_2)g_2\rho_2] + \sqrt{\frac{3(k+3}{2(2k+3)}}\tilde{\Phi} \end{aligned} \tag{9.9}$$

where

$$\beta_j = \frac{1}{2}(k+1-(-1)^j\sqrt{(k+1)(k-3)}), \;\; g_j^2 = \frac{1}{2}[(k-3)(k+1)+(-1)^j(k^2-3)\sqrt{\frac{k-3}{k+1}}],$$

$j = 1, 2$. Substituting (9.9) in (9.1) we realize that the reduced $W_3^{(1,1)}$ currents can be written in the form (9.6) with $a = \sqrt{\frac{3}{2k+3}}$ and $T_{\tilde{\Phi}} = \frac{1}{2}(\partial\tilde{\Phi})^2$. This provides us with the free field representation of the $V_3^{(1,1)}$ currents we were looking for,

$$\begin{aligned}
V^-(z) &= [(k+2)\partial\eta - \eta\sqrt{\frac{k+3}{k-3}}(g_1\partial\rho_1 - g_2\partial\rho_2)]e^{-a_1\rho_1+a_2\rho_2},\\
V^+(z) &= [(k+2)\partial^2\xi + \partial\xi\sqrt{\frac{k+3}{k-3}}(\beta_1 g_1\partial\rho_1 - \beta_2 g_2\partial\rho_2)]e^{a_1\rho_1-a_2\rho_2},\\
T_V &= -\frac{1}{2}[(\partial\rho_1)^2+(\partial\rho_2)^2] + \sum_{i=1}^{2}\gamma_i\partial^2\rho_i + T_{\xi\eta}, \quad T_{\xi\eta} = \eta\partial\xi = \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi
\end{aligned} \tag{9.10}$$

where

$$\gamma_i = \frac{(-1)^{i+1}}{4}\frac{(k+1)}{\sqrt{k^2-g}}(k-1+(-1)^{i+1}\sqrt{(k+1)(k-3)})g_i; \quad a_i = \sqrt{\frac{k+3}{k-3}}\frac{(1+\beta_i)g_i}{2k+3}$$

and we have denoted $\xi = e^{\chi}, \quad \eta = e^{-\chi}$. One can easely recognize (ξ, η) as a pair of fermionic ghosts of spins $(0, 1)$.

We are now prepared to derive the explicit form of the $V_3^{(1,1)}$ OPE algebra $(k \neq -3, -\frac{3}{2}, -1)$:

$$\begin{aligned}
V^{\mp}(z_1)V^{\mp}(z_2) &= (z_{12})^{-\frac{3}{2k+3}}V_2^{\mp}(z_2) + O(z_{12})\\
V^+(z_1)V^-(z_2) &= (z_{12})^{\frac{3}{2k+3}}[\frac{(2k+3)(k+1)}{z_{12}^3} - \frac{k+3}{z_{12}}T_V(z_2) + \frac{k+3}{2}(2W_3 - \partial T_V)] + O(z_{12})\\
T_V(z_1)V^{\mp}(z_2) &= \frac{\Delta_1^{\mp}}{z_{12}^2}V^{\mp}(z_2) + \frac{1}{z_{12}}\partial_{z_2}V^{\mp}(z_2) + O(z_{12})
\end{aligned} \tag{9.11}$$

where $\Delta_1^{\mp} = \frac{3}{2}(1 - \frac{1}{2k+3})$ are the *renormalized spins* (dimensions) of the *quantum currents* $V_1^{\pm}$ $(\equiv V^{\pm})$ and $V_2^{\pm}(z)$ are new currents of spins $\Delta_2^{\mp} = \frac{3}{2}(1 - \frac{2}{2k+3})$. For example $V_2^-(z)$ has the form

$$\begin{aligned}
V_2^-(z) &= \{[(k+2)\partial\chi + \sqrt{\frac{k+3}{k-3}}(g_1\partial\rho_1 - g_2\partial\rho_2)]^2 - \partial[(k+2)\partial\chi\\
&+ \sqrt{\frac{k+3}{k-3}}(g_1\partial\rho_1 - g_2\partial\rho_2)]\}e^{-2\chi-2a_1\rho_1+2a_2\rho_2}
\end{aligned}$$

The stress tensor $T_V(z)$ of spin 2 satisfies the standard Virasoro algebra OPE's with specific central charge

$$c_V = c_W - 1 = -6\frac{(k+1)^2}{k+3}$$

The OPE's of V_2^+, V_2^- and $V_1^\pm$ ($\equiv V^\pm$) introduce more new currents $V_l^\pm$ of spins ($L = 2k+3$)

$$\Delta_l = \frac{3}{2}l(1-\frac{l}{L}), \quad l = 1,2,\cdots$$

and with $U(1)$ charges $Q_0 = l$ (see Sect. 4 for the definition of Q_0):

$$V_l^\pm(z_1)V_{l'}^\pm(z_2) = C_{l,l'} z_{12}^{-\frac{3ll'}{L}} V_{l+l'}^\pm(z_2) + O(z_{12}) \tag{9.12}$$

The OPE's $V_l^+V_l^-$ give rise to new Q_0-neutral currents W_{l+1}, ($l = 2,\cdots,L-4$) of spins $\Delta_l = l+1$

$$V_l^+(z_1)V_l^-(z_2) = z_{12}^{\frac{3l^2}{L}}(\frac{1}{z_{12}^{3l}} + \frac{2\Delta_l}{c_V z_{12}^{3l-2}}T_V(z_2) + \cdots + \frac{d_l}{z_{12}}W_{3l-1} + O(1)) \tag{9.13}$$

For example the W_3 current that appears in the finite part of eqn. (9.11) and in the singular part of (9.13) (for $l \geq 2$) has the form

$$W_3 = B_\alpha \partial^3\rho_\alpha + B_{\alpha\beta}\partial^2\rho_\alpha\partial\rho_\beta + C_{\alpha\beta}(\partial\rho_\alpha)^2\partial\rho_\beta, \qquad \alpha,\beta = 1,2,3 \tag{9.14}$$

where we have denoted $\rho_3 = \chi$ and

$$\begin{aligned}
B_{33} &= 6B_3 = \frac{3}{2}C_{33} = -\frac{2k+5}{2}, \;\; B_{i3} = C_{3i} = (-1)^i(k+2)a_i, \; i = 1,2 \\
B_{11} &= -B_{22} = \frac{1}{2}\sqrt{\frac{k+1}{k-3}}, \;\; B_i = (-1)^i\frac{1}{6}(\gamma_i + \frac{k+1}{\sqrt{k^2-9}}g_i) \\
C_{ij} &= (-1)^{j+1}\frac{3^{|i-j|}}{6}a_j[k+4-2|i-j| + (-1)^i\sqrt{\frac{k+1}{k-3}}] \\
B_{ij} &= \frac{1}{2}\frac{g_ig_j}{(k-3)(2k+3)}[k(k+1) + (-1)^i(k+2)\sqrt{(k+1)(k-3)}], \; i \neq j
\end{aligned}$$

For generic k ($= \frac{L-3}{2}$) this W_3-current and T_V ($= W_2$) do not close the standard W_3 algebra. In the OPE of $W_3(z_1)W_3(z_2)$ the new W_4-current contributes together with the $\Lambda =: T^2 : -\frac{3}{10}\partial^2 T$, etc. It is not difficult to verify that all the W_{l+1} ($l = 1,\cdots$) form the standard W_∞-algebra which appears as a subalgebra of the *parafermionic extension of the* W_∞*-algebra* $W_\infty^{(pf)}$ *of central charge* c_V *spanned by* $V_l^\pm$ *and* W_{l+1}, $l = 1,2,\cdots$. As we shall show latter this is the case when L is positive, rational and *noninteger*. The natural question to ask is *whether exist values of* L *for which the algebra of the currents* $V_l^\pm$, W_{l+1} *closes for finite* l. The most interesting case would be if this happens for $l = 1$, i.e. when $V_1^\pm$ and T_V alone form a closed algebra. To answer this question one has to analyze the L dependence of the singularities in the OPE's (9.11), (9.12) and (9.13) (the order of the poles and cuts) and to calculate the structure constants $C_{l,l'}$ and d_l. The exponents $\pm\frac{3}{L}$ in (9.11) that gives the L-dependent singularities of the OPE's suggest to consider separately the following intervals of values of L:

Case (1): $L < 3$. No singularities in the $V^\pm V^\pm$ OPE's. Only poles of order ≤ 3 and $(-\frac{3}{|L|})$-cut in the V^+V^- OPE. Hence $V^\pm$ ($\Delta^\pm = \frac{3}{2}(1+\frac{1}{|L|})$) and T_V alone close an algebra.

Case (2): $-3 < L < 0$. Poles of order higher then 3 and $(-\frac{3}{|L|})$-cut in the V^+V^- OPE. No singular terms in $V^\pm V^\pm$. For each $|L|$ in the interval $s-1 \leq \frac{3}{|L|} < s$ ($s = 2, 3, \cdots$) the V^+V^- contains poles of order $\leq 2+s$ and involve $s-2$ new currents W_p $(p = 3, 4, \cdots s+1)$. Whether $V^\pm$, T and W_p $(p = 3, \cdots s)$ close an algebra is an open question. In the simplest case $s = 2$ the straightforward calculation based on the bosonized form (9.10) and (9.14) of $V^\pm$, T and W_3 show us that the $W_3(1)W_3(2)$ OPE introduce new W_4 current. This only proves that $V^\pm$, T and W_3 do not close an algebra in this case. It might happen that for some specific value of L (and $s = 2$) the algebra spanned by finite number of currents $V^\pm$, T_V and $W_{l+1}, l = 2, \cdots, M)$ closes. However it is more natural to expect that T_V and $W_{l+1}, l = 2, \cdots, M)$ form $V^\pm$*-extension of the* W_∞*-algebra.*

Case (3): $0 < L \leq 3$. Pole of order $[\frac{3}{L}]$ and a cut in the $V^\pm V^\pm$ OPE. Poles of order < 3 and $(\frac{3}{L})$-cut in V^+V^-. Therefore no W_{l+1} currents can appear.

($3a$) For L *rational noninteger* one has to consider all the $V_l^\pm$'s $(l = 1, 2, \cdots,)$ and T_V in order to close an algebra. The latter is an example of *purely parafermionic* W_∞ algebra. Note that $\Delta_l^\pm < 0$ for $l > L$, i.e. for $L < 1$ all the $V_L^\pm$ have negative dimensions; for $1 < L < 2$-all, but $V_1^\pm$ and for $2 < L < 3$-all but $V_1^\pm$ and $V_2^\pm$.

($3b$) The case L integer ≤ 3 provides two simple examples of quantum $V_3^{(1,1)}(L)$-algebras generated by finite number of currents: for $L = 2$ these are $V^\pm$ of $\Delta^\pm = \frac{3}{4}$ and T_V $(c_V = -\frac{3}{5})$ and for $L = 3$, T_V and $V_1^\pm$,$V_2^\pm$ of $\Delta_1^\pm = \Delta_2^\pm = 1$ and $c_V(3) = -2$. We shall give the explicit form of these two algebras later in this section.

Case (4):$L > 3$. No poles in $V^\pm V^\pm$. Poles of order ≤ 3 and $(\frac{3}{L})$-cut in V^+V^-. Therefore the Laurent modes of $V^\pm$ and T_V have to close an algebra. As in the standard PF case [3] the first of the OPE's (9.11) define the modes of $V_2^\pm$ as an infinite sum of bilinears of the $V^\pm$ modes. The OPE's $V^+_{[\frac{L}{2}]}V^-_{[\frac{L}{2}]}$ represent poles of order $L-2$ (higher than 3 for $L > 5$) and thus introduce new currents $W_{p+1}, p = 2, \cdots, L-4$.

($4a$) For L *integer* $(L > 3)$, $\Delta_l^\pm = \Delta_{L-l}^\pm$ (i.e. $\Delta_L^\pm = 0$) the $V_l^\pm V_{l'}^\pm$ subalgebra involve $L-1$ currents only, i.e. $l = 1, \cdots, L-1$. Whether the algebra of $2(L-1)+L-4$ currents $V_l^\pm, W_{p+1},(l = 1, \cdots, L-1,\ p = 1, 2, \cdots L-4)$ closes for $L > 5$ or one has to consider an infinite set of W_{p+1}, $(p = 1, 2\cdots)$, $V_l^\pm, l = 1, 2, \cdots, L-1$ is an open question. For $L = 4, 5$ the $V_l^\pm$ and T_V do form closed algebras.

($4b$) For L rational *noninteger* the OPE's $V_l^\pm V_{l'}^\pm$ introduce infinite number of PF-currents of $U(1)$-charges $l = 1, 2, \cdots$. Together with the neutral currents W_{l+1} they span a kind of *PF* W_∞-algebra. We have to note that the *difference* between the algebras ($4a$) and ($4b$) can be *summarized* in the fact that for L-integer the corresponding OPE's represent the multiplication rules of the discrete group $Z_L \otimes Z_2$ (with the identification $V_l^+ = V_{L-l}^-$), $l = 1, \cdots, L-1$ being the Z_L-charges. In the noninteger L case ($4b$) the symmetry encoded in the OPE's is $U(1) \otimes Z_2$.

Our analysis of the singularities of the OPE's (9.12) and (9.13) shows that the quantum $V_3^{(1,1)}(L)$-algebra shares many of the properties of the (both unitary and nonunitary) PF-algebras [3]. This fact allows us to apply the methods developed in ref. [3] in the derivation of the explicit (Laurent modes) form of the $V_3^{(1,1)}(L)$. We restrict ourselves to consider cases (1), ($3b$) and ($4a$) only. The allowed boundary conditions [3] (for L-integer) of the currents

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[3] We are not considering here the twisted (or C-disorder) boundary conditions: $V^\pm(ze^{2\pi i})\phi_s(0) =$

$V^{\pm}$:

$$V^{\pm}(ze^{2\pi i})\phi_s^{\eta}(0) = e^{2\pi i(bs+\eta)}V^{\pm}(z)\phi_s^{\eta}(0)$$

where $b = \frac{3}{2L}$, $\eta = 0, \frac{1}{2}$, $s = 1, \cdots L-1$ lead to the following mode expansion

$$V^{\pm}(z)\phi_s^{\eta}(0) = \sum_{m=-\infty}^{\infty} z^{\pm\frac{3s}{2L}+m-1\mp\eta} V^{\pm}_{-m\pm\eta-\frac{1}{2}+\frac{3(1\mp s)}{2L}}\phi_s^{\eta}(0) \tag{9.15}$$

The $\phi_s^{\eta}(0)$ denote certain Ramond ($\eta = \frac{1}{2}$, s-odd) and the Neveu-Schwarz ($\eta = 0$, s-even) fields. Following the arguments of ref. [3] (Sect. 4) we derive the $V_3^{(1,1)}(L)$-algebra (for $|L| > 3$) from the OPE's (9.11):

$$\begin{aligned} &\frac{2}{L+3}\sum_{p=0}^{\infty} C^p_{(-\frac{3}{L})}[V^{+}_{-\frac{3(s+1)}{2L}+m-p-\eta+\frac{1}{2}} V^{-}_{\frac{3(s+1)}{2L}+n+p+\eta-\frac{1}{2}} + V^{-}_{-\frac{3(1-s)}{2L}+n-p+\eta-\frac{1}{2}} V^{+}_{\frac{3(1-s)}{2L}+m+p-\eta+\frac{1}{2}}] \\ = \quad & -L_{m+n} + \frac{1}{2}\frac{(L-1)L}{(L+3)}(\frac{3s}{2L}+n+\eta)(\frac{3s}{2L}+n+\eta-1)\delta_{m+n,0} \end{aligned} \tag{9.16}$$

where $C^p_{(M)} = \frac{\Gamma(p-M)}{p!\Gamma(-M)}$, $m, n = 0, \pm 1, \pm 2, \cdots$ and:

$$\sum_{p=0}^{\infty} C^p_{(\frac{3}{L})}[V^{\pm}_{\frac{3(3\mp s)}{2L}-p+m+\eta-\frac{1}{2}} V^{\pm}_{\frac{3(1\mp s)}{2L}+p+n+\eta-\frac{1}{2}} - V^{\pm}_{\frac{3(3\mp s)}{2L}-p+n+\eta-\frac{1}{2}} V^{\pm}_{\frac{3(1\mp s)}{2L}+p+m+\eta-\frac{1}{2}}] = 0 \tag{9.17}$$

In fact eqns. (9.16) and (9.17) together with the Virasoro algebra of the L_n's, ($L_n = \oint z^{n+1}T(z)dz$)

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c_V}{12}n(n^2-1)\delta_{m+n,0}$$

represent the entire $V_3^{(1,1)}$-algebra for $L < -3$ only. In the case (4*a*.) (i.e. for $L > 3$) they give the subalgebra spanned by $V_1^{\pm}$ and T of the larger $V_3^{(1,1)}$-algebra which also includes $V_l^{\pm}$, $l = 2, 3, \cdots, L-1$ and certain W_p's. The algebra of the $V_l^{\pm}$'s charges ($l = 1, \cdots, L-1$):

$$V^{\pm(l)}_{m-\frac{l}{2}+\eta_l+\frac{3l(l\mp s)}{2L}}\phi_s^{\eta_l}(0) = \oint dz z^{\mp\frac{3ls}{2L}+m+l+\eta_l-1}V_l^{\pm}(z)\phi_s^{\eta_l}(0) \tag{9.18}$$

can be easely derived from the OPE's (9.12) and (9.13). Note that as a consequence of (9.12) and (9.15) the $V^{\pm}_{(2l)}$-currents have only Neveu-Schwarz boundary conditions, i.e. $\eta_{2l} = 0$, $\eta_{2l+1} = 0, \frac{1}{2}$. All algebraic relations that follows from (9.12) are either in the form (9.17) or give $V^{\pm(l)}_{-m}$ as an infinite sum of bilinears of $V^{\pm(l_1)}_{-m_1}V^{\pm(l_2)}_{-m_2}$, $l = l_1 + l_2$ as for example:

$$V^{\pm(2)}_{m+n+\frac{3(2\mp s)}{L}} = \sum_{p=0}^{\infty} C^p_{(\frac{3-L}{L})}[V^{\pm(1)}_{m-p+\frac{3(3\mp s)}{2L}} V^{\pm(1)}_{n+p+\frac{3(1\mp s)}{2L}} + V^{\pm(1)}_{n-p-1+\frac{3(3\mp s)}{2L}} V^{\pm(1)}_{m+p+1+\frac{3(1\mp s)}{2L}}] \tag{9.19}$$

$e^{2\pi i b_s}V^{\mp}(z)\phi_{s\pm 2}(0)$

To give an idea how the entire $V_3^{(1,1)}(L)$-algebra looks like we consider in more detail the simplest case $L = 4$. The $V_3^{(1,1)}(4)$-algebra is generated by $V_1^{\pm}, V_2^{\pm}$ and T_V of spins $\Delta_1^{\pm} = \frac{9}{8}, \Delta_2^{\pm} = \frac{3}{2}$ and $\Delta_T = 2$. We have to complete the algebraic relations (9.16), (9.17) of the $V_{(1)}^{\pm}, T$ subalgebra with those involving $V_{(2)}^{\pm}$. Taking $s = 2$ sector for simplicity we find

$$V_m^{+(2)}V_n^{-(2)} + V_n^{-(2)}V_m^{+(2)} = \frac{1}{2}(n^2 - \frac{1}{4})\delta_{m+n,0} - \frac{7}{9}L_{m+n} \tag{9.20}$$

$$V_{m+n}^{\pm(2)} = \sum_{p=0}^{\infty} C^p_{(-\frac{1}{4})}[V^{\pm(1)}_{m-p+\frac{3}{8}-\eta_\pm}V^{\pm(1)}_{n+p-\frac{3}{8}+\eta_\pm} + V^{\pm(1)}_{n-p-\frac{5}{8}+\eta_\pm}V^{\pm(1)}_{m+p+\frac{5}{8}-\eta_\pm}] \tag{9.21}$$

where $\eta_+ = 0$ and $\eta_- = \frac{1}{2}$. The remaining relations involving $V_{m_1}^{\pm(2)}V_{m_2}^{\pm(2)}$, $V_{m_1}^{\pm(1)}V_{m_2}^{\pm(2)}$ and $V_{m_1}^{\pm(2)}V_{m_2}^{\mp(1)}$ have a form similar to (9.17) and (9.19). Note that due to the identification $V_l^+ = V_{L-l}^-$ we have

$$V_{(2)}^+(1)V_{(1)}^-(2) = \frac{c_{2,3}}{z_{12}^{\frac{3}{2}}}V_{(1)}^+(z_2) + O(z_{12}), \quad V_{(2)}^{\pm}(1)V_{(1)}^{\pm}(2) = \frac{c_{2,1}}{z_{12}^{\frac{3}{2}}}V_{(1)}^{\mp}(z_2) + O(z_{12})$$

The eqns. (9.20) and (9.21) are an indication to consider the $V_3^{(1,1)}(4)$-algebra as “ a square root” of the Virasoro superalgebra.

Our last example is the $L = 2$ $V_3^{(1,1)}$-algebra. It is spanned by $V_1^{\pm}$ of $\Delta^{\pm} = \frac{3}{4}$ and T_V and its central charge is $c_V = -\frac{3}{5}$. This is one of the cases $(3b)$ when the OPE $V_1^{\pm}V_1^{\pm}$ has a pole and therefore the “ commutation relations” (9.17) are *not valid*. Instead we obtain:

$$\sum_{p=0}^{\infty} C^p_{(\frac{1}{2})}[V^-_{-p+m+(\eta-\frac{1}{2})-\frac{1}{4}}V^-_{p+n+(\eta-\frac{1}{2})-\frac{3}{4}} + V^-_{-p+n+(\eta-\frac{1}{2})-\frac{1}{4}}V^-_{p+m+(\eta-\frac{1}{2})-\frac{3}{4}}] = \delta_{m+n+2\eta,0} \tag{9.22}$$

and similar one for the V^+V^+’s . The eqn. (9.16) in this case take the form $(s = 2)$:

$$\frac{2}{5}\sum_{p=0}^{\infty} C^p_{(-\frac{3}{2})}[V^+_{-\frac{3}{4}+m-p-\eta}V^-_{\frac{3}{4}+n+p+\eta} + V^-_{-\frac{3}{4}+n-p+\eta}V^+_{\frac{3}{4}+m+p-\eta}] = -L_{m+n} + \frac{1}{5}(n+\eta+\frac{1}{2})(n+\eta+\frac{3}{2})\delta_{m+n,0} \tag{9.23}$$

The complicated structure of the $V_3^{(1,1)}(L)$-“commutation relations” makes the problem of the construction of irreducible highest weight representations rather difficult. One could however further extend the relation between $W_3^{(2)}$ and $V_3^{(1,1)}$-algebras on their representations. The way we have derived the explicit form of the $V_3^{(1,1)}$-generators (9.10) out of the $W_3^{(2)}$-ones (9.1):

$$G^{\pm} = V^{\pm}e^{\pm\sqrt{\frac{3}{L}}\tilde{\Phi}}, \quad T_W = T_V + \frac{1}{2}(\partial\tilde{\Phi})^2, \quad J = \sqrt{\frac{L}{3}}\partial\tilde{\Phi} \tag{9.24}$$

suggests the following form of the $W_3^{(2)}$ chiral vertex operators $\phi^{r_1 s_1}_{r_2 s_2}(z) \equiv \phi^W_{(r_i,s_i)}(z)$ in terms of the $V_3^{(1,1)}$-ones, $\phi^V_{(r_i,s_i)}(z)$ and $\tilde{\Phi}(z)$:

$$\phi^W_{(r_i,s_i)}(z) = \phi^V_{(r_i,s_i)}(z)e^{\beta_{r,s}\tilde{\Phi}} \tag{9.25}$$

Taking into account the basic OPE's that define the ϕ^W's:

$$\begin{aligned} T^W(z_1)\phi^W_{(r,s)}(z_2) &= \frac{\Delta^W_{r,s}}{z_{12}^2}\phi^W_{(r,s)}(z_2) + \frac{1}{z_{12}}\partial\phi^W_{(r,s)}(z_2) + O(z_{12}) \\ J(z_1)\phi^W_{(r,s)}(z_2) &= \frac{q_{r,s}}{z_{12}}\phi^W_{(r,s)}(z_2) + O(z_{12}), \qquad J(z_1)\phi^V_{(r,s)}(z_2) = O(z_{12}) \end{aligned}$$

and from eqn (9.24) we conclude that:

$$\beta_{r,s} = q_{r,s}\sqrt{\frac{3}{L}}, \qquad \Delta^V_{r,s} = \Delta^W_{r,s} - \frac{3}{2L}q^2_{r,s} \tag{9.26}$$

The dimensions and the charges of the fields $\phi^W_{(r,s)}$ that represent the so called "complete degenerate" highest weight representations of $W_3^{(2)}$ are given by [26] (for *rational* levels $L+3=\frac{4p}{q}$):

$$\begin{aligned} \Delta^W_{r,s} &= \frac{1}{24(L+3)}[((L+3)r_{12}-4s_{12})^2 + 3((L+3)r_1+4s_1)((L+3)r_2-4s_2) - 3(L-1)^2] - \frac{1}{8}\eta^W, \\ q_{r,s} &= \frac{1}{12}[(L+3)r_{12}-4s_{12}] \pm \frac{1}{2}\eta^W, \quad r_{12}=r_1-r_2, \quad s_{12}=s_1-s_2, \end{aligned} \tag{9.27}$$

where $\eta^W=0$, r_i-odd integers for NS-sectors, $\eta^W=\frac{1}{2}$, r_i-even integers for R-sectors and $r_i, s_i (i=1,2)$ take their values in the interval $1 \le r_i \le 2p-1$, $1 \le s_i \le 2q-1$. According to eqn. (9.26) the corresponding representations of $V_3^{(1,1)}(L)$-algebra have the following dimensions:

$$\begin{aligned} \Delta^V_{r,s} = \frac{1}{32(L+3)}&[(L-3)((L+3)r_{12}-4s_{12})^2 + 4((L+3)r_1-4s_1)((L+3)r_2-4s_2) - \\ &-4L(L-1)^2] - \frac{\eta^W}{8L}[L+3\eta^W \pm ((L+3)r_{12}-4s_{12})]. \end{aligned} \tag{9.28}$$

We have to note that $\eta^W = \frac{1}{2} - \eta^V$, $(\eta^V=\eta)$. The analog of the $W_3^{(l)}$–chiral fields ϕ_{s_0}:

$$\begin{aligned} &G^{\pm}_{-1/2}\phi_{s_0}|0\rangle = 0 \qquad \Delta^W = \frac{6}{L+3}q_{\pm}(q_{\pm} \pm \frac{L-1}{4}), \\ &q_+ = q_- = \frac{1}{6}(L+5-2s_0), \quad r_1 = 3r_2 = 3, s_1 = s_0, s_2 = 1 \end{aligned}$$

are the order-disorder parameter fields σ^η_s in the $V_3^{(1,1)}$-models:

$$V^{\pm}_{\frac{3(1\pm s)}{2L}-1/2+\eta}\sigma^\eta_s|0\rangle = 0$$

One can find their dimensions directly from (9.16):

$$\Delta^{(1/2)}_s = \frac{1}{8}\frac{L(L-1)(3s+L)(3s-L)}{(L+3)}, \qquad s-odd$$

They are a particular case of the (9.28) when

$$\begin{aligned} 2s_0 = 3s + 2L + 5, &\qquad s_0 = 1, 3, \cdots L-1 \\ r_1 = 3r_2 = 3, &\qquad s_2 = 1, \quad s_1 = s_0. \end{aligned}$$

The complete description of the representations of the $V_3^{(1,1)}(L)$-algebra (even for the case L-positive integer $L > 3$) however requires much more work. The construction of representations for the other *Cases* $(1), (2)$ and (3) is an interesting open problem.

The *purpose* of this rather detail discussion of the *quantum* $V_3^{(1,1)}(L)$-algebras was to point out the *differences* with the quantization of the W_3 and $W_3^{(2)}$ algebras and the *similarities* with the PF-algebra. The origin of all these complications is the renormalization of the spins of the $V^{\pm}$-currents

$$\Delta_1^{\pm^{quantum}} = \Delta_1^{\pm^{class}} - \frac{3}{2L}$$

which makes the singularities of the OPE-algebra (9.11) L-dependent. For certain values of L this requires to introduce new currents $V_l^{\pm}$ and $W_p^{\pm}$ in order to close the OPE-algebra. An important new phenomena is the breaking of the (classical) $U(1)$-symmetry to the discrete Z_L-symmetries for L positive integer. The typical PF feature is the replacing of the commutators or anticommutators with an infinite sum of bilinears of generators as in eqns. (9.16), (9.17), (9.18) , (9.22). One might wonder whether the $V_{n+1}^{(1,1)}$-algebras exhibit similar features. Our preliminary result shows that the renormalization of the spins of the nonlocal currents $V_{n+1}^{(1,1)}$ is a common property of all $V_{n+1}^{(1,1)}$'s: $\Delta_{(n)}^{\pm(q)} = \frac{n+1}{2}(1 - \frac{1}{2k+n+1})$. As usual the spins of the local currents W_{l+1} remain equal to the classical ones. All this indicates that the quantum $V_{n=1}^{(1,1)}$-algebras obey many of the properties of the quantum $V_3^{(1,1)}$-algebra.

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A Appendix A

Here we define a generic Lie algebra $\mathcal{G}$ in the Chevalley basis by the commutation relations

$$[h_i, h_j] = 0, \quad [h_i, E_{\pm\alpha_j}] = \pm k_{ji} E_{\pm\alpha_j}, \quad [E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} h_j \tag{A.1}$$

$i, j = 1, \cdots, rank\ \mathcal{G}$, where $h_i = \frac{2\alpha_i \cdot H}{\alpha_i^2}$ and H_i define $\mathcal{G}$ in the Cartan-Weyl basis, i.e.

$$[H_i, H_j] = 0, \quad [H_i, E_{\pm\alpha}] = \pm(\alpha)^i E_{\pm\alpha}, \quad [E_{\alpha_i}, E_{-\alpha_j}] = \frac{2\alpha \cdot H}{\alpha^2} \tag{A.2}$$

and $(\alpha)^i$ denote the i^{th} component of the root α and $k_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}$ is the Cartan matrix.

The rank $\mathcal{G}$ fundamental weights are defined by

$$\frac{2\alpha_i \cdot \lambda_j}{\alpha_j^2} = \delta_{ij}, \quad i,j = 1, ..., rank \;\; \mathcal{G} \tag{A.3}$$

and may be written in terms of simple roots as

$$\lambda_i = \mathcal{K}_{ij}\alpha_j \tag{A.4}$$

An invariant scalar product on the Lie algebra is defined by rescaling the trace of two generators in some finite dimensional representation such that

$$\mathrm{Tr}(H_i H_j) = \delta_{ij}, \quad \mathrm{Tr}(H_i E_\alpha) = 0, \quad Tr(E_\alpha E_\beta) = \frac{2}{\alpha^2}\delta_{\alpha+\beta,0} \tag{A.5}$$

A general Lie algebra $\mathcal{G}$ may be decomposed into a graded structure generated by a grading operator Q, such that

$$[Q, \mathcal{G}_{\pm i}] = \pm i\mathcal{G}_{\pm i}, \quad [\mathcal{G}_i, \mathcal{G}_j] \in \mathcal{G}_{i+j} \tag{A.6}$$

and $\mathcal{G} = \oplus_i \mathcal{G}_i$.

Throughout this paper several different gradings are used. We shall restrict ourselves to integer gradings. The most familiar is defined by

$$Q = \sum_{k=1}^{rank\mathcal{G}} \frac{2\lambda_k \cdot H}{\alpha_k^2} \tag{A.7}$$

In this case $\mathcal{G}_{\pm i}$ contain positive/negative step operators composed of i simple roots. It then follows that

$$\mathcal{G}_< = \oplus_{i<0}\mathcal{G}_i, \quad \mathcal{G}_> = \oplus_{i>0}\mathcal{G}_i \tag{A.8}$$

are nilpotent subalgebras generated by positive/negative step operators, while the zero grade $\mathcal{G}_0$ is an abelian subalgebra and is spanned by the Cartan subalgebra of $\mathcal{G}$, i.e. $\mathcal{G}_0 = U(1)^{rank\mathcal{G}}$. A general group element of $\mathcal{G}$ can be decomposed according to these nilpotent subalgebras together with the exponentiation of the abelian subalgebra $\mathcal{G}_0$ using the Gauss decomposition formula,

$$g = g_- g_0 g_+ \tag{A.9}$$

where g_- and g_+ are obtained by exponentiation of $\mathcal{G}_<$ and $\mathcal{G}_>$ respectively. Other gradings extensively used in this paper are defined by:

$$Q_j = \sum_{k\neq j}^{rank\mathcal{G}} \frac{2\lambda_k \cdot H}{\alpha_k^2} \tag{A.10}$$

The absence of the i^{th} fundamental weight in (A.10) generates a non abelian structure in the zero grade subalgebra $\mathcal{G}_0$ which is now generated by the Cartan subalgebra together with $E_{\pm\alpha_i}$, i.e. $\mathcal{G}_0 = SL(2) \otimes U(1)^{rank\mathcal{G}-1}$. The nilpotent subgroups $g_-^{(j)}$ and $g_+^{(j)}$ are generated by exponentiation of the negative and positive grades respectively (according to Q_j). The non abelian Gauss decomposition formula now reads

$$g = g_-^{(j)} g_0^{(j)} g_+^{(j)} \tag{A.11}$$

where $g_0^{(j)}$ is the $SL(2) \otimes U(1)^{rank\mathcal{G}-1}$subgroup generated by exponentiation of $\mathcal{G}_0$.

Following the same line of thought, more and more complicated non abelian structure can be introduced in $\mathcal{G}_0$ by defining grading operators with the form

$$Q_{i_1,i_2,\cdots i_l} = \sum_{k \neq i_1,\cdots i_l}^{rank\mathcal{G}} \frac{2\lambda_k \cdot H}{\alpha_k^2} \tag{A.12}$$

where $\mathcal{G}_0$ has now the form

$$\mathcal{G}_0 = g_1 \otimes g_2 \otimes \cdots \otimes g_m \otimes (U(1))^{rank\mathcal{G} - \sum_{a=1}^{m} rank g_a} \tag{A.13}$$

and the nonabelian Gauss decomposition is formally given by eqn. (A.11).

These are useful concepts in order to derive a simple and compact form for the equations of motion. Let the WZW currents be decomposed according to a gradation Q_i. From (A.11) we find,

$$\begin{aligned} J &= g^{-1}\partial g = g_+^{(j)^{-1}} \left(g_0^{(j)^{-1}} g_-^{(j)^{-1}} \partial g_-^{(j)} g_0^{(j)} + g_0^{(j)^{-1}} \partial g_0^{(j)} + \partial g_+^{(j)} g_+^{(j)^{-1}} \right) g_+^{(j)} \\ \bar{J} &= -\bar{\partial}\partial g g^{-1} = -g_-^{(j)} \left(g_-^{(j)^{-1}} \bar{\partial} g_-^{(j)} + \bar{\partial} g_0^{(j)} g_0^{(j)^{-1}} + g_0^{(j)} \bar{\partial} g_+^{(j)} g_+^{(j)^{-1}} g_0^{(j)^{-1}} \right) g_-^{(j)^{-1}} \end{aligned} \tag{A.14}$$

We now define the reduced model by giving the constant element $\epsilon_\pm$ responsible for constraining the currents in a general manner to

$$\begin{aligned} J_{constr} &= \epsilon_- + j \\ \bar{J}_{constr} &= \epsilon_+ + \bar{j} \end{aligned} \tag{A.15}$$

where j (and $\bar{j}$) contain only positive and zero (negative and zero) grades, $\epsilon_\pm$ are constant generators of grade ± 1 respectively. From the graded structure (A.14) and (A.15) and the fact that the grades are integers, we find that

$$\begin{aligned} g_0^{(j)^{-1}} g_-^{(j)^{-1}} \partial g_-^{(j)} g_0^{(j)}|_{constr} &= \epsilon_- \\ g_0^{(j)} \bar{\partial} g_+^{(j)} g_+^{(j)^{-1}} g_0^{(j)^{-1}}|_{constr} &= \epsilon_+ \end{aligned} \tag{A.16}$$

and the equations of motion $\bar{\partial} J = \partial \bar{J} = 0$ become

$$\begin{aligned} \bar{\partial} K + [K, \bar{\partial} g_+^{(j)} g_+^{(j)^{-1}}] &= 0 \\ \partial \bar{K} - [\bar{K}, g_-^{(j)^{-1}} \partial g_-^{(j)}] &= 0 \end{aligned} \tag{A.17}$$

where

$$\begin{aligned} K &= g_0^{(j)^{-1}} g_-^{(j)^{-1}} \partial g_-^{(j)} g_0^{(j)} + g_0^{(j)^{-1}} \partial g_0^{(j)} + \partial g_+^{(j)} g_+^{(j)^{-1}} \\ \bar{K} &= g_-^{(j)^{-1}} \bar{\partial} g_-^{(j)} + \bar{\partial} g_0^{(j)} g_0^{(j)^{-1}} + g_0^{(j)} \bar{\partial} g_-^{(j)} g_-^{(j)^{-1}} g_0^{(j)^{-1}} \end{aligned} \tag{A.18}$$

Imposing (A.16) into (A.17) we find the nontrivial equations to correspond to the zero grade components of eqns. (A.17), i.e.

$$\begin{aligned}\bar\partial(g_0^{(j)^{-1}}\partial g_0^{(j)}) &+ [\epsilon_-, g_0^{(j)^{-1}}\epsilon_+ g_0^{(j)}] = 0\\ \partial(\bar\partial g_0^{(j)} g_0^{(j)^{-1}}) &- [\epsilon_+, g_0^{(j)}\epsilon_- g_0^{(j)^{-1}}] = 0\end{aligned} \qquad (A.19)$$

We now point out that there is a subalgebra of $\mathcal{G}_0$, namely $\mathcal{G}_0^0$, commuting with $\epsilon_\pm$ such that

$$\bar\partial \mathrm{Tr}(g_0^{(j)^{-1}}\partial g_0^{(j)}\mathcal{H}) = \partial \mathrm{Tr}(\bar\partial g_0^{(j)} g_0^{(j)^{-1}}\mathcal{H}) = 0 \qquad (A.20)$$

where $\mathcal{H} \in \mathcal{G}_0^0$. We therefore impose an additional subsidiary constraint within the subalgebra $\mathcal{G}_0^0$,

$$J_{\mathcal{H}} = \mathrm{Tr}(g_0^{(j)^{-1}}\partial g_0^{(j)}\mathcal{H}) = 0, \qquad \bar J_{\mathcal{H}} = \mathrm{Tr}(\bar\partial g_0^{(j)} g_0^{(j)^{-1}}\mathcal{H}) = 0 \qquad (A.21)$$

For the simple case of gradation (A.10) $\mathcal{H}$ correspond to $\frac{2\lambda_j\cdot H}{\alpha_j^2}$. In terms of fields defined by

$$g_0^{(j)} = e^{\chi E_{-\alpha_j}} e^{\frac{2\lambda_j\cdot H}{\alpha_j^2}R_j + \sum_{l\neq j} H_l\phi_l} e^{\psi E_{\alpha_j}} \qquad (A.22)$$

we find

$$\begin{aligned} g_0^{(j)^{-1}}\partial g_0^{(j)} = &\left(\partial\chi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right)E_{-\alpha_j} + \sum_{l\neq j} H_l\left(\partial\phi_l + \frac{\alpha_l^2\mathcal{K}_{jl}}{\alpha_j^2\mathcal{K}_{jj}}\psi\partial\chi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right)\\ &+\frac{2\lambda_j\cdot H}{\alpha_j^2}\left(\partial R_j - \frac{1}{\mathcal{K}_{jj}}\psi\partial\chi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right) + \left(\partial\psi + \psi\partial R + \psi k_{jl}\partial\phi_l - \psi^2\partial\chi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right)E_{\alpha_j}\end{aligned}$$

and

$$\begin{aligned} \bar\partial g_0^{(j)} g_0^{(j)^{-1}} = &\left(\bar\partial\psi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right)E_{\alpha_j} + \sum_{l\neq j} H_l\left(\bar\partial\phi_l + \frac{\alpha_l^2\mathcal{K}_{jl}}{\alpha_j^2\mathcal{K}_{jj}}\chi\bar\partial\psi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right)\\ &+\frac{2\lambda_j\cdot H}{\alpha_j^2}\left(\bar\partial R_j - \frac{1}{\mathcal{K}_{jj}}\chi\bar\partial\psi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right) + \left(\bar\partial\chi + \chi\bar\partial R + \chi k_{jl}\bar\partial\phi_l - \chi^2\bar\partial\psi e^{R_j+\sum_{l\neq j}k_{jl}\phi_l}\right)E_{-\alpha_j}\end{aligned}$$

Moreover,

$$J_{\frac{2\lambda_j\cdot H}{\alpha_j^2}} = \partial R_j\Delta_j - \frac{1}{\mathcal{K}_{jj}}\tilde\psi\partial\tilde\chi e^{\sum_{l\neq j}k_{jl}\phi_l}, \qquad \bar J_{\frac{2\lambda_j\cdot H}{\alpha_j^2}} = \bar\partial R_j\Delta_j - \frac{1}{\mathcal{K}_{jj}}\tilde\chi\bar\partial\tilde\psi e^{\sum_{l\neq j}k_{jl}\phi_l}$$

where $\Delta_j = 1 + \frac{1}{2\mathcal{K}_{jj}}\tilde\psi\tilde\chi e^{\sum_{l\neq j}k_{jl}\phi_l}$. The subsidiary condition (A.21) yields,

$$\partial R_j = \frac{1}{\mathcal{K}_{jj}\Delta_j}\tilde\psi\partial\tilde\chi e^{\sum_{l\neq j}k_{jl}\phi_l}, \qquad \bar\partial R_j = \frac{1}{\mathcal{K}_{jj}\Delta_j}\tilde\chi\partial\tilde\psi e^{\sum_{l\neq j}k_{jl}\phi_l}$$

where $\tilde\psi = \psi e^{\frac{1}{2}R_j}$ and $\tilde\chi = \chi e^{\frac{1}{2}R_j}$. When inserted into (A.19), leads to the equations of motion (2.3) of the NA-Toda model described by the action (7.3) and (2.3) for the case$j = 1$.

Henceforth, given an integer gradation Q and the constant elements $\epsilon_\pm$, in the Lie algebra, we are able to decompose the currents according to Q, implement consistently the constraints (A.15) to obtain (A.16) and therefore the equations of motion (and hence the action, (see [22])). This construction highlights the fact that the first set of constraints in (2.4) are fully determined by Q and $\epsilon_\pm$. The subsidiary condition (A.21) has to be implemented by fiat since it constraints the subalgebra $\mathcal{G}_0^0$.

Conversely, since the structure of the constraints is such that allows for dynamical degrees of freedom only those contained in the zero grade subgroup $\mathcal{G}_0$ (or more precisely $\mathcal{G}_0/\mathcal{G}_0^0$), the constraints (2.4) suggest a construction of a gauged WZW action invariant under the constraint subgroup $H_+ \otimes H_-$, as in sect. 2. The gauge invariance suggests decomposing the group element as

$$g = \tilde{g}_-^{(j)} g_0^f \tilde{g}_+^{(j)} = g_-^{(j)} g_0^{(j)} g_+^{(j)} \tag{A.23}$$

where $\tilde{g}_-^{(j)}$, ($\tilde{g}_+^{(j)}$) are generated by exponentiation of negative (positive) grade generators together with those in $\mathcal{G}_0^0$,

$$\tilde{g_+^{(j)}} = g_+^{(j)} e^{\mathcal{H}(R_j)}, \quad \tilde{g_-^{(j)}} = e^{\mathcal{H}(R_j)} g_-^{(j)} \tag{A.24}$$

where R_j are nonlocal, nonphysical fields eliminated by the constraints (A.21).The g_0^f is now generated by exponentiating $\frac{\mathcal{G}}{\mathcal{G}_0^0}$. The decomposition (A.23) is important to point out the symmetry structure of the NA-Toda models as described in sect. 2.

B Appendix B

We shall derive the general solution of eqns. (3.18) for generic W_s-transformations ($3 \leq s \leq n-1$, fixed), i.e. $\epsilon^\pm = \epsilon = 0$, $\eta_p = 0$ for all $p \neq s$, ($\eta_s \equiv \eta$). Let us first consider the equations for ϵ_{ik}'s such that $i > k$. We have,

$$\begin{aligned} \epsilon_{l+m,l} &= 0, \quad l = 1, \cdots s-1, \quad l+m = n-s+3, \cdots, n+1 \\ \epsilon_{l+m,l} &= 0, \quad l = s+1, \cdots n, \quad l+m = s+2, \cdots, n+1 \end{aligned} \tag{B.1}$$

and all the remaining ϵ_{ik}'s ($i > k$) satisfy the following recursive relations

$$\epsilon_{n-p,s} + \eta W_{p+1} + \frac{k}{2}\partial\epsilon_{n-p+1,s} = 0 \quad p = 0, \cdots, n-s-1, \;\; s \neq 1 \tag{B.2}$$

$$\epsilon_{n-p,1} = \sum_{l=1}^{p-s+3} (-\frac{k}{2}\partial)^{l-1}(V^+\epsilon_{n-p+l,2}) \tag{B.3}$$

$$\epsilon_{n-p,s-r} = \sum_{l=1}^{p-s+2} (-\frac{k}{2}\partial)^{p-l-r+2}\epsilon_{n-l-r+3,s-r+1}, \tag{B.4}$$

$n-p = 2, \cdots n-s+2$; $r = 0, \cdots, s-2$. The solution of (B.2) is given by

$$\epsilon_{n-p,s} = (-\frac{k}{2}\partial)^{p+1}\eta - \sum_{l=1}^{p}(-\frac{k}{2}\partial)^{l-1}(\eta W_{p+2-l}), \quad p = 0, \cdots, n-s-1 \tag{B.5}$$

The eqn. (B.4) can be simplified to:

$$\epsilon_{n-p,s-r} = \sum_{l=1}^{p-r+2} \begin{pmatrix} l+r-2 \\ r-1 \end{pmatrix} (-\frac{k}{2}\partial)^{l-1} \epsilon_{n-p+l+r-1,s} \tag{B.6}$$

and taking into account (B.5) we find the general solution of (B.4):

$$\epsilon_{n-p,s-r} = \begin{pmatrix} p+1 \\ r \end{pmatrix} (-\frac{k}{2}\partial)^{p-r+1}\eta - \sum_{l=1}^{p-r} \begin{pmatrix} l+r-1 \\ r \end{pmatrix} (-\frac{k}{2}\partial)^{l-1}(\eta W_{p-r-l+2}), \tag{B.7}$$

$r = 0, \cdots, s-2, \;\; n-p = 2, \cdots, n-s+2$. As a consequence of (B.3) and (B.7) we obtain

$$\begin{aligned} \epsilon_{n-p,1} &= \sum_{l=1}^{p-s+3} (-\frac{k}{2}\partial)^{l-1} \{V^{+}[\begin{pmatrix} p-l+1 \\ s-2 \end{pmatrix} (-\frac{k}{2}\partial)^{p-l-s+3}\eta \\ &- \sum_{m=1}^{p-l-s+2} \begin{pmatrix} m+s-3 \\ s-2 \end{pmatrix} (-\frac{k}{2}\partial)^{m-1}\eta W_{p-l-m-s+4}]\} \end{aligned} \tag{B.8}$$

The equations for the diagonal elements ϵ_{ll} read

$$\begin{gathered} 2\epsilon_{n,n} + \epsilon_{22} + \cdots + \epsilon_{n-1,n-1} = 0, \quad \epsilon_{nn} = \epsilon_{n-1,n-1} = \cdots = \epsilon_{s+1,s+1}, \quad \epsilon_{11} = 0 \\ \epsilon_{s+1,s+1} = \epsilon_{ss} + \eta W_{n-s+1} + \frac{k}{2}\partial\epsilon_{s+1,s}, \quad \epsilon_{pp} = \epsilon_{22} + \sum_{r=3}^{p} (\frac{k}{2}\partial)\epsilon_{r,r-1}, \end{gathered} \tag{B.9}$$

$p = 3, \cdots, s$. The solution of (B.9) can be written in the following compact form

$$\begin{aligned} \epsilon_{ll} &= [\begin{pmatrix} n-l+1 \\ n-s+1 \end{pmatrix} - \frac{1}{n} \begin{pmatrix} n \\ n-s+2 \end{pmatrix}](-\frac{k}{2}\partial)^{n-s+1}\eta \\ &+ \sum_{p=0}^{n-s-1} [\frac{1}{n} \begin{pmatrix} p+s-1 \\ p+1 \end{pmatrix} - \begin{pmatrix} p+s-l \\ p \end{pmatrix}](-\frac{k}{2}\partial)^{p}(\eta W_{n-s-p+1}) \end{aligned} \tag{B.10}$$

$l = 2, \cdots, s, s+1, \cdots, n$. We next consider the recursion relations for the upper triangular part of the ϵ_{ik}'s, $(i < k)$:

$$\begin{aligned} \epsilon_{1l} &= 0, \quad l = 1, \cdots s; \quad \epsilon_{1l} = (\frac{k}{2}\partial)^{l-s-1}(\eta V^{-}), \quad l = s+1, \cdots, n+1 \\ \epsilon_{2m} &= (\frac{k}{2}\partial)^{m-2}\epsilon_{22} + (\frac{k}{2}\partial)^{m-s-1}(\eta W_n) + \sum_{l=1}^{m-s-1} (\frac{k}{2}\partial)^{m-s-l-1}[V^{+}(\frac{k}{2}\partial)^{l-1}(\eta V^{-})], \end{aligned} \tag{B.11}$$

$m = 2, \cdots, n+1$. For generic ϵ_{ml}, $(2 \leq m \leq n, \; m < l)$ we have two different formulas for $2 \leq m \leq s$ and $s < m \leq n$,

$(a) \qquad 2 \leq m \leq s, \qquad l = m+1, \cdots n+1$

$$\epsilon_{m,l} = \sum_{p=1}^{l-m} \binom{p+m-4}{m-3} (\frac{k}{2}\partial)^{p-1} \epsilon_{2,l-m-p+3} + \sum_{p=1}^{m-2} \binom{l-m+p-2}{p-1} (\frac{k}{2}\partial)^{l-m} \epsilon_{m-p+1,m-p+1}$$
$$+ \sum_{p=1}^{m-2} \binom{l-m+p-2}{p-1} (\frac{k}{2}\partial)^{l-s-p} (\eta W_{n-m+p+1}) \qquad \text{(B.12)}$$

$(b) \qquad s < m \leq n, \qquad l = m+1, \cdots n+1$

$$\epsilon_{m,l} = \sum_{p=1}^{l-m} \binom{p+m-s-2}{m-s-1} (\frac{k}{2}\partial)^{p-1} \epsilon_{s,l-m-p+s+1} + \sum_{p=1}^{m-s} \binom{l+p-m-2}{p-1} (\frac{k}{2}\partial)^{l-m} \epsilon_{m-p+1,m-p+1} \qquad \text{(B.13)}$$

The solution of eqn. (B.12) is given by ($2 \leq m \leq s$):

$$\epsilon_{ml} = \{A_{l,m}(s) + \frac{1}{n} \binom{n}{n-s+2} [(n-s+1) \binom{l-3}{m-2} - \binom{l-3}{m-3}]\} \times$$
$$\times (-1)^{l-m} (-\frac{k}{2}\partial)^{n-s-m+l+1} \eta + \sum_{p=1}^{l-m-s+1} \binom{l-s-p-1}{m-2} (\frac{k}{2}\partial)^{l-m-s-p+1} [V^{+} (\frac{k}{2}\partial)^{p-1} (\eta V^{-})] +$$
$$+ \sum_{p=0}^{n-s-1} \{\frac{1}{n} \binom{l-2}{m-2} \binom{p+s-1}{p+1} - \binom{l-3}{m-2} \binom{p+s-2}{p} - B^{p}_{l,m}(s)\} \times$$
$$\times (-1)^{l-m} (-\frac{k}{2}\partial)^{p+l-m} (\eta W_{n-s-p+1}) + \sum_{p=1}^{m-1} \binom{l-s-1}{p-1} (\frac{k}{2}\partial)^{l-s-p} (\eta W_{n-m+p+1}), \qquad \text{(B.14)}$$

where

$$A_{lm}(s) = \sum_{p=0}^{s-3} \binom{l-p-4}{l-m-1} \binom{n-p-2}{n-s+1}, \quad B^{p}_{lm}(s) = \sum_{q=0}^{s-3} \binom{l-q-4}{l-m-1} \binom{p+s-q-3}{p}$$

The corresponding solution of eqn. (B.13) ($s < m \leq n$) differs from (B.14) only by the last term which in this case is

$$\epsilon_{ml} = \{A_{l,m}(s) + \sum_{p=1}^{s-1} \binom{l-s-1}{l-m-p} (\frac{k}{2}\partial)^{l-m-p} (\eta W_{n-s+p+1})$$
$$+ \frac{1}{n} \binom{n}{n-s+2} [(n-s+1) \binom{l-3}{m-2} - \binom{l-3}{m-3}]\} (-1)^{l-m} (-\frac{k}{2}\partial)^{n-s-m+l+1} \eta$$
$$+ \sum_{p=1}^{l-m-s+1} \binom{l-s-p-1}{m-2} (\frac{k}{2}\partial)^{l-m-s-p+1} [V^{+} (\frac{k}{2}\partial)^{p-1} (\eta V^{-})]$$
$$+ \sum_{p=0}^{n-s-1} \{\frac{1}{n} \binom{l-2}{m-2} \binom{p+s-1}{p+1} - \binom{l-3}{m-2} \binom{p+s-2}{p} - B^{p}_{l,m}(s)\} \times$$
$$\times (-1)^{l-m} (-\frac{k}{2}\partial)^{p+l-m} (\eta W_{n-s-p+1}) + \sum_{p=1}^{s-1} \binom{l-s-1}{l-m-p} (\frac{k}{2}\partial)^{l-m-p} (\eta W_{n-s+p+1}), \quad \text{(B.15)}$$

The W_s transformation ($\eta \equiv \eta_s$) derived from eqn. (3.18) ($3 \leq s \leq n-1$) can be written in a compact form as

$$\begin{aligned}
\delta_\eta V^+ &= \epsilon_{22} V^+ - (\frac{k}{2}\partial)\epsilon_{21}, \quad \delta_\eta V^- = \sum_{l=s+1}^{n} \epsilon_{1l} W_{n-l+2} - \epsilon_{nn} V^- - (\frac{k}{2}\partial)\epsilon_{1,n+1}, \\
\delta_\eta W_n &= \epsilon_{21} V^- - \epsilon_{1,n+1} V^+ + (\epsilon_{22} - \epsilon_{nn}) W_n + \sum_{l=3}^{n} \epsilon_{2l} W_{n-l+2} - \frac{k}{2}\partial\epsilon_{2,n+1}, \\
\delta_\eta T &= \epsilon_{n,s-1} W_{n-s+3} + \epsilon_{n,s} W_{n-s+2} - \epsilon_{n-1,n+1} - \frac{k}{2}\partial\epsilon_{n,n+1}, \qquad \text{(B.16)}
\end{aligned}$$

and

$$\begin{aligned}
(a) & \qquad n-s+2 \leq q \leq n \\
\delta_\eta W_q &= \epsilon_{n-q+2,1} V^- + \sum_{l=2}^{n} (\epsilon_{n-q+2,l} - \delta_{l,n-q+2}\epsilon_{nn}) W_{n-l+2} - \epsilon_{n-q+1,n+1} - \frac{k}{2}\partial\epsilon_{n-q+2,n+1} \\
(b) & \qquad 2 \leq q < s \quad or \quad s \leq q < n-s+2 \\
\delta_\eta W_q &= P_q(s) + (\epsilon_{n-q+2,n-q+2} - \epsilon_{nn}) W_q + \sum_{l=n-q+3}^{n} \epsilon_{n-q+2,l} W_{n-l+2} \\
&- \epsilon_{n-q+1,n+1} - \frac{k}{2}\partial\epsilon_{n-q+2,n+1} \qquad \text{(B.17)}
\end{aligned}$$

where

$$P_q(s) = \begin{cases} \sum_{l=s-q+1}^{s} \epsilon_{n-q+2,l} W_{n-l+2}, & for \quad 2 \leq q < s \\ \sum_{l=2}^{s} \epsilon_{n-q+2,l} W_{n-l+2} + \epsilon_{n-q+2,1} V^-, & for \quad s \leq q < n-s+2 \end{cases} \qquad \text{(B.18)}$$

Taking into account the explicit form (B.10), (B.14) and (B.15) of ϵ_{ml}'s we realize the following simplifications:

$$\epsilon_{ll} - \epsilon_{nn} = \begin{pmatrix} n-l+1 \\ n-s+1 \end{pmatrix} [(-\frac{k}{2}\partial)^{n-s+1}]\eta - \sum_{p=0}^{n-s-1} \begin{pmatrix} p+s-l \\ p \end{pmatrix} (-\frac{k}{2}\partial)^p (\eta W_{n-s-p+1}) \qquad \text{(B.19)}$$

$$\begin{aligned}
C_{m,n}(s) &= \epsilon_{m,n+1} + \frac{k}{2}\partial\epsilon_{m+1,n+1} = \\
&= \{A_{n+2,m+1}(s) + \frac{1}{n}\begin{pmatrix} n \\ n-s+2 \end{pmatrix} [(n-s+1)\begin{pmatrix} n-1 \\ m-1 \end{pmatrix} - \begin{pmatrix} n-1 \\ m-2 \end{pmatrix}]]\} \times \\
&\times (-1)^{n-m+1} (-\frac{k}{2}\partial)^{2n-s-m+2}\eta + \sum_{r=1}^{n-m-s+2} \begin{pmatrix} n-s-r+1 \\ m-1 \end{pmatrix} (\frac{k}{2}\partial)^{n-m-s-r+2} [V^+ (\frac{k}{2}\partial)^{r-1} (\eta V^-)] \\
&+ \sum_{p=0}^{n-s-1} (-1)^{n-m+1} \{\frac{1}{n}\begin{pmatrix} n \\ m-1 \end{pmatrix}\begin{pmatrix} p+s-1 \\ p+1 \end{pmatrix} - \begin{pmatrix} n-1 \\ m-1 \end{pmatrix}\begin{pmatrix} p+s-2 \\ p \end{pmatrix} \\
&- B^p_{n+2,m+1}(s)\} (-\frac{k}{2}\partial)^{n+p-m+1} (\eta W_{n-s-p+1}) + D_{m,n}(s), \qquad \text{(B.20)}
\end{aligned}$$

where

$$D_{m,n}(s)=\left\{\begin{array}{ll}\sum_{p=1}^{m}\left(\begin{array}{c}n-s+1\\ p-1\end{array}\right)(\frac{k}{2}\partial)^{n-s-p+2}(\eta W_{n-m+p}), & m\leq s\\ \sum_{p=1}^{s-1}\left(\begin{array}{c}n-s+1\\ m-s+p\end{array}\right)(\frac{k}{2}\partial)^{n-m-p+1}(\eta W_{n-s+p+1}), & m>s\end{array}\right. \tag{B.21}$$

Finally, we arrive at the W_s-transformations we seek,

$$\begin{aligned}
\delta_\eta V^- &= \frac{1}{n}\left(\begin{array}{c}n\\ n-s+2\end{array}\right)V^-(-\frac{k}{2}\partial)^{n-s+1}\eta-(\frac{k}{2}\partial)^{n-s+1}(\eta V^-)\\
&\quad -\frac{1}{n}V^-\sum_{p=0}^{n-s-1}\left(\begin{array}{c}p+s-1\\ s+1\end{array}\right)(-\frac{k}{2}\partial)^p(\eta W_{n-s-p+1})+\sum_{l=s+1}^{n}W_{n-l+2}(\frac{k}{2}\partial)^{l-s-1}(\eta V^-),\\
\delta_\eta V^+ &= \frac{1}{n}V^+\{(n-s+1)\left(\begin{array}{c}n\\ n-s+2\end{array}\right)(-\frac{k}{2}\partial)^{n-s+1}\eta+\sum_{l=0}^{n-s-1}[\left(\begin{array}{c}l+s-1\\ l+1\end{array}\right)\\
&\quad -n\left(\begin{array}{c}l+s-2\\ l\end{array}\right)](-\frac{k}{2}\partial)^l(\eta W_{n-s-l+1})\}+\sum_{l=1}^{n-s+1}(-\frac{k}{2}\partial)^l\{V^+[\left(\begin{array}{c}n-l-1\\ s-2\end{array}\right)(-\frac{k}{2}\partial)^{n-l-s+1}\eta\\
&\quad -\sum_{m=1}^{n-l-s}\left(\begin{array}{c}m+s-3\\ s-2\end{array}\right)(-\frac{k}{2}\partial)^{m-1}(\eta W_{n-l-m-s+2})]\},\\
\delta_\eta T &= -(\frac{k}{2}\partial\eta)W_{n-s+2}-(n-s+1)(\frac{k}{2}\partial)(\eta W_{n-s+2})-\frac{(n-s+1)}{2}\left(\begin{array}{c}n+1\\ n-s+3\end{array}\right)\times\\
&\quad \times(-\frac{k}{2}\partial)^{n-s+3}\eta-\sum_{p=0}^{n-s-1}\{\frac{n-1}{2}\left(\begin{array}{c}p+s-1\\ p+1\end{array}\right)-(n-1)\left(\begin{array}{c}p+s-2\\ p\end{array}\right)\\
&\quad -(n-s-p)\left(\begin{array}{c}p+s-2\\ p+1\end{array}\right)-(p+1)\left(\begin{array}{c}p+s-1\\ p+2\end{array}\right)\}(-\frac{k}{2}\partial)^{p+2}(\eta W_{n-s-p+1})
\end{aligned}$$

and $\delta_\eta W_n$ is given by (B.16) with

$$\begin{aligned}
\epsilon_{21} &= \sum_{l=1}^{n-s+1}(-\frac{k}{2}\partial)^{l-1}\{V^+[\left(\begin{array}{c}n-l-1\\ s-2\end{array}\right)(-\frac{k}{2}\partial)^{n-l-s+1}\eta\\
&\quad -\sum_{m=1}^{n-l-s}\left(\begin{array}{c}m+s-3\\ s-2\end{array}\right)(-\frac{k}{2}\partial)^{m-1}(\eta W_{n-l-m-s+2})]\}\\
\epsilon_{2l} &= \frac{n-s+1}{n}\left(\begin{array}{c}n\\ n-s+2\end{array}\right)(-1)^{l-2}[(-\frac{k}{2}\partial)^{n+l-s-1}\eta]+\sum_{p=1}^{l-s-1}(\frac{k}{2}\partial)^{l-s-p-1}[V^+(\frac{k}{2}\partial)^{p-1}(\eta V^-)]\\
&\quad +\frac{1}{n}\sum_{p=0}^{n-s-1}[\left(\begin{array}{c}p+s-1\\ p+1\end{array}\right)-n\left(\begin{array}{c}p+s-2\\ p\end{array}\right)](-1)^p(\frac{k}{2}\partial)^{p+l-2}(\eta W_{n-s-p+1})+(\frac{k}{2}\partial)^{l-s-1}(\eta W_n)
\end{aligned}$$

Substituting $\epsilon_{n-q+2,1}$from (B.8), $\epsilon_{n-q+2,l}$ from (B.14) and (B.15) and $\epsilon_{ll}-\epsilon_{nn}$ and $C_{n-q+1,n}(s)$ from (B.20) in the transformation laws (B.17) and (B.18) we derive the explicit form of the η_s transformation of the W_q currents $(2\leq q<n)$.

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