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Self-Energy for Fields Obeying Higher Order Equations

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Abstract

In this work we evaluate the self-energy intergrals for fields which obey equations of motion containing the iterated Lorentz invariant D'Alembertians.

Key-words: Field theory; Quantum field theory.

This paper has been written for a special volume to be published in honour of Prof. Paulo Leal Ferreyra. Paulo has been our friend for many years. His contribution to the development of physics in Brazil and the rest of Latin-America has been very important. He has helped lot of physicists with scientific and material generosity.

We pay homage to Paulo both for his scientifc achievements as well as for his moral stature.

§I. Introduction

Higher order equations seem to be a possible description for the evolution of physical fields.[1],[4]

In view of this fact, it is useful to have the necessary tools to develop the theory.

In this work we will evaluate the self-energy integrals for the fields which obey equations of motion containing the Lorentz-invariant (iterated) D'Alembertian.

The simplest example is provided by the loop formed with "Scalar Photons".

The massless equations:

$$\Box\phi_i = j_i(\phi_1\phi_2) \qquad i = 1, 2 \tag{1}$$

give rise to the self-energy integral.

$$\Sigma(k) = \int d^\nu q \, \frac{1}{q^2} \, \frac{1}{(q-k)^2} \tag{2}$$

ν-dimension of space-time.

The integration can easily be performed with the aid of the well-known Feynman trick

$$\frac{1}{D_1 D_2} = \int_0^1 dx \, \frac{1}{[D_1 x + D_2(1-x)]^2} \tag{3}$$

Replacing eq. (3) in (2) and choosing a convenient origin in q-space, we end up with

$$\Sigma = \frac{\pi^{\frac{\nu}{2}}}{\Gamma(\nu-2)} \, \Gamma\left(2-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}-1\right)\Gamma\left(\frac{\nu}{2}-1\right)(k^2)^{\frac{\nu}{2}-2} \tag{4}$$

As was to be expected, eq. (4) depends on k^2 and no parameter with the dimension of mass appears.

§II. Higher Order Massless Fields

The self-energy integral (2) can be generalized to the case in which the equations of motion depend on interated D'Alembertians:[5]

$$\Box^n\phi_i = j_i \tag{5}$$

In this case the propagators are proportional to $(k^2)^{-n}$ and the self-energy integral is:

$$\Sigma = \int d^\nu q \, \frac{1}{q^{2n}} \, \frac{1}{(q-k)^{2n}} \tag{6}$$

The appropriate formula to use for the integral (6) is obtained by taking derivatives in eq. (3). For example, by taking n derivatives with respect to D_1 obtain:

$$(-1)^n n! \, \frac{1}{D^{n+1} D_2} = (-1)^n (n+1)! \int_0^1 dx \, \frac{x^n}{[D_1 x + D_2(1-x)]^{n+2}} \tag{7}$$

i.e.:

$$\frac{1}{D_1^{n+1} D_2} = (n+1) \int_0^1 \frac{x^n}{[D_1 x + D_2(1-x)]^{n+2}} \tag{8}$$

If we now take n derivatives with respect to D_2 we get the formula:

$$\frac{1}{D_1^{n+2} D_2^{m+1}} = \frac{(n+m+1)!}{n!m!} \int_0^1 dx \frac{x^n (1-x)^m}{[D_1 x + D_2(1-x)]^{n+m+2}} \tag{9}$$

Using (8) in (6) and changing the origin of the integration variable, we obtain:

$$\Sigma(k) = \pi^{\frac{\nu}{2}} \frac{\Gamma\left(2n - \frac{\nu}{2}\right) \Gamma\left(\frac{\nu}{2} - n\right) \Gamma\left(\frac{\nu}{2} - n\right)}{\Gamma(\nu - 2n)\Gamma(n)\Gamma(n)} (k^2)^{\frac{\nu}{2} - 2n}$$

which, of course for $n = 1$ coincides with (4).

When the fields obey equations with iterated D'Alembertians of different orders

$$\Box^\alpha \phi_1 = j_1 \tag{10}$$

$$\Box^\beta \phi_2 = j_2 \tag{11}$$

then we have to generalize (8) for arbitrary values for n and m to get

$$\Sigma(k) = \int d^\nu q \, \frac{1}{q^{2\alpha}} \, \frac{1}{(q-k)^{2\beta}}$$

$$\Sigma(k) = \frac{\pi^{\frac{\nu}{2}} \Gamma\left(\alpha + \beta - \frac{\nu}{2}\right) \Gamma\left(\frac{\nu}{2} - \alpha\right) \Gamma(\frac{\nu}{2} - \beta)}{\Gamma(\nu - \alpha - \beta)\Gamma(\alpha)\Gamma(\beta)} (k^2)^{\frac{\nu}{2} - \alpha - \beta} \tag{12}$$

§III. Massless and Klein-Gordon Fields in Interaction

A massless scalar field interacting with a Klein-Gordon field obeys the equations

$$\Box \phi_1 = j_1 \tag{13}$$

$$(\Box - \mu^2)\phi_2 = j_2 \tag{14}$$

with the appropriate expressions for j_1 and j_2. The respective propagators are q^{-2} and $(q^2 + \mu^2)^{-1}$ giving rise to the self-energy:

$$\Sigma = \int d^\nu q \, \frac{1}{(q-k)^2} \, \frac{1}{q^2 + \mu^2} \tag{15}$$

Using eq. 3 and following the customary procedures we get a hypergeometric function.

$$\Sigma = \frac{\pi^{\frac{\nu}{2}}}{\left(\frac{\nu}{2} - 1\right)} \Gamma\left(2 - \frac{\nu}{2}\right) F\left(\left(2 - \frac{\nu}{2}\right), \left(\frac{\nu}{2} - 1\right); \frac{\nu}{2}; \frac{q^2}{q^2 + \mu^2,}\right) (q^2 + \mu^2)^{\frac{\nu}{2} - 2} \tag{16}$$

If, instead of (13), we have for ϕ, an equation with iterated D'Alembertian:

$$\Box^n \phi_1 = j_1 \tag{17}$$

The self-energy integral is:

$$\Sigma = \int d^\nu q \; \frac{1}{(q-k)^{2n}} \; \frac{1}{q^2+\mu^2} \tag{18}$$

and we should use eq. (7) obtaining

$$\Sigma = \pi^{\frac{\nu}{2}} \Gamma\left(1-\frac{\nu}{2}\right) F\left(\left(n+1-\frac{\nu}{2}\right), \frac{\nu}{2}-n; \frac{\nu}{2}; \frac{q^2}{q^2+\mu^2}\right)(q^2+\mu^2)^{\frac{\nu}{2}-n-1} \tag{19}$$

In (19) (or (16)) we have now the mass-parameter of the Klein-Gordon field (cf. eq. (14)). $\sum$ is a function of q^2 and $q^2+\mu^2$

§IV. Iterated D'Alembertians

In this paragraph we will consider equations of motion containing iterated D'Alembertians for both, the massless and the massive fields.

As a first example, we shall take the equations

$$\Box^2 \phi_1 = J_1 \tag{20}$$

$$(\Box^2 - \mu^4)\phi = j_2 \tag{21}$$

The propagators are q^{-4} and $(q^4-\mu^4)^{-1}$. The self energy integral is then:

$$\Sigma = \int d^\nu q \; \frac{1}{(q-k)^4} \; \frac{1}{(q^4-\mu^4)} \tag{22}$$

We now use the identity

$$\frac{1}{q^4-\mu^4} = \frac{1}{2\mu^2}\left(\frac{1}{q^2-\mu^2} - \frac{1}{q^2+\mu^2}\right) \tag{23}$$

So that we can write (22) in the form:

$$\Sigma = \frac{1}{2\mu^2}\int d^\nu q \left(\frac{1}{q^2+\mu^2} - \frac{1}{q^2+\mu^2}\right) = \Sigma_1 - \Sigma_2 \tag{24}$$

Each of the "partial" self energies has the form (18) and the total self-energy is: $(k \to q)$

$$\begin{aligned} \Sigma \;&=\; \frac{\pi^{\frac{\nu}{2}}}{2\mu^2}\,\Gamma\left(1-\frac{\nu}{2}\right)\left\{(q^2-\mu^2)^{\frac{\nu}{2}-3} F\left(3-\frac{\nu}{2}, \frac{\nu}{2}-2; \frac{\nu}{2}; \frac{q^2}{q^2-\mu^2}\right)\right. \\ &-\; \left.(q^2+\mu^2)^{\frac{\nu}{2}-3} F\left(3-\frac{\nu}{2}; \frac{\nu}{2}-2; \frac{\nu}{2}; \frac{q^2}{q^2+\mu^2}\right)\right\} \end{aligned} \tag{25}$$

To see the general rule, we will consider a second example:

$$\Box^3 \phi_1 = j_1 \tag{26}$$

$$(\Box^3 - \mu^6)\phi_2 = j_2 \tag{27}$$

with propagators q^{-6} and $(q^6-\mu^6)^{-1}$ (respectively).

This time we use the identify

$$\frac{1}{q^6-\mu^6}=\frac{1}{3\mu^4}\left(\frac{e_1}{q^2-e_1\mu^2}+\frac{e_2}{q^2-e_2\mu^2}+\frac{e_3}{q^2-c_3\mu^2}\right) \tag{28}$$

where e_1 e_2 e_3 are the three cubic roots of unity. For the self-energy integral we have

$$\Sigma=\Sigma_1+\Sigma_2+\Sigma_3 \tag{29}$$

where

$$\Sigma_i=\frac{e_i}{3\mu^4}\int d^\nu q\,\frac{1}{(q-k)^6}\,\frac{1}{q^2-e_i\mu^2}\quad ;\quad i=1,2,3 \tag{30}$$

The integral (30) coincides with (18) for $n=3$

$$\Sigma_i=\pi^{\frac{\nu}{2}}\Gamma\left(1-\frac{\nu}{2}\right)\,\frac{e_i}{3\mu^4}\,F\left(4-\frac{\nu}{2},\frac{\nu}{2}-3;\frac{D}{2};\frac{q^2}{q^2-e_i\mu^2}\right)(q^2-e_i\mu^2)^{\frac{\nu}{2}-4} \tag{31}$$

It is now clear that we can solve a more general situation. The power of the D'Alembertian may be arbitrary and it need not be the same for the massless and for the massive fields.

We consider the equations:

$$\Box^n\phi_1=j_1 \tag{32}$$
$$(\Box^m-\mu^{2m})\phi_2=j_2 \tag{33}$$

For the propagator of the massive field ϕ_2 we use the identity:

$$\frac{1}{x^m-1}=\frac{(-1)^{m+1}}{m}\sum_{S=1}^{m}\frac{e_i}{x-e_i} \tag{34}$$

where e_S are the m-roots of unity

$$e_S=e^{i\frac{2\pi S}{m}}\qquad\qquad S=1,2\cdots m \tag{35}$$

Relation (34) allows us to write:

$$\frac{1}{q^{2m}-\mu^2 m}=\frac{(-1)^{m+1}}{m(\mu^2)^{m-1}}\sum_{S=1}^{m}\frac{e_5}{q^2-e_S\mu^S} \tag{36}$$

The self-energy integral corresponding to (32) (33) is

$$\Sigma=\int d^\nu q\,\frac{1}{(q-k)^{2n}}\,\frac{1}{q^{2m}-\mu^{2n}} \tag{37}$$

Using (36) we reduce (37) to a sum of "partial" self-energies:

$$\Sigma_S=\frac{(-1)^{m+1}}{m(\mu^2)^{m-1}}\,e_S\int d^\nu q\,\frac{1}{(q-k)^{2n}}\,\frac{1}{q^2-e_S\mu^2} \tag{38}$$

The result of the integration in (38) is given by (18) and (19)

$$\Sigma_S=\frac{(-1)^{m+1}e_S}{m(\mu^2)^{m-1}}\,\pi^{\frac{\nu}{2}}\Gamma\left(1-\frac{\nu}{2}\right)\cdot F\left(n+1-\frac{\nu}{2},\frac{\nu}{2}-n;\frac{\nu}{2};\frac{q^2}{q^2-e_S\mu^2}\right)\cdot(q^2-e_S\mu^2)^{\frac{\nu}{2}-n-1} \tag{39}$$

§V. Arbitrary Polinomials in $\Box$

The method followed in §IV for the consideration of a massive field can be generalized to the case where the Lorentz invariant equation of motion is given by a polynomial in the D'Alembertian operator.

In fact, any polynomial in $\Box$ can be algebraically factorized into Klein-Gordon factors[6] i.e.:

$$\Box^n + a_1\Box^{n-1} + \cdots + a_n = (\Box - \mu_1^2)(\Box - \mu_2^2)(\Box - \mu_3^2)\cdots(\Box - \mu_n^2) \tag{40}$$

where μ_S^2 $(S = 1, 2 \cdots n)$ are the n(complex) roots of the polynomial.

The corresponding propagator has the form:

$$P(q^2) = \frac{1}{q^2 + \mu_1^2}\,\frac{1}{q^2 + \mu_2^2}\cdots\frac{1}{q^1 + \mu_n^2} \tag{41}$$

The factorization of $P(q^2)$ into Klein-Gordon propagators allows us to write:

$$P(q^2) = \frac{\alpha_1}{q^2 + \mu_1^2} + \frac{\alpha_2}{q^2 + \mu_2^2} + \cdots + \frac{\alpha_n}{q^2 + \mu_n^2} \tag{42}$$

where $\alpha_S (S = 1, 2 \cdots n)$ are appropriate μ-dependent constans.

In this way, the self-energy integral for a field obeying an equation which is a polynomial in $\Box$ interacting with a massless field obeying (23) can be reduced to the type of integral given by (38). The solution is then a sum of "partial" self energies given by (39) where the roots $e_S\mu^2$ have been substituted by the roots μ_S^2.

By using a similar method we can treat the interaction between two fields obeying equations of motion with arbitrary-possibly different, polynomials in $\Box$. Both polynomial are first factorized as in (40). Then both propagators are written as in (42). Finally, the total self-energy integral is decomposed into partial self-energies consisting of convolutions of each of the terms of one propagator with each of the terms of the other propagator[7].

§VI. Discussion

The possible use of higher order equations of motion for the description of natural processes give rise to the necessity of developping tools for the evaluation of diagrams corresponding to the pertubative solution of the evolution of the fields.

In the present work we were interested in the evaluation of the second order self-energy integrals when the intervening fields obey higher order equations.

Our presentation goes from simpler to more involved cases. We started with massless fields (§II) and increasing complication in the respective powers of the D'Alembertian. ($\Box$).

In §III we considered a Klein-Gordon field interacting with a massless field, the power of $\Box$ for the latter being arbitrary.

In §IV we treated a massless field whose equation contains different powers of $\Box$ interacting with a field of the "Klein Gordon type" in whose equation the D'Alembertian has an arbitrary power. Finally, in §V we show that it is possible to treat more general cases. For example, a massless field with arbitrary power of $\Box$ in its equation of motion in interaction with another field whose equation is determined by any polynomial in $\Box$. The solution in this case depends on the factorization of the polynomial in Klein-Gordon factors.

It is also possile to treat the interaction of two fields obeying equations with two different (arbitrary) polynomials of $\Box$. Both polynomials are factorized into Klein-Gordon factors. Then each "Klein Gordon propagator" for the fields gives rise to a convolution self-energy integral with another field another partial "Klein-Gordon propagator" for the other field. If the polynomials have degree n and m (respectively) the total self-energy is expressed as a sum of m times n "elementary" self-energies.

Bibliography

1. - D.G. Boulware: "Quantization of Higher Derivative Theories of Gravity". In Quantum Theory of Gravity ed. S. Christensen, Adam Hilger Ltd (1984).

2. - C.G. Bollini and J.J. Giambiagi: "Generalized Klein Gordon Equation from Supersymmetry." Phys. Rev. D23, 3316 (1985).

3. - C.G. Bollini and J.J. Giambiagi: "Gauge Superfields in Six Dimensions" Phys. Rev. D 39, 1169 (1989).

4. - C.G Bollini and J.J. Giambiagi: "Higher Order Field-Equations." Rev. Mex de Fisica **36**, 23 (1990).

5. - C.G. Bollini and J.J. Giambiagi: "Arbitrary, powers of D'Alembertians and the Huyghen's principle J. Math. Phys. **34**, 610 (1993).

6. - D. Barci, C.G. Bollini, L. Oxman and M. Rocca. "Hygher Order Equations and Constituent Fields J. of Modern Physics.

7. - C.G. Bollini and J.J. Giambiagi "Convolutions of Particle Green Functions" II Nuovo Cimento **105 A**, 1081 (1992).