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REINHARDT DOMAINS OF HOLOMORPHY IN BANACH SPACES

by

Mário C. MATOS* and Leopoldo NACHBIN[1]**

[1]Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Instituto de Matemática, Estatística e Ciência da Computação
Universidade Estadual de Campinas
Caixa Postal 6065
13081 - Campinas, SP - Brasil

**Department of Mathematics
University of Rochester
Rochester, N.Y., 14627
U.S.A.

ABSTRACT. In a Banach Space with unconditional Schauder basis it is not true in general that all holomorphic functions are representable by multiple power series (the series of monomials) but, for Reinhardt domains, every domain of existence is the domain of existence of a holomorphic function representable by a multiple power series and, in fact, the domain of convergence of such a series.

Key-words: Holomorphy types; Domains of holomorphy.

I. INTRODUCTION

In 1908, at the International Congress of Mathematics in Rome, D. Hilbert outlined a theory of holomorphic functions in infinitely many variables. For him a holomorphic function had an expansion in series of the form

$$f(x_1,x_2,\ldots) = \sum_{\substack{n_1,\ldots,n_k=0 \\ k=1,2,\ldots}}^{\infty} c_{n_1 n_2 \ldots n_k} x_1^{n_1} \ldots x_k^{n_k}$$

with the series being absolutely convergent on $|x_1| \leq \varepsilon_1, \ldots, |x_k| \leq \varepsilon_k, \ldots$. His results were published in 1909 (see [17]) and they were concerned with analytic continuation and composition of holomorphic functions. After the works of M. Frēchet ([7],[8],[9],[10],[11]) and R. Gâteaux ([12],[13],[14]) it became clear that holomorphic functions in infinite dimensional vector spaces should be studied as representable by power series of n-homogeneous polynomials (obtained from n-linear functions) since this implicit point of view had a far more range than the explicit representation through the variables as it was proposed by D. Hilbert. In a series of papers, A.D. Michal, a student of Frēchet , and his own students A.H. Clifford, R.S. Martin, I.G. Highbert, A.E. Taylor developed the theory of holomorphic mappings along this line of reasoning and stablished definitely the equivalence of the concept of holomorphic mapping between normed spaces (i.e., a mapping represented by a power series of homogeneous polynomials in a neighborhood of each point) with the existence of a Frēchet derivative at each point and with continuity plus the existence of a Gâteaux derivative at each point. See [15],[20], [22] and [27]. After that the research in Infinite Dimensional Holomorphy was worked out by considering the concept of holomorphic mapping under this point of view. In 1978 P.J. Bolland and S. Dineen (see [3]) brought back the multiple power series representation when they studied holomorphic functions on nuclear locally convex spaces with a basis. See also [5] and Dineen's book [6] for a good exposition of results related to this line of reasoning. In [21] M.C. Matos characterized all the holomorphic functions in open subsets of Banach spaces with unconditional basis which are representable by multiple power series (of monomials) and made applications concerning continuous functions in $[0,2\pi]$ having absolutely convergent Fourier series. His approach is connected with the notion of holomorphy type introduced by L. Nachbin in [24].

In this article we give examples of Banach spaces with unconditional basis in which there are holomorphic functions (even polynomials) *not representable* by multiple power series. It is known that in open subsets of ℓ_1 all holomorphic functions *are representable* by multiple power series (see [21] and [26]) and that in open subsets of c_0 these functions representable by multiple power series coincide with the nuclear holomorphic functions (see [21]). However, in general we prove here that, for domains of Reinhardt, domains of existence coincide with domains of existence of holomorphic functions representable by multiple power series. We also show that these are the domains of convergence of multiple power series, as well as the modularly decreasing logarithmically convex domains. Partial results in this direction were obtained by G.I. Katz in [19].

II. HOLOMORPHIC FUNCTIONS REPRESENTABLE BY MULTIPLE POWER SERIES

We consider throughout this article a complex Banach space E with a normalized unconditional Schauder basis $(b_j)_{j=1}^{\infty}$. Hence every z in E can be writen, in a unique way, as the sum of a series $\sum_{j=1}^{\infty} z_j b_j$, where $z_j \in \mathbb{C}$ is called the j-th component of z and we denote by φ_j the continuous linear functional on E defined by $\varphi_j(z) = z_j$ for every $j \in \mathbb{Z}_+ = \{1,2,\ldots\}$. In order to simplify the notations we write $I = \mathbb{N}^{(\mathbb{Z}_+)}$, i.e., I denotes the set of all sequences $(\alpha_j)_{j=1}^{\infty} = \alpha$ of natural numbers having only a finite number of terms different from zero. If $z \in E$ and $\alpha \in I$ we denote by z^α the complex number $z_{j_1}^{\alpha_{j_1}} \ldots z_{j_n}^{\alpha_{j_n}}$, where $\alpha_{j_1}, \ldots, \alpha_{j_n}$ are the non-zero terms of α.

A *multiple power series around a point* $a \in E$ is a series of the form

$$\sum_{\alpha \in I} c_\alpha (z - a)^\alpha \tag{1}$$

where $c_\alpha \in \mathbb{C}$ for every $\alpha \in I$. This series *converges to the value* $f(z)$ *at the point* $z \in E$ if for every $\varepsilon > 0$ there is a finite subset J_ε of I such that

$$\left| \sum_{\alpha \in J} c_\alpha (z - a)^\alpha - f(z) \right| < \varepsilon \tag{2}$$

for every finite subset J of I containing J_ε. In this case $f(z)$ is called the *sum of the series at the point* z and we also write

$$f(z) = \sum_{\alpha \in I} c_\alpha (z - a)^\alpha .$$

If B is a subset of E and the series converges to $f(z)$ at each $z \in B$, we say that the multiple power series *converges pointwise to* f *in* B. The *convergence is uniform over* B if (2) holds for every $z \in B$ with J_ε independent of $z \in B$.

Since I is a denumerable set, once a linear order is fixed in I, the multiple power series around a evaluated at a point $z \in E$ is a numerical series and the convergence defined by (2) is the unconditional (or, equivalently, absolute) convergence of this series. We recall that for a finite dimensional E the set of points where a multiple power series around a converges either reduces to $\{a\}$ or contains $\{a\}$ in its interior. As it was pointed out in [21] this does not happen when E is not finite dimensional. In this case the set of points of E where $\sum_{\alpha \in I} z^\alpha$ converges is formed by those z in E such that $(z_j)_{j=1}^{\infty} \in \ell_1$ and $|z_j| < 1$ for all $j \in \mathbb{Z}_+$. Hence, if $E \neq \ell_1$, this set has empty interior and does not reduce to $\{0\}$. If $E = \ell_1$ it is clear that this set contains the open unit ball of center 0. We say that a multiple power series around a *has a domain of convergence* D if $a \in D$ and D is the interior of the set of all points where the series converges pointwise (it is easy to see that D is connected). It was proved in [21], by using Baire's Theorem and a result of M. Zorn, that if a multiple power series around a has a domain of convergence D, then the pointwise sum f of this series defines a holomorphic function in D and the series converges uniformly and absolutely in a neighborhood of a. Therefore, if U is an open subset of E and f is a complex function defined in U such that there is a multiple power series around x converging to f pointwise in a neighborhood of x for every $x \in U$, then f is holomorphic in U.

We refer the reader to the books of L. Nachbin [24], J. Mujica [23] and S. Dineen [6] for the notations and basic results of th Theory of Infinite Dimensional Holomorphy.

We recall that we can always replace the norm of E by an equivalent one satisfying

$$\|x\| = \sup\{\|\sum_{j=1}^{k} \lambda_j x_j b_j\|;\ \lambda_j \in \mathbb{C},\ |\lambda_j| \leq 1, j,k \in \mathbb{Z}_+\}. \tag{3}$$

In this article the norm of E satisfies (3) and it is clear that $\|x\| = \||x|\|$ for every $x \in E$, when we set $|x| = \sum_{j=1}^{\infty} |x_j| b_j \in E$. If

P belongs to the vector space $P(^nE)$ of all continuous n-homogeneous polynomials in E and P is the pointwise sum of a multiple power series $\sum_{\alpha\in I} c_\alpha z^\alpha$ in a neighborhood of 0, then it follows that P is the pointwise sum of the same series in E. Hence we may define

$$\tilde{P}(z) = \sum_{\alpha\in I} |c_\alpha| z^\alpha \qquad (\forall z \in E), \qquad (4)$$

and get $\tilde{P} \in P(^nE)$. It is easy to show that

$$\nu(P) = \|\tilde{P}\| = \sup_{\|x\|\leq 1} |\tilde{P}(x)| = \sup_{\|x\|\leq 1} \sum_{\alpha\in I} |c_\alpha| |x|^\alpha \qquad (5)$$

defines a norm in the vector space $P_\nu(^nE)$ of all P in $P(^nE)$ which are representable pointwise in E by the sum of a multiple power series. This normed space is complete and

$$\|P\| \leq \nu(P) \qquad (\forall P \in P_\nu(^nE)). \qquad (6)$$

In [21] it was proved that $P_\nu(^n\ell_1) = P(^n\ell_1)$ with

$$\nu(P) \leq e^n\|P\| \qquad (\forall P \in P(^n\ell_1)). \qquad (7)$$

Since E has unconditional basis it follows that every continuous linear functional in E is an element of $P_\nu(^1E)$. Hence, if $P_f(^nE)$ denotes the vector space of all finite sums of n-th powers of elements of E', it is clear that $P_f(^nE) \subset P_\nu(^nE)$ for every $n \in \mathbb{N}$. The closure $P_c(^nE)$ of $P_f(^nE)$ under the norm $\|.\|$ of $P(^nE)$ is the Banach space of all n-homogeneous polynomials of compact type in E.

2.1. THEOREM. If $n \in \mathbb{N}$ then $\mathcal{P}_\nu({}^nE) \subset \mathcal{P}_c({}^nE)$ and this inclusion is continuous.

PROOF. Since $\mathcal{P}_f({}^nE) \subset \mathcal{P}_\nu({}^nE)$ and (6) holds for every $P \in \mathcal{P}_f({}^nE)$, it is enough to show that $\mathcal{P}_f({}^nE)$ is dense in $\mathcal{P}_\nu({}^nE)$ for the norm ν. If $P \in \mathcal{P}_\nu({}^nE)$ we have

$$P(x) = \sum_{\alpha \in J_n} c_\alpha x^\alpha \qquad (\forall x \in E),$$

where $J_n = \{\alpha \in I \ ; \ |\alpha| = \sum_{j=1}^{\infty} \alpha_j = n\}$ and the series converges uniformly and absolutely in the closed ball $\overline{B}_\rho(0)$ for some $\rho > 0$. Let $(\alpha^{(j)})_{j=1}^{\infty}$ be an enumeration of J_n. Hence, for every $\varepsilon > 0$, there is $m_\varepsilon > 0$ such that

$$\sup_{\|x\| \leq \rho} \sum_{j>m} |c_{\alpha^{(j)}}| \, |x|^{\alpha^{(j)}} < \varepsilon$$

for each $m \geq m_\varepsilon$. If $y \in E$ is such that $\|y\| = 1$ we may write $y = \rho^{-1} z$ with $z \in \overline{B}_\rho(0)$ and

$$\sup_{\|y\| = 1} \sum_{j>m} |c_{\alpha^{(j)}}| \ |y|^{\alpha^{(j)}} < \frac{\varepsilon}{\rho^n} \qquad (8)$$

for every $m \geq m_\varepsilon$. If we set

$$Q_m(x) = \sum_{j=1}^{m} c_{\alpha^{(j)}} x^{\alpha^{(j)}} \qquad (\forall x \in E)$$

we have $Q_m \in \mathcal{P}_f({}^nE)$ and, by (8),

$$\nu(P - Q_m) \leq \frac{\varepsilon}{\rho^n} \quad \text{for every } m \geq m_\varepsilon.$$

This shows that $P_f(^nE)$ is ν-dense in $P_\nu(^nE)$. Q.E.D.

2.2. COROLLARY. (1) $P_\nu(^n\ell_2) \neq P(^n\ell_2)$ for every $n \in \mathbb{N}$, $n > 1$

(2) $P_\nu(^m\ell_p) \neq P(^m\ell_p)$ if $p > 1$ and $m > p$.

(3) $P_c(^n\ell_1) = P(^n\ell_1)$ for every $n \in \mathbb{N}$.

PROOF. If $P_\nu(^nE) = P(^nE)$ it follows from 2.1 $P_c(^nE) = P(^nE)$. By a result of R. Alencar [1], if E is reflexive, $P(^nE)$ is reflexive. S.B. Chae proved in [4] that $P(^n\ell_2)$ is not reflexive for every $n \in \mathbb{N}$, $n > 1$. By a result of R. Aron $P(^m\ell_p)$ is not reflexive is $p > 1$ and $m > p$. Hence (1) and (2) follow . It is clear that (3) follows from 2.1 and (7).

Q.E.D.

In [21] it was proved that $(P_\nu(^nE))_{n=0}^\infty$ is a holomorphy type ν in Nachbin's sense (see [24]) and that, for every open subset U of E, *the vector space* $H_\nu(U)$ *of all holomorphic functions of type* ν *in* U (i.e., holomorphic functions in U such that $\limsup_{n\to\infty}[\nu(\frac{1}{n}\hat{d}^n f(x))]^{1/n} < +\infty$ for every $x \in U$) *coincides with the vector space of all functions in* U *which are representable pointwise by a multiple power series around* x *in a neighborhood of* x *for every* $x \in U$. Hence it is clear that Corollary 2.2 and (7) imply the following result.

2.3. COROLLARY. (1) $H_\nu(U) = H(U)$ if $E = \ell_1$.

(2) $H_\nu(U) \neq H(U)$ if $E = \ell_p$ with $p > 1$.

The following theorem will be used in next paragraph.

2.4. THEOREM. If D is the domain of convergence of a multiple series around $a \in E$, then its pointwise sum f is in $H_\nu(D)$.

PROOF. With no loss of generality we may consider $a = 0$. Let D be the domain of convergence of $\sum_{\alpha\in I} c_\alpha x^\alpha$ and let f be its pointwise sum in D. Since we always have pointwise absolute convergence in D it is clear that $|x| \in D$ for every $x \in D$. We have to show that f is representable pointwise in a neighborhood of $b \in D$ by a multiple power series around b. We take $\rho > 0$ such that $B_\rho(b)$ and $B_\rho(|b|)$ are contained in D. If $z \in B_\rho(b)$ it is clear that $\|z - b\| = \| |z - b| \| < \rho$ and $|z - b| + |b| \in B_\rho(|b|) \subset D$. Hence

$$\sum_{\beta\in I} \left| \sum_{\alpha\geq\beta} c_\alpha \binom{\alpha}{\beta} b^{\alpha-\beta} \right| |z - b|^\beta \leq$$

$$\leq \sum_{\beta\in I} \sum_{\alpha\geq\beta} |c_\alpha| \binom{\alpha}{\beta} |b|^{\alpha-\beta} |z - b|^\beta =$$

$$= \sum_{\alpha\in I} \sum_{\beta\leq\alpha} |c_\alpha| \binom{\alpha}{\beta} |b|^{\alpha-\beta} |z - b|^\beta =$$

$$= \sum_{\alpha\in I} |c_\alpha| (|b| + |z - b|)^\alpha < + \infty$$

for every $z \in B_\rho(b)$. This means that

$$f(z) = \sum_{\beta\in I} \Big(\sum_{\alpha\geq\beta} c_\alpha \binom{\alpha}{\beta} b^{\alpha-\beta} \Big) (z - b)^\beta$$

for every $z \in B_\rho(b)$ with absolute convergence of the series.

Q.E.D.

III. REINHARDT OPEN SETS OF HOLOMORPHY

A subset S of E is called a *Reinhardt set* if for every $x \in S$ and $\theta = (\theta_j)_{j=1}^{\infty} \in \mathbb{R}^{\mathbb{Z}_+}$ we have $x_\theta = \sum_{j=1}^{\infty} e^{i\theta_j} x_j b_j \in S$.

A *modularly decreasing subset* S of E is one such that for every $x \in S$ and $y \in E$ with $|y_j| \leq |x_j|$ for $j \in \mathbb{Z}_+$, we have $y \in S$. It is clear that in this case S is also a Reinhardt subset of E.

For $x,y \in E$, $r > 0$, $s > 0$, $r+s=1$ we can show that

$$|x|^r |y|^s = \sum_{j=1}^{\infty} |x_j|^r |y_j|^s b_j$$

defines an element of E. Since $\max\{|x_j|, |y_j|\} = \frac{1}{2}(|x_j|+|y_j|) + \frac{1}{2}\big||x_j|-|y_j|\big|$ and $|x_j|^r |y_j|^s \leq \max\{|x_j|,|y_j|\}$ for every $j \in \mathbb{Z}_+$, it follows that $|x|^r|y|^s \in E$. A subset U of E is called *logarithmically convex* if for every $x,y \in U$, $r > 0$, $s > 0$, $r+s=1$ we get $|x|^r|y|^s \in |U| = \{|u|;\ u \in U\}$.

3.1. PROPOSITION. If D is the domain of convergence of a multiple power series around 0, then D is modularly decreasing and logarithmically convex.

PROOF. It is clear that D is modularly decreasing. Let $\sum_{\alpha\in I} c_\alpha x^\alpha$ be a multiple power series around 0 having D as its domain of convergence. If $x,y \in D$ we consider $\varepsilon > 0$ such that $B_\varepsilon(|x|)$ and $B_\varepsilon(|y|)$ are contained in D. Hence if $t \in B_\varepsilon(0)$ we get

$$\sum_{\alpha\in I} |c_\alpha| \, [\,|x|+|t|\,]^\alpha < +\infty \quad \text{and} \quad \sum_{\alpha\in I} |c_\alpha| \, [\,|y|+|t|\,]^\alpha < +\infty .$$

From the Holder's inequality we have, for $r > 0$, $s > 0$, $r+s = 1$, $t \in B_\varepsilon(0)$,

$$|x_j|^r |y_j|^s + |t_j| = |x_j|^r |y_j|^s + |t_j|^r |t_j|^s \leq (|x_j|+|t_j|)^r (|y_j|+|t_j|)^s$$

and

$$\sum_{\alpha\in I} [|c_\alpha| (|x|+|t|)^\alpha]^r [|c_\alpha|(|y|+|t|)^\alpha]^s \leq$$

$$\leq [\sum_{\alpha\in I} |c_\alpha|(|x| + |t|)^\alpha]^r [\sum_{\alpha\in I} |c_\alpha| (|y| + |t|)^\alpha]^s < +\infty.$$

Hence

$$\sum_{\alpha\in I} |c_\alpha| |x|^r |y|^s + t|^\alpha \leq \sum_{\alpha\in I} |c_\alpha|^{r+s} (|x|^r |y|^s + |t|^r |t|^s)^\alpha \leq$$

$$\leq \sum_{\alpha\in I} [|c_\alpha| (|x| + |t|)^\alpha]^r [|c_\alpha| (|y| + |t|)^\alpha]^s < +\infty.$$

This means that the series converges absolutely at $|x|^r |y|^s + t$ for every $t \in B_\varepsilon(0)$. Hence $|x|^r |y|^s \in D$.

Q.E.D.

If U is an open Reinhardt subset of $\mathbb{C}^n$ and $0 \in U$, it is known that U is domain of holomorphy if, and only if, U is modularly decreasing and logarithmically convex. This is also equivalent to U being the domain of convergence of a multiple power series around 0. We shall prove a similar result for $U \subset E$. For this we need a few auxiliary concepts and results.

If K is a subset of E we set

$$\tilde{K} = \{x \in E;\ |x|^\alpha \leq \sup_{t\in K} |t|^\alpha \quad \forall\alpha \in I\}.$$

If A is a subset of an open subset U of E we denote

$$\widetilde{A}(\nu,U) = \{x \in U,\ |\widetilde{P}(x)| \leq \sup_{t \in A} |\widetilde{P}(t)| = \|\widetilde{P}\|_A \quad \forall P \in \mathcal{P}_\nu(E)\}$$

where $\mathcal{P}_\nu(E) = \bigoplus_{k=0}^{\infty} \mathcal{P}_\nu({}^kE)$ is the set of all polynomials in E which are pointwise representable in E by a multiple power series around 0. It is clear that $\widetilde{K}(\nu,U) \subset \widetilde{K}(\nu,E) \subset \widetilde{K}$ for every $K \subset U$.

3.2. THEOREM. If U is a modularly decreasing, logarithmically convex open subset of E it is possible to find an increasing sequence $(C_j)_{j=1}^{\infty}$ of open Reinhardt subsets of E such that $U = \bigcup_{j=1}^{\infty} C_j$ and

$$d_U(\widetilde{C}_j(\nu;U)) \geq \frac{1}{j} > 0$$

for every $j = 1,2,\ldots$. Here $d_U(A)$ means the distance of A to the boundary of U.

PROOF. We start by considering the following sets

$$A_j = \{x \in U\ ;\ d_U(x) > \frac{1}{j}\}$$

$$B_j = \{x \in A_j\ ;\ \|x\| < j \text{ and } d_{A_j}(x) > \frac{1}{j}\}$$

$$C_j = \{x \in B_j\ ;\ \sup_{n \geq j} \|T_n(x) - x\| < d_{B_j}(x)\}$$

where $T_n(x) = \sum_{j=1}^{n} x_j b_j$ and we set $E_n = T_n(E)$. It is easy to see that these sets are open and (i) $U = \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} C_j$, (ii) $T_n(C_j)$ is contained in $B_j \cap E_n$ if $n \geq j$, (iii) $B_j \cap E_n$ is relatively compact in $A_j \cap E_n$ for every j and n. Now we may use property (3) of the norm of E (see paragraph 2) to show that $\|x_\theta - y\| = \|x - y_{-\theta}\|$ for all $x,y \in E$ and $\theta = (\theta_j)_{j=1}^{\infty} \in \mathbb{R}^{\mathbb{Z}_+}$. This fact implies that A_j, B_j and C_j are Reinhardt open sets, since U is also a Reinhardt set. If $x \in A_k$, $y \in E$ and $|y_j| \leq |x_j|$ for $j = 1,2,\ldots$, then there is $r > \frac{1}{k}$ such that $x + B_r(0) \subset U$. Since U is modularly decreasing, we use property (3) of the norm of E to show that $y + B_r(0) \subset U$. This implies that $y \in A_k$ and A_k is modularly decreasing. Now, if $x,y \in A_k$, there is $r > \frac{1}{k}$ such that $|x| + B_r(0)$ and $|y| + B_r(0)$ are contained in U. Since U is logarithmically convex, we have $(|x| + |t|)^r(|y| + |t|)^s \in U$ for $t \in B_r(0)$, $r > 0$, $s > 0$ and $r+s = 1$. By Holder's inequality we get

$$|x_j|^r |y_j|^s + |t_j| = |x_j|^r |y_j|^s + |t_j|^r |t_j|^s \leq (|x_j|+|t_j|)^r (|y_j|+|t_j|)^s$$

for $j = 1,2,\ldots$. Since U is modularly decreasing we conclude that $|x|^r |y|^s + t \in U$ for every $t \in B_r(0)$. Hence $|x|^r |y|^s \in A_k$ and A_k is logarithmically convex. If $x \in \tilde{C}_j(\nu,U)$ we choose $k \geq j$ such that $x \in C_k$. We shall prove that

$$T_n x \in \widetilde{B_j \cap E_n}(\nu, U \cap E_n) \quad \text{for } n \geq k \qquad (*)$$

We consider $n \geq k$ and $P_n \in \mathcal{P}_\nu(E_n)$. It is clear that $P_n \circ T_n \in \mathcal{P}_\nu(E)$. As in the proof of Theorem 2.1. it is possible to show that every element of $\mathcal{P}_\nu(E)$ has its multiple power series around 0 converging absolutely and uniformly over every bounded subset of E. Since C_n is bounded in E, it follows that the multiple power series of $P_n \circ T_n$ around 0 converges uniformly and absolutely over C_n. If $\varepsilon > 0$ there is J_ε contained in I, J_ε finite such that

$$\sup_{t \in C_n} \left| \sum_{\alpha \in J_\varepsilon} |c_\alpha| t^\alpha - \sum_{\alpha \in I} |c_\alpha| t^\alpha \right| < \varepsilon$$

if $P_n \circ T_n(t) = \sum_{\alpha \in I} c_\alpha t^\alpha$ for each $t \in E$. We denote $P(t) = \sum_{\alpha \in J_\varepsilon} c_\alpha t^\alpha$ for $t \in E$. Hence we have proved that $P \in P_\nu(E)$ and $\|\widetilde{P} - \widetilde{P_n \circ T_n}\|_{C_n} <$ $< \varepsilon$. Since $C_j \cup \{x\} \subset C_k \subset C_n$ we have

$$|\widetilde{P}_n(T_n(x))| = |\widetilde{P_n \circ T_n}(x)| \leq |\widetilde{P}(x)| + \varepsilon \leq$$

$$\leq \|\widetilde{P}\|_{C_j} + \varepsilon \leq \|\widetilde{P_n \circ T_n}\|_{C_j} + 2\varepsilon \leq$$

$$\leq \|\widetilde{P}_n\|_{B_j \cap E_n} + 2\varepsilon .$$

It follows that (*) holds true, since $\varepsilon > 0$ is arbitrary. Now we consider the natural topological isomorphism between E and $\mathbb{C}^n$ given by $\psi: x \in E_n \longrightarrow \psi(x) = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Since $U \cap E_n$ and $A_j \cap E_n$ are modularly decreasing and logarithmically convex in E_n, we also have $\psi(U \cap E_n)$ and $\psi(A_j \cap E_n)$ with the same properties in $\mathbb{C}^n$. Since $\psi(B_j \cap E_n)$ is a relatively compact subset of $\psi(A_j \cap E_n)$ it is possible to find a finite set R of points $(r_1, \ldots, r_n)$, with $r_1 > 0, \ldots, r_n > 0$, and

$$\psi(B_j \cap E_n) \subset X = \bigcup_{r \in R} \overline{D}_r(0) \subset \psi(A_j \cap E_n)$$

where $\overline{D_r(0)} = \{z \in \mathbb{C}^n ; |z_\ell| \leq r_\ell , \ \ell = 1, \ldots, n\}$. Since $\widetilde{X}$ is the smallest logarithmically, modularly decreasing set containing X and $\psi(A_j \cap E_n)$ has this property, we get

$$\widetilde{\psi(B_j \cap E_n)}(\nu, \psi(U \cap E_n)) \subset \widetilde{X} \subset \psi(A_j \cap E_n).$$

But we know that $\tilde{X}$ is compact and $\psi[\widetilde{B_j \cap E_n}(\nu, U \cap E_n)]$ is a subset of $\widetilde{\psi(B_j \cap E_n)}(\nu, \psi(U \cap E_n))$. Hence $\psi^{-1}(\tilde{X})$ is a compact subset of $A_j \cap E_n$ and we get

$$\widetilde{B_j \cap E_n}(\nu, U \cap E_n) \subset \psi^{-1}(\tilde{X}) \subset A_j \cap E_n \subset A_j .$$

This and (*) imply that $d_U(T_n(x)) > \frac{1}{j}$ for all $n > k$. Hence $d_U(x) \geq \frac{1}{j}$. This shows that $d_U(\tilde{C}_j(\nu, U)) \geq \frac{1}{j}$ for every $j = 1, 2, \ldots$

Q.E.D.

3.3. THEOREM. If U is a Reinhardt domain such that $0 \in U$ and it is the union of an increasing sequence $(C_j)_{j=1}^{\infty}$ of open Reinhardt sets with $d_U(\tilde{C}_j(\nu, U)) > 0$ for every $j = 1, 2, \ldots,$ then U is the domain of existence of some $f \in H_\nu(U)$ and we may write

$$f(x) = \sum_{\alpha \in I} c_\alpha x^\alpha \qquad (\forall x \in U)$$

where the $c_\alpha \in \mathbb{C}$ for $\alpha \in I$ are uniquely determined.

PROOF. Let D be a denumerable dense subset of U. For every $x \in D$ we denote by B_x the largest open ball of center x contained in U. Let $(x_j)_{j=1}^{\infty}$ be a sequence formed by the elements of D in such a way that every $x \in D$ appears in it infinitely many times. If we set $A_j = \tilde{C}_j(\nu, U)$ we have $d_U(A_j) > 0$ and $B_x \not\subset A_j$ for every $x \in D$ and $j \in \mathbb{Z}_+$. If necessary we replace $(A_j)_{j=1}^{\infty}$ by a subsequence in order to obtain $(y_j)_{j=1}^{\infty}$ in U such that $y_j \in B_{x_j}$, $y_j \notin A_j$, $y_j \in A_{j+1}$ for each $j \in \mathbb{Z}_+$. Since $y_j \notin A_j$ there is $f_j \in P_\nu(E)$ such that

$$|\tilde{f}(y_j)| > 1 > \|\tilde{f}_j\|_{C_j} .$$

If necessary we take suficiently high powers of f_j and get by induction a sequence $(g_j)_{j=1}^{\infty}$ in $P_\nu(E)$ such that

$$\|\tilde{g}_j\|_{C_j} < \frac{1}{2^j}$$

$$|\tilde{g}_j(y_j)| > j + 1 + |\sum_{k<j} \tilde{g}_k(y_j)|$$

for $j = 1,2,\ldots$. It follows that, for each $\ell = 1,2,\ldots$, we have $\sum_{j=1}^{\infty} \|\tilde{g}_j\|_{C_\ell} < +\infty$. Hence $g = \sum_{j=1}^{\infty} \tilde{g}_j \in H(U)$. On the other hand, since $|\tilde{g}_j(y_\ell)| \geq \ell$ for all $\ell \leq j$, we have $|g(y_\ell)| \geq \ell$ for $\ell = 1,2,\ldots$. Since C_ℓ is a Reinhardt set we get $|x| \in C_\ell$ whenever $x \in C_\ell$. Hence, for $x \in C_\ell$, there is $M_\ell \geq 0$ such that

$$\sum_{j=1}^{\infty} |\tilde{g}_j(|x|)| \leq M_\ell < +\infty$$

$$\|$$

$$\sum_{j=1}^{\infty} \sum_{\alpha\in I} |\frac{1}{\alpha!} D^\alpha g_j(0)| \, |x|^\alpha .$$

It follows that

$$g(|x|) = \sum_{\alpha\in I} (\sum_{j=1}^{\infty} |\frac{1}{\alpha!} D^\alpha g_j(0)|) \, |x|^\alpha$$

$$= \sum_{\alpha\in I} |\frac{1}{\alpha!} D^\alpha g(0)| \, |x|^\alpha \leq M_\ell$$

for every $x \in U$. Here $D^\alpha g(0)$ has the usual meaning concerning partial derivatives. This shows that $g \in H_\nu(U)$

$$g(x) = \sum_{\alpha \in I} \frac{1}{\alpha!} D^\alpha g(0) x^\alpha$$

for every $x \in U$ and g is not bounded in $(y_j)_{j=1}^\infty$. Now we prove that g in not bounded in B_x for every $x \in D$. If $x \in D$ is given we consider $(j_k)_{k=1}^\infty$ in $\mathbb{N}$ strictly increasing such that $x = x_{j_k}$ for $k = 1,2,\ldots$. Hence $y_{j_k} \in B_x$ for $k = 1,2,\ldots$ and g is not bounded in B_x. Now we show that U is the domain of existence of g. If there were open subsets V and W in E and $h \in \mathcal{H}(V)$ such that V is connected and $V \not\subset U$, $\phi \neq W \subset U \cap V$, $h = g$ in W, we would consider $a \in V \cap \partial U \cap \partial W$ and $r > 0$ such that $B_{2r}(a) \subset V$ and we would choose $x \in D \cap W \cap B_r(a)$. Then $d_U(x) < r$, $B_x \subset B_{2r}(a) \subset V$. Since B_x is connected and contained in $V \cap V$, we would have $B_x \subset W_o$, where W_o is the connected component of $U \cap V$ containing W. Hence h would not be bounded in B_x, hence in $B_{2r}(a)$. Since we may choose $r > 0$ arbitrarily, h would not be locally bounded at a, a contradiction.

Q.E.D.

An open subset U of E is called an *open set of ν-holomorphy* if it is not possible to find open subsets V and W such that V is connected, $V \not\subset U$, $\phi \neq W \subset U \cap V$ and for every $f \in \mathcal{H}_\nu(V)$ there is $f_1 \in \mathcal{H}(V)$ such that $f = f_1$ in W.

3.4. THEOREM. If U is a Reinhardt domain in E such that $0 \in U$, then the following statements are equivalent.

(1) U is the domain of convergence of a multiple power series around 0.

(2) U is modularly decreasing and logarithmically convex.

(3) U is the domain of existence of some $f \in \mathcal{H}_\nu(U)$.

(4) U is a domain of existence of some $g \in \mathcal{H}(U)$.

(5) U is a domain of ν-holomorphy.

(6) U is a domain of holomorphy

(7) U is pseudo-convex

(8) U is union of an increasing sequence $(C_j)_{j=1}^{\infty}$ of open Reinhardt subsets of U such that $d_U(\tilde{C}_j(\nu,U)) > 0$ for every $j = 1,2,\ldots$.

PROOF. The equivalence of (4), (6) and (7) is due to a more general theorem proved by Gruman and Kiselman (see [16]). It is obvious that (3) implies (4), (3) implies (5) and (5) implies (6). The implications (2) ⇒ (8) and (8) ⇒ (3) we proved respectively in Theorem 3.2 and Theorem 3.3 . In [19] Katz has proved that (7) implies (2). Since (1) implies (2) by 3.1., we only have to show that (8) ⇒ (1). If we assume (8) we proved in Theorem 3.3 that U is the domain of existence of some $g \in H_\nu(U)$ and

$$g(x) = \sum_{\alpha\in I} \frac{1}{\alpha!} D^\alpha g(0) x^\alpha \qquad (\forall x \in U)$$

This shows that the domain of convergence D of this multiple power series contains U. The sum of this series defines an element $h \in H_\nu(D)$ such that $h = g$ in U. Since U is the domain of existence of g, it follows that U = D.

Q.E.D.

REFERENCES

[1] R.L. ALENCAR - Aplicações nucleares e integrais e a propriedade de Radon Nikodym - Doctoral Dissertation , IME , Universidade de São Paulo, S.P., Brasil - 1982

[2] R.L. ALENCAR , R. ARON & S. DINEEN - A reflexive Space of Holomorphic Functions in Infinitely Many Variables, Proc. of Am. Math. Society, vol. 90, N. 3, March 1984.

[3] P.J. BOLAND and S. DINEEN - Holomorphic functions on fully nuclear spaces. B.S.M.F , 106, 1978 , p 311-336

[4] S.B. CHAE - Holomorphic germs on Banach spaces. Ann. Inst. Fourier, 21, 3, 1971, p. 107-141

[5] S. DINEEN - Analytic functionals on fully nuclear spaces. Studia Math., 73 , 1982, p. 11-32

[6] S. DINEEN - Complex Analysis in Locally convex spaces. Mathematics Studies, 57. North-Holland, Amsterdam. 1981

[7] M. FRÉCHET - Une définition fonctionnelle des polynômes. Nouv. Ann. Math., 9, 1909, p. 145-162

[8] M. FRÉCHET - Sur les fonctionnelles continues. Ann. Sc. de l'École Norm. Sup. 3, 27, 1910, p. 193-216

[9] M. FRÉCHET - Sur les fonctionnelles bilinéaires. T.A.M.S., 16, 1915, p. 215-234

[10] M. FRÉCHET - Les transformations ponctuelles abstraites. C.R.A. Sc., Paris, 180, 1925, p. 1816-1817

[11] M. FRÉCHET - Les polynômes abstraits . Jour. Math. Pures Appl., (9), (8) 1929, p. 71-92

[12] R. GÂTEAUX - Sur les fonctionnelles continue et les fonctionnelles analytiques. C.R.A. Sc., Paris, 157, 1913, p. 325-327

[13] R. GÂTEAUX - Fonctions d'une infinité des variables indépendantes. B.S.M. Fr., 47, 1919, p. 70-96

[14] R. GÂTEAUX - Sur diverses questions de calcul fonctionnel. B.S.M. Fr., 50, 1922, p. 1-37

[15] L.M. GRAVES - Topics in functional calculus. B.A.M.S. ,41, 1935, p. 641-662

[16] L. GRUMAN and C.O. KISELMAN. Le problème de Levi dans les espaces de Banach à base. C.R.A. Sc., Paris, 274 , 1972, p. 821-824

[17] D. HILBERT. Wesen und Zieleiner Analysis der unendlich vielen unabhängigen Variabeln. Rend. del Circolo Mat. di Palermo, 27, 1909, p. 59-74

[18] L. HÖRMANDER - An introduction to complex analysis in several variables.Van Nostrand, 1966

[19] G.I. KATZ - Analytic continuation in infinite dimension and Reinhardt sets - Doctoral Dissertation - University of Rochester, Rochester, N.Y., U.S.A., 1974

[20] R.S. MARTIN - Contributions to the theory of functionals. Doctoral Dissertation. California Institute of Technology. U.S.A. 1932

[21] M.C. MATOS - On holomorphy in Banach Spaces and Absolute Convergence of Fourier Series. Relatório Interno nº 341. IMECC-UNICAMP, Brazil, 1985 (To appear in Portugaliae Mathematica).

[22] A.D. MICHAL and A.H. CLIFFORD. Fonctions analitiques implicites dans des espaces vectoriels abstraits. C.R.A. Sc., Paris, 197, 1933, p. 735-737

[23] J. MUJICA - Complex Analysis in Banach spaces. Mathematics studies 120. North-Holland, Amsterdam. 1986

[24] L. NACHBIN - Holomorphic Functions, Domains of Holomorphy and local properties. Mathematics Studies 1, North-Holland, Amsterdam, 1970.

[25] L. NACHBIN - Topology on spaces of holomorphic mappings. Ergebrisse der Mathematik und ihrer Grenzgebiete, Band 47. Springer-Verlag. New York, 1969

[26] R.A. RYAN - Holomorphic mappings on ℓ_1-(To appear)

[27] A.E. TAYLOR - Analytic functions in general analysis. Ann. Sc. Normale Sup. Pisa, (2), 6, 1937, p. 277-292