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$SU(2,R)_q$ Symmetries of Non-Abelian Toda Theories

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Abstract

The classical and quantum algebras of a class of conformal NA-Toda models are studied.It is shown that the $SL(2,R)_q$ Poisson brackets algebra generated by certain chiral and antichiral charges of the nonlocal currents and the global $U(1)$ charge appears as an algebra of the symmetries of these models.

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The (Non Abelian) NA-Toda theories are singled out among all 2-D conformal models (CFT's) by their important role in the construction of exact solutions for both self-dual 4-D Yang-Mills theories (axial symmetric instantons) [1, 2, 3] and string theory (black hole backgrounds)[4, 5, 6]. One expects that their quantum counterparts will provide an appropriate statistical mechanical tools for describing the critical behaviour of the $SU(n)$ gauge theory. The progress in the quantization of large class of 2-D CFT's based on the Abelian Toda models and their algebra of symmetries - W_n mininal models, W_n-gravities, etc, [7, 8, 9, 10, 11, 21] suggests that similar algebraic strategy of quantization takes place for the NA-Toda as well. Few important steps in this direction concerning the construction of conserved currents and their (*non-local and non-Lie*) classical (Poisson brackets) algebras for the particular case of B_2 NA-Toda [6, 12] have been realized.

This letter is devoted to the construction of classical and quantum algebras of symmetries of the first few members ($n = 1, 2, 3$) of the following family of A_n-NA-Toda models:

$$\begin{aligned} L &= -\frac{k}{2\pi}\{\frac{1}{2}\eta_{ik}\partial\phi_i\bar{\partial}\phi_k - (\frac{2}{k})^2\sum_{i=1}^{n-1} e^{\phi_{i-1}+\phi_{i+1}-2\phi_i}) + \frac{1}{2\Delta}e^{-\phi_1}(\partial\psi\bar{\partial}\chi + \partial\chi\bar{\partial}\psi) - \\ &- \frac{1}{4\Delta}e^{-\phi_1}[\bar{\partial}\phi_1(\chi\partial\psi - \psi\partial\chi) - \partial\phi_1(\chi\bar{\partial}\psi - \psi\bar{\partial}\chi)]\} \end{aligned} \tag{1}$$

where $\eta_{ik} = 2\delta_{i,k} - \delta_{i,k+1} - \delta_{i,k-1}$, $\Delta = 1 + \frac{n+1}{2n}e^{-\phi_1}\psi\chi$ and $\partial = \partial_\tau + \partial_\sigma$, $\bar{\partial} = \partial_\tau - \partial_\sigma$, $\phi_0 = \phi_n = 0$. They represent the (non-compact) $SL(2,R)/U(1)$-parafermions interacting with the A_{n-1}-abelian Toda model. One can derive (1) as an effective Lagrangean for the gauged $H_- \setminus A_n/H_+$-WZW model [1], $H_\pm = N^\pm_{(1)} \otimes H_0^{0(1)}$ where $N^\pm_{(1)}$ are nilpotent subgroups of A_n spanned by $E_{[\alpha]_1}$ or $E_{-[\alpha]_1}$ ($[\alpha]_1$- all positive roots but α_1) and $H_0^{0(1)} = exp\{R(z,\bar{z})\lambda_1 H\}$. This is equivalent to consider a specific Hamiltonian reduction of the A_n-WZW model by imposing the following set of constraints:

$$\begin{aligned} &J_{-\alpha_i} = \bar{J}_{\alpha_i} = 1,\ i = 2,\ldots,n;\ \ J_{\lambda_1 H} = \bar{J}_{\lambda_1 H} = 0 \\ &J_{-[\alpha]} = \bar{J}_{[\alpha]} = 0,\ \text{ for } \alpha \text{ nonsimple positive roots} \end{aligned} \tag{2}$$

(i.e. $J_{-\alpha_1}$ and $\bar{J}_{\alpha_1}$ are left *unconstrained*). Taking into account the residual gauge symmetries (that keep (2) unchanged) we find the reduced form of the conserved currents:

$$\begin{aligned} J &= V^+E_{-\alpha_1} + \sum_{i=2}^{n} E_{-\alpha_i} + V^-E_{\alpha_1+\cdots+\alpha_n} + \sum_{i=2}^{n} W_{n-i+2}E_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_n} \\ \bar{J} &= \bar{V}^+E_{\alpha_1} + \sum_{i=2}^{n} E_{\alpha_i} + \bar{V}^-E_{-\alpha_1-\cdots-\alpha_n} + \sum_{i=2}^{n} \bar{W}_{n-i+2}E_{-\alpha_i-\alpha_{i+1}-\cdots-\alpha_n} \end{aligned} \tag{3}$$

The well known fact is that the remaining currents $V^\pm$, W_{n-i+2} (and $\bar{V}^\pm$, $\bar{W}_{n-i+2}$) appears as conserved currents of the reduced model (1). The spins of these currents calculated with respect to the improved stress tensor ($T = W_2$):

$$T^{imp} = \frac{1}{k+n+1}Tr : J^aJ^a : + \sum_{i=2}^{n}\lambda_i \cdot H\partial J_i - \frac{(n-1)}{2}\lambda_1 \cdot H\partial J_1$$

[1] see for more details our forthcoming paper [13]

are given by $s^{\pm} = s(V^{\pm}) = \frac{(n+1)}{2}$, $s_i = s(W_{n-i+2}) = n - i + 2$. Note that our Lagrangean (1) is invariant under a global $U(1)$ gauge transformation : $\psi' = e^{\alpha}\psi$, $\chi' = e^{-\alpha}\chi$ and $\phi_i' = \phi_i$ (α is a constant). The corresponding *nonchiral* $U(1)$-current :

$$J_{\mu} = -\frac{k}{4\pi}(\chi\partial_{\mu}\psi - \psi\partial_{\mu}\chi + \psi\chi\partial_{\mu}\phi_1)\frac{e^{-\phi_1}}{\Delta} \tag{4}$$

($\partial\bar{J} + \bar{\partial}J = 0$ where $J = \frac{1}{2}(J_0 + J_1)$, $\bar{J} = \frac{1}{2}(J_0 - J_1)$) completes the list of the conserved currents of the NA-Toda models given by (1). Starting from the WZW currents $J = \frac{k}{2}g^{-1}\partial g$, $\bar{J} = -\frac{k}{2}\bar{\partial}gg^{-1}$ ($g = e^{\chi_{[\alpha]}E_{-[\alpha]}}e^{\phi_i\lambda_i H}e^{\psi_{[\alpha]}E_{[\alpha]}}$) and solving the constraints and the gauge fixing conditions in terms of the physical fields $\psi = \psi_1 e^{-\frac{n+1}{2n}R}$, $\chi = \chi_1 e^{-\frac{n+1}{2n}R}$ and $\phi_i, (i = 2, 3, \cdots, n)$ only one can *in principle* derive the explicit form of the remaining currents. We find that, for example, $V^+, \bar{V}^-$ and $T(\bar{T})$ are given by :

$$V^+ = \frac{k}{2}\frac{e^{-\phi_1 + \frac{n+1}{2n}R}\partial\chi}{\Delta} \qquad \bar{V}^- = \frac{k}{2}\frac{e^{-\phi_1 + \frac{n+1}{2n}R}\bar{\partial}\psi}{\Delta}$$

$$T(z) = \frac{1}{2}\eta_{ik}\partial\phi_i\partial\phi_k + \sum_{i=1}^{n-1}\partial^2\phi_i + \partial\chi\partial\psi\frac{e^{-\phi_1}}{\Delta} + \frac{n-1}{2}\partial(\frac{\psi\partial\chi}{\Delta}e^{-\phi_1}) \tag{5}$$

and $\bar{T} = T(\partial, \psi, \chi \longrightarrow \bar{\partial}, \chi, \psi)$. The $R(z, \bar{z})$ is the nonlocal field that denotes the solution of the following system of equations :

$$\partial R = \frac{e^{-\phi_1}}{\Delta}\psi\partial\chi \qquad , \bar{\partial}R = \frac{e^{-\phi_1}}{\Delta}\chi\bar{\partial}\psi \tag{6}$$

which are nothing but the constraint equations $J_{\lambda_1 H} = \bar{J}_{\lambda_1 H} = 0$. One can eliminate R from (5) by solving eqn (6) and thus introducing certain *nonlocal* terms in $V^{\pm}$ and $\bar{V}^{\pm}$. The *nonlocality* of the $V^{\pm}$ and $\bar{V}^{\pm}$ is one of the *main features* of the NA-Toda models (1) originated from the additional *PF-type constraint* $J_{\lambda_1 H} = \bar{J}_{\lambda_1 H} = 0$. Note that all the others W_{n-i+2}-currents are *local* . To derive their explicit form, as well as the form of V^- and $\bar{V}^+$ for arbitrary n is quite a difficult problem. For $n = 1$ we obtain:

$$V^-_{n=1}(z) = \frac{k}{2}e^{-R}\partial\psi \ , \ \bar{V}^+_{n=1}(z) = \frac{k}{2}e^{-R}\bar{\partial}\chi \tag{7}$$

(both of spin 1) and for $n = 2$ the remaining nonlocal current (of spin $\frac{3}{2}$) are given by

$$V^-_{n=2}(z) = (\frac{k}{2})^2 e^{-\frac{3}{4}R}\left(\partial^2\psi + \frac{1}{16}\psi(\partial R)^2 - \psi(\partial\phi_1)^2 - \psi\partial^2\phi_1 - \frac{1}{4}\psi\partial^2 R - \frac{1}{2}\partial\psi\partial R\right) \tag{8}$$

and $\bar{V}^+_{n=2} = V^-_{n=2}$ ($\psi \rightarrow \chi$, $\partial \rightarrow \bar{\partial}$).

The simplest method [16] for deriving the chiral algebra of symmetries of (1) spanned by $V^{\pm}, W_{n-i+2}$, ($i = 2, 3, \cdots, n$) consists in imposing the constraints (2) and the gauge fixing conditions directly in the A_n-gauge transformations: $\delta J = [\epsilon, J] - \frac{k}{2}\partial\epsilon$. This leads to the following system of first order differential equations

$$-\frac{k}{2}\partial\epsilon_{ik} = J_{ij}\epsilon_{jk} - \epsilon_{ij}J_{jk}, \qquad (ik) \neq \{(2,1), (p, n+1)(1, n+1)\} \tag{9}$$

for $i,j,k = 1,2,\cdots,n+1$. The equations for $(i,k) = \{(2,1),(p,n+1),(1,n+1)\}$

$$\delta J_{ik} = \epsilon_{ij} J_{jk} - J_{ij}\epsilon_{jk} - \frac{k}{2}\partial\epsilon_{ik}$$

$(J_{12} = V^+, J_{1,n+1} = V^-, J_{p,n+1} = W_{n-p+2}, p = 2,3,\cdots,n-1)$ gives the transformation law for the remaining currents we are looking for. The problem is to solve (9) for the redundant ϵ_{ik}'s in terms of the independent parameters $\epsilon_{12} = \epsilon^-, \epsilon_{n+1,1} = \epsilon^+, \epsilon_{n+1,n} = \epsilon, \eta_{n-p+2} = \epsilon_{n+1,p}, p = 2,3,\cdots,n-1$ and the currents $V^\pm, W_{n-i+2}$. The transformations generated by the nonlocal currents $V^\pm$ are given by ($\tilde{\epsilon}^\pm = -\frac{k}{2}\epsilon^\pm$):

$$\begin{aligned}\delta_{\epsilon^\pm} V^- &= \frac{n+1}{nk^2}\int \epsilon(\sigma-\sigma')[\tilde{\epsilon}^+(\sigma')V^-(\sigma') - \tilde{\epsilon}^-(\sigma')V^+(\sigma')]V^-(\sigma)d\sigma' \\ &- \sum_{s=0}^{n-2}(\frac{k}{2})^{s-1} W_{n-s}\partial^s\tilde{\epsilon}^-(\sigma) + (\frac{k}{2})^{n-1}\partial^n\tilde{\epsilon}^-(\sigma) \end{aligned} \tag{10}$$

and similar for $\delta_{\epsilon^\pm}V^+$. Reminding the relation between the infinitesimal transformations and the currents Poisson brackets:

$$\delta_{\epsilon^\pm} I(\sigma) = \int d\sigma' \epsilon^\pm(\sigma')\{V^\mp(\sigma'), I(\sigma)\}$$

$(I = V^+$ or $V^-)$ we deduce from (10) the form of the algebra of the nonlocal currents $V^\pm$:

$$\begin{aligned}\{V^\pm(\sigma), V^\pm(\sigma')\} &= -\frac{n+1}{nk^2}\epsilon(\sigma-\sigma')V^\pm(\sigma)V^\pm(\sigma') \\ \{V^+(\sigma), V^-(\sigma')\} &= \frac{n+1}{nk^2}\epsilon(\sigma-\sigma')V^+(\sigma)V^-(\sigma') + (\frac{k}{2})^{n-1}\partial^n_{\sigma'}\delta(\sigma-\sigma') \\ &- \sum_{s=0}^{n-2}(\frac{k}{2})^{s-1}W_{n-s}(\sigma')\partial^s_{\sigma'}\delta(\sigma-\sigma') \end{aligned} \tag{11}$$

Leaving the general solution of (9) to our forthcoming paper [13], we consider here a few particular cases $n = 1,2,3$. The algebra of the symmetries $V_2^{(1,1)}$ of the A_1-NA-Toda model is generated by $V^\pm_{n=1}$ (of spin 1) only (i.e. eqn. (11) for $n = 1$). It is identical to the semiclassical limit of the PF-algebra [17] studied in ref. [18]. In the $n = 2$ case the corresponding $V_3^{(1,1)}$ algebra contains appart from (11), which now reads,

$$\{V^+(\sigma), V^-(\sigma')\} = \frac{3}{2k^2}\epsilon(\sigma-\sigma')V^+(\sigma)V^-(\sigma') - \frac{2}{k}T(\sigma')\delta(\sigma-\sigma') + \frac{2}{k}\partial_{\sigma'}\delta(\sigma-\sigma')$$

$(T = W_2)$ the PB's of T and $V^\pm$

$$\{T(\sigma), V^\pm(\sigma')\} = sV^\pm(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \delta(\sigma-\sigma')\partial_{\sigma'}V^\pm(\sigma') \tag{12}$$

with $s = \frac{3}{2}$ and the usual Virasoro subalgebra

$$\{T(\sigma), T(\sigma')\} = 2T(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \partial_{\sigma'}T(\sigma')\delta(\sigma-\sigma') - \frac{k^2}{2}\partial^3_{\sigma'}\delta(\sigma-\sigma'). \tag{13}$$

The $n = 3$ case appears to be a *nonlocal* and *nonlinear* (quadratic terms) extension of the Virasoro algebra (13) with two spin $s = 2$ nonlocal currents $V^{\pm}_{n=3}$ and one local W_3 of spin $s = 3$. The $V_4^{(1,1)}$-algebra combines together the features of the PF-algebra and the W_3-one. Apart from the (11), (12) and (13) (with central charge $-2k^2$) we have two new PB's

$$\begin{aligned}\{w_3(\sigma), V^{\pm}(\sigma')\} &= \mp\frac{5k}{3}(\partial^2_{\sigma'}\delta(\sigma-\sigma'))V^{\pm}(\sigma') \mp \frac{5k}{2}(\partial_{\sigma'}\delta(\sigma-\sigma'))\partial_{\sigma'}V^{\pm}(\sigma')\\ &\pm\delta(\sigma-\sigma')(\frac{2}{3k}TV^{\pm} - k\partial^2_{\sigma'}V^{\pm})\\ \{w_3(\sigma), w_3(\sigma')\} &= 4(\partial_{\sigma'}\delta(\sigma-\sigma'))(V^+V^- + \frac{1}{6}T^2)(\sigma') + 2\delta(\sigma-\sigma')\partial_{\sigma'}(V^+V^- + \frac{1}{6}T^2)(\sigma')\\ &-\frac{3k^2}{4}(\partial_{\sigma'}\delta(\sigma-\sigma'))\partial^2_{\sigma'}T(\sigma') - \frac{5k^2}{4}(\partial^2_{\sigma'}\delta(\sigma-\sigma'))\partial_{\sigma'}T(\sigma')\\ &-\frac{k^2}{6}\delta(\sigma-\sigma')\partial^3_{\sigma'}T(\sigma') - \frac{5k^2}{6}(\partial^3_{\sigma'}\delta(\sigma-\sigma'))T(\sigma') + \frac{k^4}{6}\partial^5_{\sigma'}\delta(\sigma-\sigma')\end{aligned} \tag{14}$$

The method we have used in the derivation of the $V^{(1,1)}_{n+1}$-algebras ($n = 1, 2, 3$) allows us to find the corresponding fields (ψ, χ, ϕ_1)-transformations. The conformal transformations have the form :

$$\begin{aligned}\delta_{\epsilon}\phi_i &= \frac{i(i-n)}{2}\partial\epsilon + \epsilon\partial\phi_i, \quad \delta_{\bar\epsilon}\phi_i = \frac{i(i-n)}{2}\bar\partial\bar\epsilon + \bar\epsilon\bar\partial\phi_i\\ \delta_{\epsilon}\psi &= \frac{(1-n)}{2}(\partial\epsilon)\psi + \epsilon\partial\psi, \quad \delta_{\bar\epsilon}\psi = \bar\epsilon\bar\partial\psi\\ \delta_{\epsilon}\chi &= \epsilon\partial\chi, \qquad \delta_{\bar\epsilon}\chi = \frac{(1-n)}{2}(\bar\partial\bar\epsilon)\chi + \bar\epsilon\bar\partial\chi\end{aligned} \tag{15}$$

$i = 1, \cdots, n-1$. The transformation generated by V^+ are quite simple but highly nonlocal:

$$\begin{aligned}\{V^+(\sigma), \psi(\sigma')\} &= -\frac{n+1}{4nk}\epsilon(\sigma-\sigma')V^+(\sigma)\psi(\sigma') + \frac{1}{k}e^{\frac{n+1}{2n}R(\sigma')}\delta(\sigma-\sigma')\\ \{V^+(\sigma), \chi(\sigma')\} &= \frac{n+1}{4nk}\epsilon(\sigma-\sigma')V^+(\sigma)\chi(\sigma'); \quad \{V^+(\sigma), \phi_i(\sigma')\} = 0\end{aligned} \tag{16}$$

For $n > 1$, the corresponding V^--transformation are much more complicated than the V^+-ones.

The complete algebra of the symmetries of the NA-Toda models (1) contains together with the two (chiral) $V^{(1,1)}_{n+1}$ and (antichiral) $\bar V^{(1,1)}_{n+1}$ (spanned by $\bar V^{\pm}, \bar W_{n-i+2}$) algebras, the PB's of the global charge $Q_0 = \int J_0 d\sigma$ of the *nonchiral* $U(1)$ current:

$$\{Q_0, V^{\pm}(\sigma)\} = \pm V^{\pm}(\sigma), \{Q_0, \bar V^{\pm}(\sigma)\} = \pm\bar V^{\pm}(\sigma), \{Q_0, W_{n-i+2}(\sigma)\} = \{Q_0, \bar W_{n-i+2}(\sigma)\} = 0 \tag{17}$$

as well. To describe the full structure of this larger algebra one has to calculate the PB's of the chiral with antichiral currents. As usual, the chiral and antichiral charges of the *local* currents *do commute* :

$$\{L^{(n-i+2)}_{m_1}, \bar L^{(n-j+2)}_{m_2}\} = 0,$$

where $L_m^{(n-i+2)} = \int d\sigma W_{(n-i+2)}\sigma^{m+n-i+1}$ (and the same for $\bar{L}$). The new phenomena occurs with the PB's of certain nonlocal charges. Our *main observation* is that $Q^+ = \int d\sigma V^+(\sigma)$ and $\bar{Q}^- = \int d\sigma \bar{V}^-(\sigma)$ have *nonvanishing* PB's:

$$\{Q^+, \bar{Q}^-\} = \frac{k\pi}{2}\int_{-\infty}^{\infty} d\sigma \partial_\sigma e^{\frac{n+1}{n}R-\phi_1-ln\Delta} \tag{18}$$

In order to prove eqn. (18) we realize V^+ and $\bar{V}^-$ (given by eqn. (5)) in terms of fields ψ, χ, ϕ_i, their space (∂_σ) derivatives and their conjugate momenta $\Pi_\psi, \Pi_\chi, \Pi_{\phi_i}$. The rest is straightforward calculation based on the canonical PB's $\{\rho_i(\sigma), \Pi_{\rho_j}(\sigma')\} = \delta_{ij}\delta(\sigma-\sigma')$ where $\rho_i = (\psi, \chi, \phi_i)$. Note that the field in the exponent of thr r.h.s. of (18):

$$\varphi = R - \frac{n}{n+1}(\phi_1 + ln\Delta) \tag{19}$$

is related to the $U(1)$-current $J_\mu = \frac{k}{2\pi}\epsilon_{\mu\nu}\partial^\nu(\varphi + \frac{n}{n+1}\phi_1)$. Since the topological conserved current $I_\mu = \frac{k}{2\pi}\frac{n}{n+1}\epsilon_{\mu\nu}\partial^\nu\phi_1$ has vanishing PB's with $V^\pm$ and $\bar{V}^\pm$ one can redefine the $U(1)$-charge as follows:

$$H_1 = Q_0 - \int I_0 d\sigma = -\frac{k}{2\pi}(\varphi(\infty) - \varphi(-\infty)),$$
$$\{H_1, Q^+\} = Q^+, \qquad \{H_1, \bar{Q}^-\} = -\bar{Q}^- \tag{20}$$

Then eqn. (18) takes the following suggestive form:

$$\{Q^+, \bar{Q}^-\} = q_{(n)}^\delta(q_{(n)}^{H_1} - q_{(n)}^{-H_1})$$

where $q_{(n)} = e^{-\frac{n+1}{n}(\frac{\pi}{k})}$ and $\delta = -\frac{k}{2\pi}(\varphi(\infty) + \varphi(-\infty))$. As a consequence of the PB's of $\varphi(\sigma)$ with V^+ and $\bar{V}^-$ we find that $\{\delta, Q^+\} = \{\delta, \bar{Q}^-\} = 0$. Finally introducing new charges E_1, F_1 (instead of Q^+ and $\bar{Q}^-$):

$$E_1 = \sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1+\delta}{2}}}{(q^2-1)^{\frac{1}{2}}}Q^+, \qquad F_1 = \sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1+\delta}{2}}}{(q^2-1)^{\frac{1}{2}}}\bar{Q}^-$$

we realize that the PB's algebra (18), (20) of $Q^+, \bar{Q}^-$ and Q_0 takes the standard form of the q-deformed $SL(2,R)$-algebra (n-arbitrary):

$$\{E_1, F_1\} = \frac{q_{(n)}^{H_1} - q_{(n)}^{-H_1}}{q_{(n)} - q_{(n)}^{-1}}, \qquad \{H_1, E_1\} = E_1, \qquad \{H_1, F_1\} = -F_1. \tag{21}$$

With the explicit form (7),(8) of V^- and $\bar{V}^+$ (for $n = 1, 2$ only) at hand we can calculate the PB's of their charges

$$Q^- = \int \sigma^{n-1}V^-(\sigma)d\sigma, \qquad \bar{Q}^- = \int \sigma^{n-1}\bar{V}^-(\sigma)d\sigma$$

as well as the mixed PB's $\{Q^\pm, \bar{Q}^\pm\}$. The result is:

$$\{E_0, F_0\} = \frac{q_{(n)}^{H_0} - q_{(n)}^{-H_0}}{q_{(n)} - q_{(n)}^{-1}}, \qquad \{Q^\pm, \bar{Q}^\pm\} = 0, \quad n = 1, 2 \tag{22}$$

where(we are omiting the index (n) in q)

$$E_0 = \sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1-\delta}{2}}}{(1-q^2)^{\frac{1}{2}}}Q^-, \qquad F_0 = \sqrt{\frac{2}{k\pi}}\frac{q^{\frac{1-\delta}{2}}}{(1-q^2)^{\frac{1}{2}}}\bar{Q}^+, \qquad H_0 = -H_1$$

The PB's (21) and (22) can be written in a compact form :

$$\begin{aligned} \{H_i, E_j\} &= \kappa_{ij}E_j \qquad \{H_i, F_j\} = -\kappa_{ij}F_j, \qquad i,j = 0,1 \\ \{E_i, F_j\} &= \delta_{ij}\frac{q_n^{H_i} - q_n^{-H_i}}{q_n - q_n^{-1}}, \qquad \kappa_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned} \tag{23}$$

which is known to be the *centerless affine* $SL(2,R)_q$ PB's algebra in the principal gradation [15] and [14] (the Serre relations are omited). The *conclusion* is that the *classical q-deformed affine (for $n = 1,2$) $SL(2,R)$ PB's algebra (23), generated by certain nonlocal charges appears as the algebra of the Noether symmetries of the NA-Toda models (1).* Note that the deformation parameter $q_n = e^{-\frac{n+1}{n}(\frac{\pi}{k})}$ is a function of the classical coupling constant k and of the rank n of the underlying A_n-algebra .

To make the above statement complete we have to demonstrate that the field equations of (1):

$$\begin{aligned} \partial\bar{\partial}\phi_i &= (\frac{2}{k})^2 e^{\phi_{i+1}+\phi_{i-1}-2\phi_i} - \frac{(n-i)}{n}e^{-\phi_1}\frac{\partial\chi\bar{\partial}\psi}{\Delta^2} \\ \bar{\partial}(\frac{\partial\chi}{\Delta}e^{-\phi_1}) &= -\frac{n+1}{2n}\frac{\bar{\partial}\psi\partial\chi}{\Delta^2}\chi e^{-2\phi_1}, \qquad \partial(\frac{\bar{\partial}\psi}{\Delta}e^{-\phi_1}) = -\frac{n+1}{2n}\frac{\bar{\partial}\psi\partial\chi}{\Delta^2}\psi e^{-2\phi_1} \end{aligned} \tag{24}$$

admit solutions such that $\varphi(\infty, t_0) \neq \varphi(-\infty, t_0)$, i.e. with nontrivial $U(1)$-charge $H_1 \neq 0$. If no such solutions exist, (i.e. $\varphi(\infty, t_0) = \varphi(-\infty, t_0)$) then all PB's of the chiral and antichiral nonlocal charges *vanish identically.*

Our construction of the solutions of eqns. (24) is based on the following *observation*: the change of the field variables ϕ_i, ψ, χ into $\varphi_i, V^+, \bar{V}^-$ given by

$$\begin{aligned} \varphi_i &= \phi_{i-1} + \frac{n-i+1}{n}R + (i-1)\ln(V^+\bar{V}^-), \quad \phi_0 = 0 \\ \psi V^+ &= (\frac{k}{2})e^{\frac{n+1}{n}\varphi_1}\partial\varphi_1, \quad \chi\bar{V}^- = (\frac{k}{2})e^{\frac{n+1}{n}\varphi_1}\bar{\partial}\varphi_1 \end{aligned} \tag{25}$$

(the last two equations reflect the definition (5) of $V^+, \bar{V}^-$ and (6) of R) *maps* the eqns. (24) into the following system of equations :

$$\begin{aligned} \partial\bar{\partial}\varphi_l &= (\frac{2}{k})^2 e^{\varphi_{l-1}+\varphi_{l+1}-2\varphi_l}, \quad l = 1, \cdots, n-1 \\ \partial\bar{\partial}\varphi_n &= (V^+\bar{V}^-)^n e^{-2\varphi_n+\varphi_{n-1}}, \quad \partial\bar{V}^- = \bar{\partial}V^+ = 0 \end{aligned} \tag{26}$$

The general solution of eqns. (26) can be found by slight modification of the Gervais-Bilal method [9], realizing the $\varphi_i, (i = 1, \cdots, n)$ and $V^+, \bar{V}^-$ in terms of $n+1$-independent functions $f_l(t+\sigma), \bar{f}_l(t-\sigma), l = 1, 2, \cdots, n+1$:

$$e^{\varphi_1} = (\frac{k}{2})^{-n} f_l\bar{f}_l, \quad \cdots \quad e^{\varphi_p} = (\frac{1}{2})^{p(p-n-1)}\frac{1}{p!}f_{l_1,\cdots l_p}\bar{f}_{l_1,\cdots l_p}$$

$$(V^+)^n = \epsilon_{l_1,l_2,\cdots l_{n+1}} f_{l_1} f_{l_2}^{(1)} \cdots f_{l_{n+1}}^{(n)}; \quad (\bar{V}^+)^n = \epsilon_{l_1,l_2,\cdots l_{n+1}} \bar{f}_{l_1} \bar{f}_{l_2}^{(1)} \cdots \bar{f}_{l_{n+1}}^{(n)} \tag{27}$$

where $f_{l_1,\cdots l_p}$ are rank p antisymmetric tensors. For example, $f_{l_1,l_2} = f_{l_1} f'_{l_2} - f'_{l_1} f_{l_2}$. Then the solution of eqns. (24) is given by:

$$e^R = (\frac{k}{2})^{-n} f_l \bar{f}_l, \qquad e^{\phi_1} = \frac{1}{2}(\frac{k}{2})^{1-n}(f_{lm}\bar{f}_{lm})(f_p\bar{f}_p)^{\frac{1-n}{n}}(V^+\bar{V}^-)^{-1}, etc.$$

$$\psi = (\frac{k}{2})^{\frac{1-n}{2}}(f_l\bar{f}_l)^{\frac{1-n}{2n}}(f'_p\bar{f}_p)(V^+)^{-1}, \qquad \chi = (\frac{k}{2})^{\frac{1-n}{2}}(f_l\bar{f}_l)^{\frac{1-n}{2n}}(f_p\bar{f}'_p)(\bar{V}^-)^{-1} \tag{28}$$

Therefore the field φ from eqn. (19) whose assymptotics are under investigation takes the form:

$$\varphi = -\frac{n}{n+1} ln(\frac{k}{2})^2 \left(\frac{(f'_l\bar{f}'_l)(f_p\bar{f}_p) - \frac{n-1}{2n}(f'_l\bar{f}_l)(f_p\bar{f}'_p)}{(f_m\bar{f}_m)^2(V^+\bar{V}^-)} \right) \tag{29}$$

We next consider the following ansatz

$$f_l = \alpha_l e^{(t+\sigma)a_l}, \qquad \bar{f}_l = \bar{\alpha}_l e^{(t-\sigma)\bar{a}_l}, \qquad \sum_{l=1}^{n+1}(a_l - \bar{a}_l) = 0 \tag{30}$$

and for convenience we choose the following parametrization for the a_l's:

$$a_1 - \bar{a}_1 = b_1 + b_2 + \cdots + b_n; \qquad a_p - \bar{a}_p = -b_{p-1}, \quad b_n > b_{n-1} > \ldots > b_1 \tag{31}$$

where $p = 2, 3, \ldots, n+1$. Under all these conditions we calculate the limits $\sigma \longrightarrow \pm\infty$ of eqn. (29) at $t = 0$:

$$\varphi(\infty, 0) = -\frac{n}{n+1}\ln((\frac{k}{2})^2\frac{n+1}{2n}a_1\bar{a}_1 A), \quad \varphi(-\infty, 0) = -\frac{n}{n+1}\ln((\frac{k}{2})^2\frac{n+1}{2n}a_{n+1}\bar{a}_{n+1}A),$$

$A = V^+\bar{V}^-(t=0)$. Since $a_1\bar{a}_1 \neq a_{n+1}\bar{a}_{n+1}$ the solutions (30) are an example of solutions of (24) with $H_1 \neq 0$.

The quantization of the classical $V_{n+1}^{(1,1)}$-algebras represents certain new features all related to the nonlocal terms $\epsilon(\sigma)$ in the r.h.s. of (11). We find more convenient to directly apply the procedure of *quantum* Hamiltonian reduction to the A_n-WZW models instead of quantizing the results of the classical Hamiltonian reduction. The method we are going to use is an appropriate generalization of the derivation of the parafermionic algebra [17] from the affine $SU(2)$-one (or $SL(2,R)$ for the noncompact PF's) by imposing the constraint $J_3 = 0$. Following the arguments of ref. [17] we define the quantum (compact)V_2-algebra as

$$V_2 = \{SU(2)_k, J_3 = 0\}$$

Therefore its generators $\psi^\pm$ represent the $J_3 = \sqrt{\frac{k}{2}}\partial\phi$ independent part of the $\hat{S}U(2)_k$-ones, namely, $J^\pm$,

$$J^\pm = \psi^\pm e^{\mp\alpha\phi}, \qquad T = T_V + \tfrac{1}{2}(\partial\phi)^2$$
$$J_3(z_1)\psi^\pm(z_2) = O(z_{12}), \quad \phi(z_1)\phi(z_2) = -ln z_{12} + O(z_{12}) \tag{32}$$

Taking into account the $SU(2)$ OPE's:

$$J_3(z_1)J^{\pm}(z_2) = \pm\frac{i}{z_{12}}J^{\pm}(z_2) + O(z_{12})$$

and eqns. (32) we find $\alpha = i\sqrt{\frac{2}{k}}$. Another consequence of eqn. (32) is that the dimensions of $\psi^{\pm}$ are $\Delta^{\pm} = 1 - \frac{1}{k}$ (we have used that $\Delta_{J^{\pm}} = 1$). Finally the construction (32), the $\phi(z_1)\phi(z_2)$ –OPE and the $SU(2))_k$–OPE's leads to the following V_2-algebra OPE's:

$$\begin{aligned} \psi^{\pm}(z_1)\psi^{\pm}(z_2) &= z_{12}^{-\frac{2}{k}}\psi^{\pm}_{(2)}(z_2) + O(z_{12}) \\ \psi^{+}(z_1)\psi^{-}(z_2) &= z_{12}^{\frac{2}{k}}(\frac{k}{z_{12}^2} + (k+2)T_V(z_2) + O(z_{12})), \end{aligned} \qquad (33)$$

which is nothing but the well known PF-algebra [17]. Although the PF (V_2)-algebra (33) is *by construction* the quantum version of the classical PB's algebra (see eqn. (11) for $n = 1$):

$$\begin{aligned} \{V^{\pm}(\sigma), V^{\pm}(\sigma')\} &= -\epsilon(\sigma-\sigma')V^{\pm}(\sigma)V^{\pm}(\sigma') \\ \{V^{-}(\sigma), V^{+}(\sigma')\} &= \epsilon(\sigma-\sigma')V^{+}(\sigma)V^{-}(\sigma') + \partial_{\sigma'}\delta(\sigma-\sigma') \end{aligned} \qquad (34)$$

the discrepancy between the dimensions $\Delta^{\pm} = 1 - \frac{1}{k}$ of $\psi^{\pm}$ and $\Delta^{\pm}_V = 1$ of $V^{\pm}$ requires more precise definition of the relation of eqn. (33) and (34). The exact *statement* is as follows: Let $V^{\pm} = \frac{1}{k}\psi^{\pm}$ and the $V^{\pm}$ PB's are defined as certain limit of the OPE's (33):

$$\{V^a(z_1), V^b(z_2)\} = \lim_{k\to\infty}\frac{k}{2\pi i}(V^a(z_1)V^b(z_2) - V^b(z_2)V^a(z_1)) \qquad (35)$$

($a, b = \pm$). Then the $k \longrightarrow \infty$ limit of the OPE's (33) reproduces the PB's algebra (34). The proof is straightforward. Applying twice the OPE's (33) we obtain

$$z_{12}^{\frac{2}{k}}(V^{\pm}(z_1)V^{\pm}(z_2) - e^{-\frac{2\pi i}{k}\epsilon(z_{12})}V^{\pm}(z_2)V^{\pm}(z_1)) = \frac{1}{k^2}O(z_{12})$$

$$z_{12}^{\frac{-2}{k}}(V^{-}(z_1)V^{+}(z_2) - e^{\frac{2\pi i}{k}\epsilon(z_{12})}V^{+}(z_2)V^{-}(z_1)) = \frac{1}{k}(\frac{1}{z_{12}^2 + i0} - \frac{1}{z_{21}^2 + i0}) + \frac{k+2}{k^2}O(z_{12}) \quad (36)$$

where the identity $i\pi\epsilon(z_{12}) = ln\frac{z_{12}+i0}{z_{21}+i0}$ has been used. The $k \to \infty$ limit of (36) reproduces the PB's (34) of the classical V_2-algebra [2]. The conclusion is that the nonlocal PB's algebra (34) is *semiclassical limit* ($k \to \infty$) of the PF-algebra and that *the quantization* of the nonlocal currents $V^{\pm}$ requires *renormalization* of their (classical) spins: $\Delta^{\pm}_q = \Delta^{\pm}_{cl} - \frac{1}{k}$. For k-positive integers the global $Z_2 \otimes U(1)$ symmetry of the classical A_1-NA-Toda model is broken to the $Z_2 \otimes Z_k$ of the quantum theory.

The quantization of the $V_3^{(1,1)}$-algebra is based on the following *observation* : the $A_2^{(1,1)}$-NA-Toda model is equivalent to the $U(1)$-*reduced* Bershadsky-Polyakov $A_2^{(2)}$-NA-Toda model

[2] The noncompact case $SL(2,R)/U(1)$ corresponds to the change $\phi \longrightarrow i\phi$, which turns out to be equivalent to $k \to -k$ in the OPE's, spins, etc.

(BP) [19, 16]. In fact, the set of constraints (and gauge fixing conditions) (2), (3) for $n = 2$ appears to be the image of the BP-ones [19]

$$J_{-\alpha_2} = \bar{J}_{\alpha_2} = 0, \quad J_{-\alpha_1-\alpha_2} = \bar{J}_{\alpha_1+\alpha_2} = 1 \tag{37}$$

($J_{-\alpha_1} = 0$ is the gauge fixing condition for the constraint$J_{-\alpha_2} = 0$) and the additional constraint

$$J_{(\lambda_1-\lambda_2)\cdot H} = \bar{J}_{(\lambda_1-\lambda_2)\cdot H} = 0 \tag{38}$$

under specific Weyl reflection $\omega_{\alpha_1}(\alpha) = \alpha_1 - (\alpha\cdot\alpha_1)\alpha$. The constraint (38) imposed on the $U(1)$ current transforms $W_3^{(2)}$-algebra (the symmetry of the original BP-model) into the nonlocal algebra $V_3^{(2)} \equiv V_3^{(1,1)}$:

$$V_3^{(2)} = \{W_3^{(2)}; J_{(\lambda_1-\lambda_2)\cdot H} = 0\} \tag{39}$$

The *statement* is that $A_2^{(2)}$ and $A_2^{(1,1)}$-models have identical algebras of symmetries $V_3^{(2)} = V_3^{(1,1)}$ (see eqns. (11)) and their Lagrangeans :

$$\begin{aligned}
\mathcal{L}_2^{(2)} &= -\frac{k}{2\pi}\left(\partial\varphi_0\bar{\partial}\varphi_0 + \frac{e^{\varphi_0}\bar{\partial}\psi_0\partial\chi_0}{1+\frac{3}{4}e^{\varphi_0}\psi_0\chi_0} - e^{-2\varphi_0}(1+\psi_0\chi_0 e^{\varphi_0})\right)\\
\mathcal{L}_2^{(1,1)} &= -\frac{k}{2\pi}\left(\partial\varphi\bar{\partial}\varphi + \frac{e^{-\varphi}\bar{\partial}\psi\partial\chi}{1+\frac{3}{4}e^{-\varphi}\psi\chi} - e^{-2\varphi}\right)
\end{aligned}$$

are related by the following change of the variables:

$$\psi = \chi_0 e^{\varphi_0}(1+e^{\varphi_0}\psi_0\chi_0)^{-\frac{1}{4}},\ \chi = \psi_0 e^{\varphi_0}(1+e^{\varphi_0}\psi_0\chi_0)^{-\frac{1}{4}},\ \varphi = \varphi_0 - \frac{1}{2}ln(1+e^{\varphi_0}\psi_0\chi_0)$$

i.e. $\mathcal{L}_2^{(2)} = \mathcal{L}_2^{(1,1)}$ + total derivative. [3] This fact, together with the OPE's of the $W_3^{(2)}$-currents $G^{\pm}, T_W, J(\equiv J_{(\lambda_1-\lambda_2)\cdot H})$ (of spins $\Delta_{G^\pm} = \frac{3}{2}, \Delta_T = 2, \Delta_J = 1$) (see ref. [19])

$$\begin{aligned}
&J(z_1)G^{\pm}(z_2) = \pm\frac{1}{z_{12}}G^{\pm}(z_2) + O(z_{12});\ \ J(z_1)J(z_2) = \frac{(2k+3)}{3z_{12}} + O(z_{12}),\\
&G^{\pm}(z_1)G^{\pm}(z_2) = O(z_{12}), \text{ etc.,}
\end{aligned} \tag{40}$$

lead us to the following relation between $G^{\pm}, T_W$, $J = \sqrt{\frac{(2k+3)}{3}}\partial\tilde{\phi}$ and the $V_3^{(1,1)}$-currents $V^{\pm}, T_V$:

$$G^{\pm} = V^{\pm}e^{\pm\sqrt{\frac{3}{2k+3}}\tilde{\phi}}, \quad T_W = T_V + \frac{1}{2}(\partial\tilde{\phi})^2. \tag{41}$$

Remind that according to (39) we have to impose

$$J(z_1)V^{\pm}(z_2) = O(z_{12}) = J(z_1)T_V(z_2). \tag{42}$$

and that the OPE's (40) are compatible with the bosonization of the $U(1)$ current J if the following OPE

$$\tilde{\phi}(z_1)\tilde{\phi}(z_2) = \ln z_{12} + O(z_{12}) \tag{43}$$

[3]The detailed proof is present in our forthcoming paper [13]

takes place. As a consequence of eqns. (40), (41), (42) and (43) we find that the spins of the *quantum currents* $V^{\pm}$ are *renormalized* ($\Delta^{\pm}_{cl} = \frac{3}{2}$)

$$\Delta^{\pm}_{q} = \frac{3}{2} - \frac{3}{2(2k+3)}$$

and that the $V^{\pm}$ and T_V-OPE's (that define the quantum $V_3^{(1,1)}$) have the form ($k = -3, -\frac{3}{2}, -1$):

$$\begin{aligned} V^{\pm}(z_1)V^{\pm}(z_2) &= z_{12}^{-\frac{3}{2k+3}} V^{\pm}_{(2)}(z_2) + O(z_{12}) \\ V^{+}(z_1)V^{-}(z_2) &= z_{12}^{\frac{3}{2k+3}} \left(\frac{(2k+3)(k+1)}{z_{12}^3} - \frac{k+3}{z_{12}} T_V(z_2) \right) + O(z_{12}) \\ T_V(z_1)V^{\pm}(z_2) &= \frac{\Delta^{\pm}}{z_{12}^2} V^{\pm}(z_2) + \frac{1}{z_{12}} \partial V^{\pm}(z_2) + O(z_{12}) \end{aligned} \quad (44)$$

The $T_V(z_1)T_V(z_2)$ has the standard form of the Virasoro algebra OPE with central charge $c_V = c_W - 1 = -6\frac{(k+1)^2}{(k+3)}$. The $V_3^{(1,1)}$-algebra (44) is quite similar to the standard PF-algebra [17] and for L-positive integers ($L > 3$) the OPE's (44) involve more currents $V_l^{\pm}$, $l = 1, 2, \cdots, L-1$, of dimensions $\Delta_l^{\pm} = \frac{3}{2}\frac{l}{L}(L-l)$, $L = 2k+3$. Following the arguments of ref. [17] we define (Laurent) mode expansion for the currents $V^{\pm}$:

$$V^{\pm}(z)\phi_s^{\eta}(0) = \sum_{m=-\infty}^{\infty} z^{\pm\frac{3s}{2L}+m-1\mp\eta} V^{\pm}_{-m\pm\eta-\frac{1}{2}+\frac{3(1\mp s)}{2L}} \phi_s^{\eta}(0) \quad (45)$$

where $\phi_s^{\eta}(0)$ denote certain Ramond ($\eta = 1/2, s - odd$) and Neveu-Schwarz ($\eta = 0, s - even$) fields, $s = 1, 2, \cdots L-1$. Then the OPE's (44) give rise to the following *PF-type " commutation relations"* for the $V_3^{(1,1)}(L)$-algebra ($|L| > 3$)

$$\begin{aligned} &\frac{2}{L+3} \sum_{p=0}^{\infty} C^p_{(-\frac{3}{L})} \left(V^{+}_{-\frac{3(s+1)}{2L}+m-p-\eta+\frac{1}{2}} V^{-}_{\frac{3(s+1)}{2L}+n+p+\eta-\frac{1}{2}} + V^{-}_{-\frac{3(1-s)}{2L}+n-p+\eta-\frac{1}{2}} V^{+}_{\frac{3(1-s)}{2L}+m+p-\eta+\frac{1}{2}} \right) \\ &= -L_{m+n} + \frac{(L-1)L}{2(L+3)} (\frac{3s}{2L} + n + \eta)(\frac{3s}{2L} + n + \eta - 1)\delta_{m+n,0} \end{aligned} \quad (46)$$

where $C^p_{(M)} = \frac{\Gamma(p-M)}{p!\Gamma(-M)}$, $m, n = 0, \pm 1, \pm 2, \cdots$ and

$$\sum_{p=0}^{\infty} C^p_{(\frac{3}{L})} \left(V^{\pm}_{\frac{3(3\mp s)}{2L}-p+m+\eta-\frac{1}{2}} V^{\pm}_{\frac{3(1\mp s)}{2L}+p+n+\eta-\frac{1}{2}} - V^{\pm}_{\frac{3(3\mp s)}{2L}-p+n+\eta-\frac{1}{2}} V^{\pm}_{\frac{3(1\mp s)}{2L}+p+m+\eta-\frac{1}{2}} \right) = 0 \quad (47)$$

In the particular cases $L = 2, 3$ the OPE's $V^{\pm}V^{\pm}$ have also a pole , which makes eqn. (47)*nonvalid.* The simplest example of such $V_3^{(1,1)}$ algebra $L = 2$ is spanned by $V^{\pm}$ of $\Delta_L^{\pm} = \frac{3}{4}$ and T_V only. Its central charge is $c_V(L = 2) = -\frac{3}{5}$. The relations (47) are now substituted by:

$$\sum_{p=0}^{\infty} C^p_{(\frac{1}{2})} \left(V^{-}_{-p+m+\eta-\frac{3}{4}} V^{-}_{p+n+\eta-\frac{5}{4}} + V^{-}_{-p+n+\eta-\frac{3}{4}} V^{-}_{p+m+\eta-\frac{5}{4}} \right) = \delta_{m+n+2\eta,0}$$

and similar one for V^+V^+'s. Again as in the $n=1$ case one can verify that certain limit of the OPE's (44) reproduces the classical PB's $V_3^{(1,1)}$-algebra (11).

The relation (41) between $W_3^{(2)}$ and $V_3^{(2)}$ currents leads to the following form of the $W_3^{(2)}$-(chiral) vertex operators $\phi^W_{(r_i,s_i)}(z)$, $i=1,2$ in terms of the $V_3^{(1,1)}$-ones $\phi^V_{(r_i,s_i)}$ and $\tilde{\phi}$:

$$\phi^W_{(r_i,s_i)} = \phi^V_{(r_i,s_i)} exp(q_{(r_i,s_i)}\sqrt{\frac{3}{L}}\tilde{\phi}) \tag{48}$$

The construction (48) is a consequence of (41), (43), the OPE's

$$\begin{aligned} T^W(z_1)\phi^W_{(r,s)}(z_2) &= \frac{\Delta^W_{r,s}}{z_{12}^2}\phi^W_{(r,s)}(z_2) + \frac{1}{z_{12}}\partial\phi^W_{(r,s)}(z_2) + O(z_{12}) \\ J(z_1)\phi^W_{(r,s)}(z_2) &= \frac{q_{r,s}}{z_{12}}\phi^W_{(r,s)}(z_2) + O(z_{12}), \end{aligned}$$

which define $\phi^W_{(r,s)}$ as $W_3^{(2)}$ primary fields and the fact that $\phi^V_{(r,s)}$ are J-neutral, i.e. $J(z_1)\phi^V_{(r,s)}(z_2) = O(z_{12})$. Finally we realize that the dimensions of the $V_3^{(1,1)}$-primary fields $\phi^V_{(r,s)}$ are related to the $\phi^W_{(r,s)}$-dimensions and charges by the following formula:

$$\Delta^V_{(r,s)} = \Delta^W_{(r,s)} - \frac{3}{2L}q^2_{(r,s)} \tag{49}$$

Taking into account the explicit values of $\Delta^W_{(r,s)}$ and $q_{(r,s)}$ for the class of " completely degenerate" highest weight representations of $W_3^{(2)}$ (which for rational levels $L+3=\frac{4p}{q}$ have been calculated in ref. [19]) we find that the conformal dimensions $\Delta^V_{(r,s)}$ of the " degenerate" representations of $V_3^{(1,1)}$ are given by :

$$\begin{aligned} \Delta^V_{(r,s)} = \frac{1}{32(L+3)}\left((L-3)((L+3)r_{12}-4s_{12})^2 + 4L((L+3)r_1-4s_1)((L+3)r_2-4s_2)\right) - \\ -\frac{4L(L-1)^2}{32(L+3)} - \frac{\eta^W}{8L}[L+3\eta^W \pm ((L+3)r_{12}-4s_{12})], \end{aligned} \tag{50}$$

$1\le r_i\le 2p-1$, $1\le s_i\le 2q-1$, $r_{12}=r_1-r_2$ where $\eta^W=0$, r_i-odd integers for the NS-sector, $\eta^W=1/2$, r_i-even integers for the Ramond-sector and $\eta^W=\frac{1}{2}-\eta$.

The parafermionic features of the V_2 and $V_3^{(1,1)}$-algebras rises the question whether the quantum $V_{n+1}^{(1,1)}$ -algebras share these properties. Our preliminary calculations of the *renormalized* spins of the *nonlocal* currents $V^\pm_{(n)}$ (for the $A_n^{(1,1)}$-model)

$$\Delta_n^\pm = \frac{n+1}{2}(1-\frac{1}{2k+n+1})$$

shows that this is indeed the case. An interesting open question is about the *quantum counterpart* of the classical $\hat{S}L(2,R)_q$-PB's algebra (21), (22), (23). Although we have no satisfactory answer to this question, the particular case $n=1$, $k=2$ (critical Ising model) provides a promissing hint. The quantum nonlocal charges Q^+ and $\bar{Q}^-$ coincide in this case with the Ramond sector's zero modes ψ_0, $\bar{\psi}_0$ of the Ising fermions. Due to the double degeneracy of the lowest energy state $|\sigma_\pm\rangle$

$$\psi_0|\sigma_\pm >= \frac{1}{\sqrt{2}}|\sigma_\mp > \qquad \bar{\psi}_0|\sigma_\pm >= \mp i\frac{1}{\sqrt{2}}|\sigma_\mp >$$

their commutator does not vanish (see Sect. 6 of ref. [20]),

$$[\psi_0, \bar{\psi}_0]|\sigma_\pm >= i\Gamma|\sigma_\pm > \tag{51}$$

where Γ is the fermion parity operator . The nonvanishing commutator of the "left" and "right " fermionic zero modes *is not in contradiction with the holomorphic factorization* of the critical Ising model. What is important is that the *anticommutator* $[\psi_0, \bar{\psi}_0]_+ = 0$ indeed vanishes.

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