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λ-Point Transition in Quantum q-Gases

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Abstract

We show that Bose-Einstein condensation is present in highly deformed q-gases, the critical temperature being higher than for the usual ideal boson gas. The specific heat C_V has a λ-point transition behaviour not exhibited by non deformed ideal gases.

Key-words: Quantum groups; Statistical Mechanics; Bose-Einstein condensation; Superfluidity.

Quantum Groups [1–3] have emerged as an appealing non-trivial generalization of Lie Algebras and Lie groups. As Heisenberg algebras play an important role in a wide range of problems, deformed Heisenberg algebras [4] have been attracting increasing interest.

Our main motivation to discuss deformed quantum gases [5–8] comes from the role played by the theory of ideal gases in many different physical phenomena as superfluidity, blackbody radiation, phonons in a cristal lattice etc. [9].

In a previous letter [8] we have analysed the high-temperature (or low-density) approximation for a quantum q-gas at large q limit. In this letter we are going to study the phenomenon of condensation and the behaviour of the specific heat for that system. We find that the presence of deformation increases the critical temperature. Also, the specific heat shows a λ-point discontinuity instead of a cusp singularity, as exhibited in non-deformed ideal gases.

Let A_i, A_i^+ and N_i be respectively the annihilation, creation and occupation number operators of particles in level i, with energy ω_i satisfying the algebra [4, 10]

$$\begin{aligned} & A_i A_j^+ - q^{2\delta_{ij}} A_j^+ A_i = \delta_{ij} \\ & [N_i, A_j^+] = \delta_{ij} A_j^+ \quad , \quad [N_i, A_j] = -\delta_{ij} A_j \ . \end{aligned} \tag{1}$$

The Hamiltonian for an ideal q-gas can then be defined as [5–8]:

$$H = \sum_i \omega_i A_i^+ A_i = \sum_i \omega_i [N_i] \ , \tag{2}$$

with $[n] = (q^{2n} - 1)/(q^2 - 1)$.

The grand canonical partition function is given by:

$$Z = Tr \exp[-\beta(H - \mu N)] = \exp(-\beta\Omega) \ , \tag{3}$$

where $\beta = 1/kT$, with k the Boltzmann constant and N is the total number operator

$$N = \sum_i N_i \ . \tag{4}$$

μ is the chemical potential and Ω is the grand canonical potential. For the above system Z factorizes and the grand canonical potential is given by a sum over single level partition functions

$$\Omega = -\frac{1}{\beta} \sum_i \ln Z_i^0(\omega, \beta, \mu) \ , \tag{5}$$

where

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i [n] - \mu n)} \ . \tag{6}$$

As in the usual approach, we enclose the system in a large volume V and the sum is then replaced by an integral over the p space:

$$\sum_i \to \frac{V}{h^3} \int d^3 p \ , \tag{7}$$

where in the case of a non-relativistic q-boson the energy is defined as $\omega_i = \vec{p}^{\ 2}/2m$.

We find the pressure $P = -\Omega/V$ and the density of states $n = \partial P/\partial\mu$ to be:

$$
\begin{aligned}
P(T,z) &= \beta^{-1}\Lambda^{-3}Y_q(z) \\
n(T,z) &= \Lambda^{-3}y_q(z)\ ,
\end{aligned}
\tag{8}
$$

where the fugacity $z = \exp(\beta\mu)$ and $\Lambda = (\frac{h^2\beta}{2\pi m})^{1/2}$. The functions $Y_q(z)$ and $y_q(z)$ are respectively:

$$
Y_q(z) = \frac{4}{3\pi^{1/2}}\int_0^\infty dx\ x^{3/2}\ \frac{\sum_{n=0}^{\infty}[n]z^n e^{-[n]x}}{\sum_{n=0}^{\infty}z^n e^{-[n]x}} \tag{9.a}
$$

$$
y_q(z) = \frac{4}{3\pi^{1/2}}\int_0^\infty dx\, x^{3/2}\left\{\frac{\sum_{n=0}^{\infty}[n]nz^n e^{-[n]x}}{\sum_{n=0}^{\infty}z^n e^{-[n]x}} - \frac{\left(\sum_{n=0}^{\infty}[n]z^n e^{-[n]x}\right)\left(\sum_{n=0}^{\infty}nz^n e^{-[n]x}\right)}{\left(\sum_{n=0}^{\infty}z^n e^{-[n]x}\right)^2}\right\}. \tag{9.b}
$$

In the high-temperature (or low-density) approximation this system has been analiticaly investigated both for $q \cong 1$ [5–7] (see ref. [6] for a detailed discussion) and large q [8]. Clearly, in this regime it is not possible to access the Bose-Einstein condensation phenomenon.

Let us now study the Bose-Einstein condensation for the highly deformed case, where, as it will be justified later, the series (9) above can be approximated by their first three terms. As usual, when $z \to 1$ (or $T \to T_c$, T_c being the critical temperature) we have to take into account the zero-point energy and single out its contribution in (9). In addition, inspection of eq. (6) clearly shows that when $\omega_i = 0$ the effect of the deformation is cancelled showing that we cannot, for the zero-point energy case, cut the series into a polynomial. Keeping n constant we now consider lower temperatures: $n\Lambda^3$ then increases and so does z, until $z = 1$. This happens when $T = T_c^q$, defined as $n\Lambda_c^3 = y_q(1)$ or

$$
T_c^q = \frac{h^2}{y_q(1)^{2/3}2\pi mk}\ n^{2/3}\ . \tag{10}
$$

Comparing T_c^q to the critical temperature for non-deformed gases of the same density n, we find

$$
\frac{T_c^q}{T_c} = \left(\frac{2.61}{y_q(1)}\right)^{2/3}\ . \tag{11}
$$

Figure 1 shows T_c^q/T_c as a function of the deformation parameter q, where $y_q(1)$ has been numerically calculated keeping only the first three terms of series (9b). As we have already mentioned, the validity of this approximation will be discussed later.

Analogously to the non-deformed case [9] the basic equations are:

$$
P(T,z) = \beta^{-1}\Lambda^{-3}Y_q(z) \tag{12.a}
$$

$$
n(T,z) = \frac{1}{V}\ \frac{z}{1-z} + \Lambda^{-3}y_q(z) \tag{12.b}
$$

where the first term on the right-hand side of (12b), which is due the contribution of the zero energy, is relevant only for $T \leq T_c^q$. In this region z remains equal to one, as is the usual case.

The specific heat per particle C_V, defined as

$$\frac{C_V}{k} = \frac{1}{kn} \left.\frac{\partial \tilde{e}}{\partial T}\right|_n , \tag{13}$$

where $\tilde{e}$ is the energy density (internal energy per volume), can be computed as usual and has the form

$$\frac{C_V}{k} = \frac{15}{4} (\Lambda^3 n)^{-1} Y_q(z) - \frac{9}{4} \frac{y_q(z)}{z y_q'(z)} \quad T > T_c \tag{14.a}$$

$$\frac{C_V}{k} = \frac{15}{4} (\Lambda^3 n)^{-1} Y_q(1) \quad T < T_c \, . \tag{14.b}$$

with $y_q'(z) = \partial y_q(z)/\partial z$.

Let us describe our numerical results. Tables I and II show the values of $Y_q(1), y_q(1)$ and $y_q'(1)$ for $q = 3$ and 4, considering the upper limit of the sums in (9) as being $n = 2, 3, 4$ and 5. We can see that for $q > 3$ the approximation of keeping only the first three terms of the series is valid with an accuracy of at least 10^{-3}. Finally, in fig. 2 we see the behaviour of the specific heat as a function of T/T_c^q for $q = 4$. C_V shows a λ-point transition which is a feature of interesting phenomena like, e.g., superfluidity. We must note that as q approaches 1, in order to have the same order of accuracy (10^{-3}) more terms have to be added to the series in eq. (9). Also, the discontinuity in C_V diminishes and disappears for $q = 1$, becoming then the usual cusp singularity.

In spite of the specific heat $T^{3/2}$ behaviour instead of T^3, for $T < T_c$, as it is the case for H^{II} [11], deformed q-gases seem to describe H^{II} better than the usual ideal boson gas, which does not present a λ-point transition. Besides, the q-gas we have considered is the simplest one and the model can be improved by choosing a different deformed Hamiltonian.

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CAPTION FOR FIGURES AND TABLES

Figure 1 - T_c^q/T_c behaviour as a function of the deformation parameter q. (T_c is the critical temperature for the non deformed case).

Figure 2 - The specific heat C_V/k as a function of T/T_c presents a λ-point transition when $q > 1$.

Table I and II - Values of Y_q, y_q and y'_q with different upper limits n in eqs. 9.

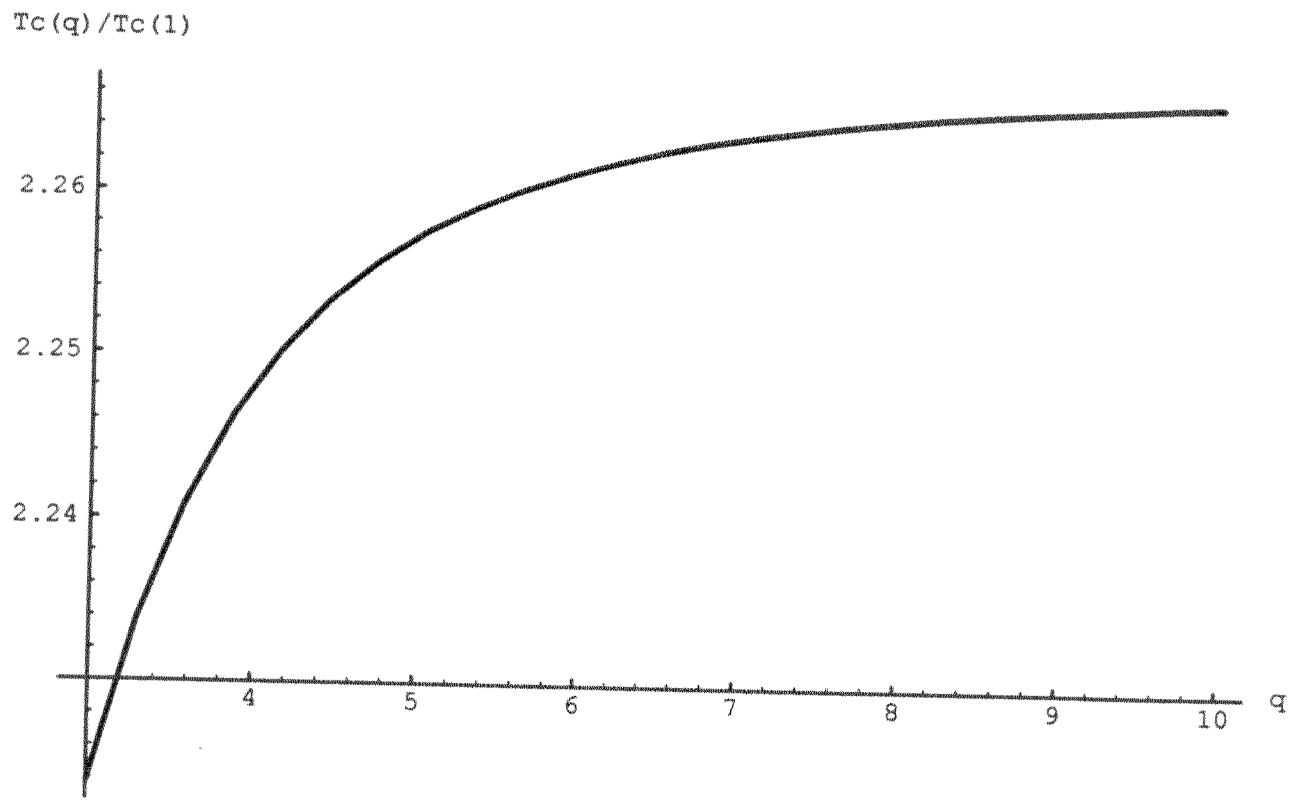

Figure 1

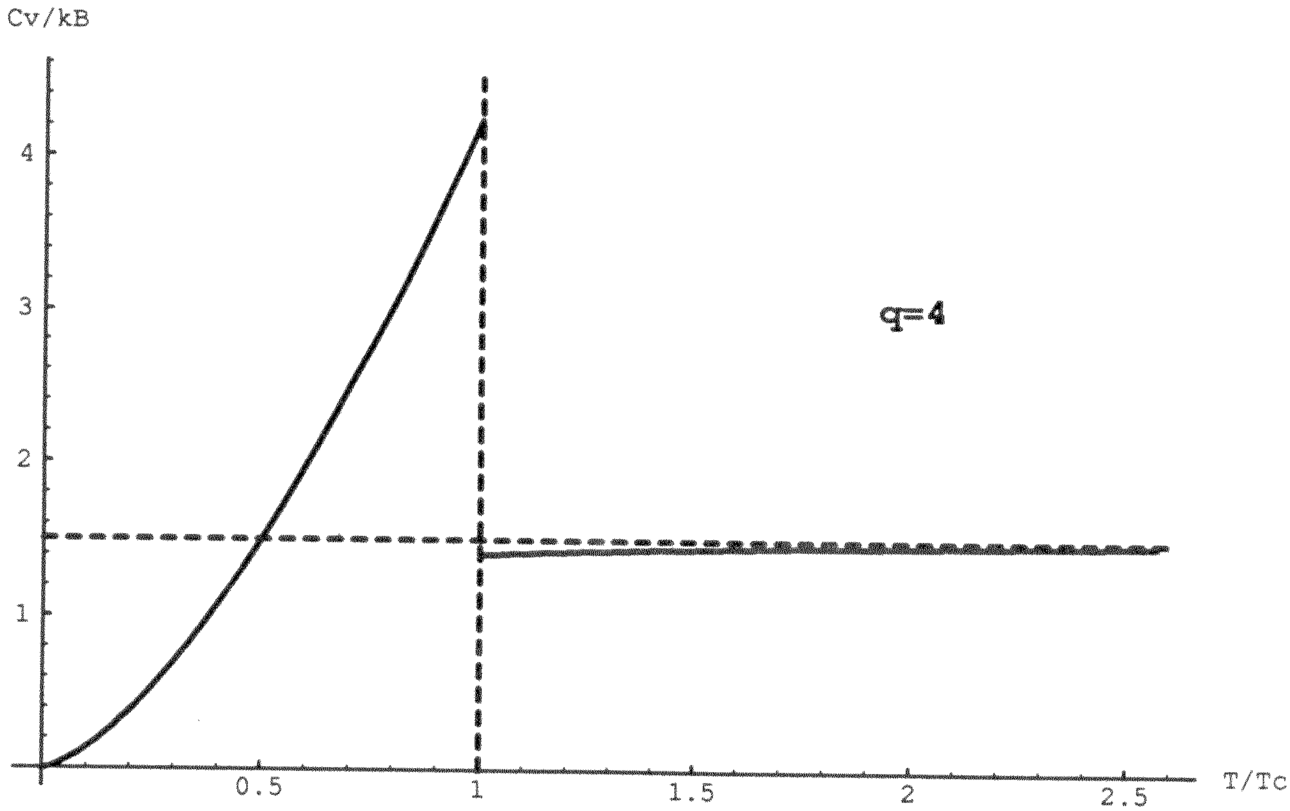

Figure 2

Table I

$q = 3$

n	2	3	4	5
$Y_q(1)$	0.882905	0.883290	0.883300	0.883301
$y_q(1)$	0.787659	0.788110	0.788438	0.788439
$y'_q(1)$	0.631581	0.632749	0.632799	0.632801

Table II

$q = 4$

n	2	3	4	5
$Y_q(1)$	0.874085	0.874157	0.874158	0.874158
$y_q(1)$	0.774906	0.775046	0.775048	0.775048
$y'_q(1)$	0.616266	0.616478	0.616482	0.616482

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