CBPF-NF-018/02

Rotating Cylindrical Shell Source for Lewis Spacetime

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Abstract

Rotating thin-shell-like sources for the stationary cylindrically symmetric vacuum solutions (Lewis) are constructed and studied. It is found, by imposing the non existence of timelike curves in the exterior of the shell, and that the source satisfies the weak, dominant and strong energy condictions that the parameters, commonly denoted by a and σ, are restricted to $0 \leq \sigma \leq 1/4$ when $a > 0$, or $1/4 \leq \sigma \leq 1/2$ when $a < 0$.

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1 Introduction

The Lewis metric represents the most general stationary cylindrically symmetric solutions to Einstein vacuum equations [1, 2]. However, as it is well known, in the case when all the parameters of the metric are real, the so called Weyl class, the Cartan scalars corresponding to this metric are the same as those of the static Levi-Civita spacetime [3] and therefore both metrics, Levi-Civita and Lewis (Weyl class), are locally indistinguishable [4]. This situation is also reflected by the fact that a coordinate transformation exists [5], which casts one of the metrics into the other. Although the consequences implied by this transformation are physically inadmissible (e.g. periodic time), the transformation itself is mathematically regular with a non-vanishing Jacobian. Comments above put in evidence the very peculiar character of the stationarity of Lewis metric and the difficulties in the understanding of the physical meaning of its parameters, which has been brought out before [6]. It is our endeavour with this work to delve deeper into this question . With this purpose we shall construct shell-like sources for the Lewis metric. These shells will be studied and matched to the Lewis spacetime. Doing so, the parameters of the exterior metric will be related to physical properties of the source, and their ranges of validity somehow restricted by energy conditions.

The paper is organized as follows: in Sec. II we discuss about the global and local properties of the spacetime, both, inside and outside a cylindrical source. Then, we show that the solutions with $a > 0,\ \sigma \leq \frac{1}{4}$ or $a < 0,\ \sigma \geq \frac{1}{4}$ are free of CTC's far from the axis of symmetry , where a and σ are two of the four free parameters appearing in the Lewis vacuum solutions, with a and σ being usually related to the angular defect and the mass per unit length, respectively. In Sec. III, using Taub's method [7] we construct cylindrical shell-like sources, by taking the rotating Minkowski spacetime as the interior of the shell, while in Sec. IV we write the surface energy momentum tensor, obtained in Sec. III, in its canonical form by solving the corresponding eigenvalue problem, and then impose the three energy conditions, weak, strong and dominant. It is found that these conditions can be satisfied for the solutions with $a > 0,\ 0 \leq \sigma \leq \frac{1}{4}$ or $a < 0,\ \frac{1}{4} \leq \sigma \leq \frac{1}{2}$. In Sec. V we calculate the vorticity of the shell as well as the energy density per unit length, which will bring out further the role of different parameters in the stationarity of the spacetime. The paper is closed by Sec. VI, in which our main conclusions are presented.

2 The Lewis metric and its local and global properties

The general stationary cylindrically symmetric vacuum solutions of the Einstein field equations, the Lewis metric, are usually given by

$$ds^2 = f dt^2 - 2k dt d\phi - e^{\mu}(d\varrho^2 + dz^2) - l d\phi^2, \tag{2.1}$$

where f, l, k and μ are functions of ϱ only, being given by

$$\begin{aligned} f &= a\varrho^{4\sigma} - \frac{\gamma^2 \varrho^{2(1-2\sigma)}}{a}, \quad l = \frac{\varrho^2}{f} - \Omega^2 f, \\ k &= -\Omega f, \qquad e^{\mu} = \varrho^{4\sigma(2\sigma-1)}, \quad \Omega \equiv b + \frac{\gamma \varrho^{2(1-2\sigma)}}{af}, \end{aligned} \tag{2.2}$$

where $a\ (\neq 0)$, b, γ and σ are the four free parameters of the solutions (Note that in this paper we use the notations slightly different from the ones used in [1, 2]). When these parameters are all real, the corresponding solutions are usually referred to the Weyl class, and when they are complex, the corresponding solutions are usually referred to the Lewis class. In this paper we shall restrict ourselves only to the Weyl class.

Setting $b = 0 = \gamma$, the Lewis solutions (with $a > 0$) reduce to the Levi-Civita solutions [2, 3], which represent the gravitational field produced by a cylindrically symmetric static source, with a being related to the angular defect and σ the mass per unit length of the cylinder [8],[9]. In order to give a geometrical meaning to the radial coordinate ϱ we first transform it into a proper radius r by defining

$$\varrho^{2\sigma(2\sigma-1)} d\varrho = dr, \tag{2.3}$$

so obtaining

$$\varrho = R^{1/\Sigma}, \quad R = \Sigma r, \quad \Sigma \equiv 4\sigma^2 - 2\sigma + 1. \tag{2.4}$$

With (2.4) the metric (2.1) becomes

$$ds^2 = F dt^2 - 2K dt d\phi - dr^2 - H dz^2 - L d\phi^2, \tag{2.5}$$

where

$$\begin{aligned} F &= aR^{4\sigma/\Sigma} - \frac{\gamma^2}{a} R^{2(1-2\sigma)/\Sigma}, \quad H = R^{4\sigma(2\sigma-1)/\Sigma}, \\ L &= \frac{(1-b\gamma)^2}{a} R^{2(1-2\sigma)/\Sigma} - ab^2 R^{4\sigma/\Sigma}, \\ K &= -\frac{\gamma(1-b\gamma)}{a} R^{2(1-2\sigma)/\Sigma} - abR^{4\sigma/\Sigma}. \end{aligned} \tag{2.6}$$

Since now the radial coordinate r defines the proper distance, without loss of generality, we shall consider the solutions only in the region $r \in [0, \infty)$. It can be shown too that the above solutions are singular at $r = 0$ (or equivalently $R = 0$) except for the cases $\sigma = 0, 1/2, \pm\infty$. In the latter cases, the Riemann tensor vanishes in the region $r \in (0, \infty)$, and the spacetime is (locally) flat. The singularity at $r = 0$ is usually considered as representing the existence of some kind of source [10]. However, this kind of interpretation is not completely satisfactory, until some physically acceptable source is found. It is in this vein that in the following we shall look for shell-like sources for the above solutions. That is, we shall consider the Lewis solutions valid only in the region outside of a rotating cylindrically symmetric thin shell, say, located on the hypersurface $r = r_0$, and then join them to a rotating flat region in the interior of the shell. By this way, we can consider the Lewis vacuum field as produced solely by the rotating thin shell. If the matter on the shell satisfies the weak, strong and dominant energy conditions [11], then we shall consider it as physically acceptable source of the Lewis vacuum solutions. However, we are aware of the fact that a great deal of "exotic" (but still seeming physically meaningful) scenarios may produce violation of any of the energy conditions above, and therefore caution should be exerted before ruling out the correspondig sources.

Let us start by noticing that the spacetime inside and outside the shell must satisfy several physical and geometrical conditions [2, 8, 12]. In general, checking those conditions is not trivial. As a matter of fact, only when the symmetry axis is free of curvature singularities, we know how to impose them. When it is singular, it is still not clear which conditions should be required [13, 14]. Fortunately, in the present case since the region inside the shell is assumed to be flat, the axis is regular. Then, we impose the following conditions: (i) There must exist an axially symmetric axis, which is usually characterized by the condition,

$$X \equiv \|\xi^{\mu}_{(\phi)}\xi^{\nu}_{(\phi))}g_{\mu\nu}| = | g_{\phi\phi} | \to 0, \tag{2.7}$$

as $r \to 0^+$, where we have chosen the radial coordinate such that the axis is located at $r = 0$, ϕ denotes the angular coordinate, with the hypersurfaces $\phi = 0$ and $\phi = 2\pi$ being identical,and $\xi^{\mu}_{(\phi)}$ is the Killing vector along $d\phi$ (ii) The spacetime near the symmetry axis must be locally flat, which can be expressed as [2],

$$\frac{X_{,\alpha}X_{,\beta}g^{\alpha\beta}}{4X} \to 1, \tag{2.8}$$

as $r \to 0^+$ and the comma stands for partial differentiation. Note that solutions that fail to satisfy this condition are sometimes accepted. For example, when the left-hand side of Eq.(2.8) approaches a finite constant, the singularity on the axis can be related to a cosmic string [15].

These are the conditions that the metric inside of the shell has to satisfy.

For the spacetime outside the shell we impose the conditions: (iii) *No closed timelike curves* (CTC's). In cylindrical spacetimes, CTC's are rather easily introduced [11]. While the physics of the CTC's is not yet clear [16], we shall not consider this possibility here and simply require that [17]

$$\xi^{\mu}_{(\phi)}\xi^{\nu}_{(\phi)}g_{\mu\nu} < 0, \tag{2.9}$$

holds in all the region outside the shell. Since we are studying axisymmetric solutions of the form (2.1), in which ϕ is restricted to $[0, 2\pi]$, the (t, ϕ) surfaces thus have cylindrical topology. This imposes restrictions on the permissible transformations on (t, ϕ). It is showed in [13] that the only allowed transformations are

$$t' = Yt, \ \phi' = \phi + Zt \tag{2.10}$$

where Y and Z are constants. Under these transformations $g_{\phi\phi} = g_{\phi'\phi'}$, hence the conditions (2.9) remains the same for (2.10). (iv) *Asymptotical flatness.* Since the source now is a thin shell located at a finite distance from the axis, we require that the spacetime be asymptotically flat as $r \to +\infty$. We note that because of the cylindrical symmetry, the spacetime can never be asymptotically flat in the axial direction. Therefore, in the following whenever we mention asymptotical flatness we mean in radial direction. (v) *No spacetime singularities.* The spacetime outside the shell must be free of any singularities. By this way, we are sure that the Lewis vacuum spacetime is indeed produced only by the rotating thin shell. Otherwise, the singularities may represent additional sources, being a possibility that we do not consider in this paper.

It can be shown that conditions (iv) and (v) are satisfied by the Lewis vacuum solutions, while the condition (iii) given by Eq.(2.9) becomes

$$L(R) = \frac{(1-b\gamma)^2}{a} R^{4\sigma/\Sigma}[R^{2(1-4\sigma)/\Sigma} - R_1^{2(1-4\sigma)/\Sigma}] > 0, \tag{2.11}$$

where

$$R_1 \equiv \left|\frac{ab}{1-b\gamma}\right|^{\Sigma/(1-4\sigma)}. \tag{2.12}$$

From the above it is simple to see that Eq.(2.11) should hold in the following four cases , far from the axis,

$$\begin{aligned}
&(a) \quad a > 0,\ \sigma < \frac{1}{4},\ b\gamma \neq 1,\ R \in (R_1, \infty);\\
&(b) \quad a > 0,\ \sigma = \frac{1}{4},\ b\gamma \neq 1,\ a^2b^2 < (1-b\gamma)^2,\ R \in (0, \infty);\\
&(c) \quad a < 0,\ \sigma = \frac{1}{4},\ b \neq 0,\ a^2b^2 > (1-b\gamma)^2,\ R \in (0, \infty);\\
&(d) \quad a < 0,\ \sigma > \frac{1}{4},\ b \neq 0,\ R \in (R_1, \infty).
\end{aligned} \tag{2.13}$$

3 Matching Lewis spacetime to a cylindrical rotating shell source

In 1937, van Stockum constructed a rotating cylindrically symmetric dust fluid as the sources of the Lewis vacuum solutions [18], and showed that only the solutions (of the Weyl class) with $a0$ and $0 \leq \sigma < \frac{1}{4}$ can be produced by such a dust fluid.

In this paper, we shall consider an infinitely thin cylindrical shell of anisotropic rotating fluid with a finite radius and we match it to the exterior Lewis spacetime given by Eqs.(2.5) and (2.6). For the interior of the shell we assume a rotating Minkowski spacetime, since it is the only spacetime deprived of energy density. In order to do the matching we only require the continuity of the metric coefficients across the shell [7], so allowing us to obtain the most general rotating thin shell. Using the same coordinate system as in Eq.(2.5), the rotating Minkowski spacetime with angular velocity ω can be written in the form,

$$ds^2_- = (1-\omega^2 r^2)dt^2 - 2\omega r^2 dt d\phi - dr^2 - dz^2 - r^2 d\phi^2. \quad (3.1)$$

Indices $-$ and $+$ refer to interior and exterior spacetimes of the shell, respectively. Clearly the conditions (2.7) and (2.8) are satisfied by the metric (3.1). In addition to these two conditions, we also require that the Killing vector $\xi^\mu_{(t)} = \delta^\mu_t$ remains timelike in the interior region of the shell,

$$\xi^\mu_{(t)}\xi^\nu_{(t)} g^-_{\mu\nu} = 1 - \omega^2 r^2 > 0, \quad (0 \le r \le r_0). \quad (3.2)$$

On the other hand, without loss of generality, we make a reparametrization of t and z,

$$t \to \frac{t}{A}, \quad z \to \frac{z}{B}, \quad (3.3)$$

where A and B are constants. Then, the solutions of Eqs.(2.5) and (2.6) become,

$$ds^2_+ = F dt^2 - 2K dt d\phi - dr^2 - H dz^2 - L d\phi^2, \quad (3.4)$$

with

$$\begin{aligned} F &= \frac{1}{A^2}\left[a R^{4\sigma/\Sigma} - \frac{\gamma^2}{a} R^{2(1-2\sigma)/\Sigma}\right], \quad H = \frac{1}{B^2} R^{4\sigma(2\sigma-1)/\Sigma}, \\ L &= \frac{(1-b\gamma)^2}{a} R^{2(1-2\sigma)/\Sigma} - a b^2 R^{4\sigma/\Sigma}, \\ K &= -\frac{1}{A}\left[ab R^{4\sigma/\Sigma} + \frac{\gamma(1-b\gamma)}{a} R^{2(1-2\sigma)/\Sigma}\right]. \end{aligned} \quad (3.5)$$

On the hypersurface $r = r_0$, the first junction condition requires that

$$g^+_{\mu\nu}|_{r_0} = g^-_{\mu\nu}|_{r_0}. \quad (3.6)$$

From the 00-component of Eq.(3.6), we find that

$$F(R_0) = \frac{\gamma^2}{aA^2} R_0^{4\sigma/\Sigma}[R_2^{2(1-4\sigma)/\Sigma} - R_0^{2(1-4\sigma)/\Sigma}] = 1 - \omega^2 r_0^2 > 0, \quad (3.7)$$

where

$$R_0 = \Sigma r_0, \quad R_2 \equiv \left|\frac{a}{\gamma}\right|^{\Sigma/(1-4\sigma)}. \quad (3.8)$$

Eq.(3.7) further restricts the validity of the Lewis solutions as representing the vacuum gravitational field outside a cylindrical source and/or the range of validity of the coordinates of (2.1).

As a matter of fact, in the cases (a) and (d) given in Eq.(2.13), $F(R)$ is always negative when R is sufficiently large. Therefore, in these two cases the condition $F(R_0) > 0$ is possible only in certain range of R. A closer investigation shows that the four cases given in Eq.(2.13) have to be further restricted to

$$\begin{aligned}
&(a) \quad a > 0, \quad \sigma < \frac{1}{4}, \quad b\gamma < \frac{1}{2}, \quad R_1 < R_0 < R_2;\\
&(b) \quad a > |\gamma|, \quad \sigma = \frac{1}{4}, \quad -(a+\gamma) < b < a - \gamma, \quad b\gamma \neq 1, \quad R_0 > 0;\\
&(c) \quad -|\gamma| < a < 0, \quad \sigma = \frac{1}{4}, \quad \gamma - |a| < b < \gamma + |a|, \quad b\gamma \neq 0, \quad R_0 > 0;\\
&(d) \quad a < 0, \quad \sigma > \frac{1}{4}, \quad b\gamma \neq 0, \quad b\gamma < \frac{1}{2}, \quad R_1 < R_0 < R_2.
\end{aligned} \tag{3.9}$$

These conditions are sufficient to ensure the absence of CTC's outside the source and the timelike nature of the Killing vector $\xi^{\mu}_{(t)}$,inside the shell.

From the above expressions we can see that when $\sigma \neq 1/4$ the rotating shell can be present only in between the two cylinders $R = R_1$ and $R = R_2$. It is remarkable to note that in the cases (c) and (d) where $a < 0$, the static limit $b = 0 = \gamma$ is forbidden by the first junction condition. Moreover, in the cases (a) and (d), there always exists a point $R = R_2$, where

$$F(R) = \begin{cases} \geq 0, & R \leq R_2, \\ < 0, & R > R_2. \end{cases} \tag{3.10}$$

That is, (as expected) the Killing vector $\xi^{\mu}_{(t)} = \delta^{\mu}_t$ changes from time-like in the region $R \in [R_0 R_2)$ to space-like in the region $R \in (R_2, \infty)$, thereby restricting the range of R (for the whole spacetime) to $(0, R_2)$.

It should be also noted that the conditions (3.9) are valid not only for the case where a thin shell is the sole source of the Lewis metric, but also for the case where the *whole* interior region $r \leq r_0$ is all filled with matter. Then, any kind of matching between a cylindrical stationary source and the Lewis vacuum spacetime is satisfied by (3.9).

Considering the other components of Eq.(3.6), we obtain

$$\begin{aligned}
A &= \frac{R_0^{1/\Sigma}}{r_0}, \quad B = R_0^{2\sigma(2\sigma-1)/\Sigma},\\
a &= \frac{2(1-b\gamma)^2 R_0^{2(1-2\sigma)/\Sigma}}{r_0^2 \pm \Omega_0},\\
\omega r_0 &= -\frac{\gamma r_0^2}{(1-b\gamma)R_0^{1/\Sigma}} - \frac{2b(1-b\gamma)R_0^{1/\Sigma}}{r_0^2 \pm \Omega_0}.
\end{aligned} \tag{3.11}$$

with

$$\Omega_0 \equiv \left[r_0^4 + 4b^2(1-b\gamma)^2 R_0^{2/\Sigma}\right]^{1/2}. \tag{3.12}$$

Taub [7] showed that if (3.6) is satisfied then the first derivatives of the metric are in general discontinuous across $r = r_0$, giving rise to a shell of matter. Following him, we first introduce the quantity $b_{\mu\nu}$ via the relations

$$g^{+}_{\mu\nu,\lambda}|_{r_0} - g^{-}_{\mu\nu,\lambda}|_{r_0} = n_\lambda b_{\mu\nu}, \tag{3.13}$$

where n_λ is the normal to the hypersurface $r = r_0$, directed outwards and given by $n_\lambda = \delta^r_\lambda$. Then, in terms of $b_{\mu\nu}$, the energy-momentum tensor (EMT), $T_{\mu\nu}$, of the shell is given by [7],

$$T_{\mu\nu} = \tau_{\mu\nu}\delta(r - r_0), \tag{3.14}$$

where $\delta(r - r_0)$ denotes the Dirac delta function and $\tau_{\mu\nu}$ the surface EMT, given by

$$\tau_{\mu\nu} = \frac{1}{16\pi}[b(ng_{\mu\nu} - n_\mu n_\nu) + n_\lambda(n_\mu b^\lambda_\nu + b^\lambda_\mu)n_\nu - nb_{\mu\nu} - n_\lambda n_\delta b^{\lambda\delta} g_{\mu\nu}], \tag{3.15}$$

with $n \equiv n_\lambda n^\lambda$, and $b \equiv b^\lambda_\lambda$. It can be shown that in the present case the non-vanishing components of $b_{\mu\nu}$ are

$$b_{tt} = 2\omega^2 r_0 + F'_0, \quad b_{t\phi} = 2\omega r_0 - K'_0, \quad b_{zz} = -H'_0, \quad b_{\phi\phi} = 2r_0 - L'_0, \tag{3.16}$$

where a prime stands for differentiation with respect to r. Substituting the above expressions into Eq.(3.15), we find that the surface EMT can be written in the form,

$$\tau_{\mu\nu} = \rho t_\mu t_\nu + q(t_\mu \phi_\nu + \phi_\mu t_\nu) + p_z z_\mu z_\nu + p_\phi \phi_\mu \phi_\nu, \tag{3.17}$$

where

$$\begin{aligned}
\rho &= \frac{1}{16\pi R_0}\left[1 \mp (1 - 4\sigma)J(r_0)\right], \\
q &= \frac{4\sigma - 1}{16\pi R_0}\left[J^2(r_0) - 1\right]^{1/2}, \quad p_z = \frac{(1 - 2\sigma)\sigma}{4\pi R_0}, \\
p_\phi &= \frac{1}{16\pi R_0}\left[1 - 4\sigma + 8\sigma^2 \mp (1 - 4\sigma)J(r_0)\right],
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
t_\mu &= \delta^t_\mu, \quad z_\mu = \delta^z_\mu, \quad \phi_\mu = \omega r_0 \delta^t_\mu + r_0 \delta^\phi_\mu, \\
t_\mu t^\mu &= -z_\mu z^\mu = -\phi_\mu \phi^\mu = 1, \quad t_\mu z^\mu = t_\mu \phi^\mu = z_\mu \phi^\mu = 0,
\end{aligned} \tag{3.19}$$

with

$$J(r_0) \equiv \frac{\Omega_0}{r_0^2}. \tag{3.20}$$

The upper sign “$-$” in Eq.(3.18) corresponds to the case $a0$, and the lower sign “$+$” corresponds to the case $a0$.

4 Physical Interpretation of the Surface Energy-Momentum Tensor and the Energy Conditions

In order to have the physical interpretation for each term appearing in Eq.(3.17), we need first to cast the surface EMT in its canonical form, that is, solving the eigenvalue problem [14],

$$\tau^\mu_\nu \xi^\nu = \lambda \xi^\mu. \tag{4.1}$$

Before doing so, we note that when $\sigma = 1/4$, which corresponds to the cases (b) and (c) classified in Eq.(3.9), we have $q = 0$ and the surface EMT of Eq.(3.17) is already in its canonical form (the same is true when $b = 0$).

Then, the three unit vectors t_μ, z_μ and φ_μ are the corresponding eigenvectors of Eq.(4.1). Thus, now ρ can be considered as representing the energy density of the matter shell, and p_z and p_φ the principal pressures along the two spacelike eigen-directions, defined, respectively, by z_μ and φ_μ. It can be also shown that the corresponding EMT satisfies all the three energy conditions [11]. Therefore, it is concluded that *the Lewis vacuum solutions with $\sigma = 1/4$ for both of the two cases $a > 0$ and $a < 0$, can be produced by physically acceptable rotating thin shell.*

Thus, in the following we need only to consider the cases (a) and (d) of Eq.(3.9). In the latter cases, the system of equations (4.1) will possess nontrivial solutions only when the determinant $\det|\tau^\mu_\nu - \lambda\delta^\mu_\nu| = 0$, which can be written as [14]

$$\lambda(p_z - \lambda)\left[\lambda^2 - (\rho - p_\varphi)\lambda + q^2 - \rho p_\varphi\right] = 0. \tag{4.2}$$

Clearly, the above equation has four roots, $\lambda = 0,\ p_z,\ \lambda_\pm$, where

$$\lambda_\pm = \frac{1}{2}\left[(\rho - p_\varphi) \pm D^{1/2}\right], \quad D \equiv (\rho + p_\varphi)^2 - 4q^2. \tag{4.3}$$

It can be shown that the eigenvalue $\lambda = 0$ corresponds to the eigenvector $\xi^\mu_1 = n^\mu$, where n^μ is the normal vector to the hypersurface $r = r_0$. The eigenvalue $\lambda = p_z$ corresponds to the eigenvector $\xi^\mu_2 = z^\mu$, which represents the pressure of the shell in the z-direction. On the other hand, substituting Eq.(4.3) into Eq.(4.1), we find that the corresponding eigenvectors are given, respectively, by

$$\xi^\mu_\pm = (\lambda_\pm + p_\varphi)u^\mu + q\varphi^\mu. \tag{4.4}$$

In the rest of this section we shall only consider the case $a > 0$ in details. For the case $a < 0$ we present only the final results since the analysis is similar to the $a > 0$ case. Thus assuming $a > 0$, we find that

$$\begin{aligned}
\lambda_+ + p_\varphi &= \frac{1}{16\pi R_0}\left[\Sigma - (1-4\sigma)J(r_0) + \sqrt{D}\right],\\
D &= \frac{1}{(8\pi R_0)^2}\left[(1-4\sigma)^2 + \Sigma^2 - 2\Sigma(1-4\sigma)J(r_0)\right].
\end{aligned} \tag{4.5}$$

Following [14], we shall further distinguish the three subcases: (1) $D > 0$; (2) $D = 0$; and (3) $D < 0$.

4.1 $D > 0$

From Eq.(4.5) we can see that the condition $D > 0$ can be written as

$$(1-4\sigma)J(r_0) < \frac{\Sigma}{2} + \frac{(1-4\sigma)^2}{2\Sigma}, \ \ (D > 0). \tag{4.6}$$

As can be seen from Eq.(4.3), now the two roots $\lambda_{\pm}$ and the two eigenvectors $\xi^{\mu}_{\pm}$ are all real and satisfy the relations,

$$\begin{aligned} (\lambda_+ + p_\varphi)(\lambda_- + p_\varphi) &= q^2, \\ \frac{\xi^\mu_\pm \xi^\nu_\pm g_{\mu\nu}}{D^{1/2}(\lambda_\pm + p_\varphi)} &= \pm 1, \\ \xi^\mu_+ \xi^\nu_- g_{\mu\nu} &= 0. \end{aligned} \tag{4.7}$$

From these expressions we can see that when $\lambda_+ + p_\varphi > 0$, the eigenvector ξ^μ_+ is timelike, and ξ^μ_- is spacelike, while when $\lambda_+ + p_\varphi < 0$, the two vectors exchange their roles. Let us first consider the case where $\lambda_+ + p_\varphi > 0$. **Case A.1.1** $\lambda_+ + p_\varphi > 0$: This condition can be written as

$$\Sigma - (1 - 4\sigma)J(r_0) + \sqrt{D} > 0, \ \ (\lambda_+ + p_\varphi > 0). \tag{4.8}$$

Setting

$$\begin{aligned} E^\mu_{(0)} &\equiv \frac{\xi^\mu_+}{\left[D^{1/2}(\lambda_+ + p_\varphi)\right]^{1/2}}, \quad E^\mu_{(1)} \equiv n^\mu, \quad E^\mu_{(2)} \equiv z^\mu, \\ E^\mu_{(3)} &\equiv \frac{\xi^\mu_-}{\left[D^{1/2}(\lambda_- + p_\varphi)\right]^{1/2}}, \ \ (\lambda_+ + p_\varphi > 0), \end{aligned} \tag{4.9}$$

we find that $E^\mu_{(a)}$, $(a = 0, 1, 2, 3)$ form an orthogonal basis, i.e., $E^\lambda_{(a)} E_{(b)\lambda} = \eta_{ab}$, with $\eta_{ab} = \text{diag.}\{1,\ -1,\ -1,\ -1\}$.

Then, in terms of these unit vectors, the surface EMT given by Eq.(3.17) takes the form

$$\tau^{\mu\nu} = \rho_{(0)} E^\mu_{(0)} E^\nu_{(0)} + p_{(2)} E^\mu_{(2)} E^\nu_{(2)} + p_{(3)} E^\mu_{(3)} E^\nu_{(3)}, \tag{4.10}$$

where ¶

$$\begin{aligned} \rho_{(0)} &= \frac{(\lambda_+ + p_\varphi)}{2q^2}\left\{D^{1/2} p_\varphi - \left[p_\varphi(\rho + p_\varphi) - 2q^2\right]\right\} \\ &= \frac{1}{2}\left[D^{1/2} + (\rho - p_\varphi)\right] = \frac{\sqrt{D}}{2} + \frac{\sigma(1-2\sigma)}{8\pi R_0}, \\ p_{(3)} &= \frac{(\lambda_- + p_\varphi)}{2q^2}\left\{D^{1/2} p_\varphi + \left[p_\varphi(\rho + p_\varphi) - 2q^2\right]\right\} \\ &= \frac{1}{2}\left[D^{1/2} + (p_\varphi - \rho)\right] = \frac{\sqrt{D}}{2} - \frac{\sigma(1-2\sigma)}{8\pi R_0}, \\ p_{(2)} &= p_z = \frac{\sigma(1-2\sigma)}{4\pi R_0}, \quad (\lambda_+ + p_\varphi > 0). \end{aligned} \tag{4.11}$$

¶Note the typos in the expressions given by Eq.(39) in [14]. After the corrections, they should be given by the first parts of Eq.(4.11) in each of their corresponding expressions.

Thus, in terms of its tetrad components the surface EMT can be cast in the canonical form,

$$\left[\tau_{(a)(b)}\right] = \begin{bmatrix} \rho_{(0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_{(2)} & 0 \\ 0 & 0 & 0 & p_{(3)} \end{bmatrix}, \tag{4.12}$$

where $\tau_{(a)(b)} \equiv \tau_{\mu\nu} E^{\mu}_{(a)} E^{\nu}_{(b)}$.

This corresponds to the Type I fluid defined in [11]. Hence, in this case the surface EMT can be considered as representing an anisotropic fluid with its surface energy density given by $\rho_{(0)}$, measured by observers whose four-velocity is given by $E^{\mu}_{(0)}$, and the principal pressures in the directions $E^{\mu}_{(2)}$ and $E^{\mu}_{(3)}$, given respectively by $p_{(2)}$ and $p_{(3)}$. Certainly, this interpretation is valid only when the surface EMT satisfies some physical conditions, such as, the weak, dominant, and/or strong energy conditions [11]. In this paper, we shall not consider the exactly physical nature of the matter content of the shell, but impose these three energy conditions. If they are satisfied, then we shall consider the shell as physically acceptable.

It can be shown that in this subcase, all the three energy conditions are satisfied for the range of σ given by

$$0 \leq \sigma < \frac{1}{4}, \tag{4.13}$$

by properly choosing the radius of the rotating thin shell such that the following condition is fulfilled,

$$\sqrt{D} \geq \frac{\sigma(1-2\sigma)}{4\pi R_0}. \tag{4.14}$$

On the other hand, it can be also shown that Eq.(4.8) is automatically satisfied, once Eqs.(4.6) and (4.13) are fulfilled.

Therefore, it is concluded that *all the Lewis vacuum solutions with $a > 0$ and $0 \leq \sigma < 1/4$ can be produced by a physically acceptable rotating shell.* **Case A.1.2** $\lambda_+ + p_\varphi < 0$: This condition can be written as

$$\Sigma - (1-4\sigma)J(r_0) + \sqrt{D} < 0, \quad (\lambda_+ + p_\varphi < 0). \tag{4.15}$$

Since now ξ^{μ}_{-} is time-like, the orthogonal basis can be chosen as

$$E^{\mu}_{(a)} \equiv \left\{ \frac{\xi^{\mu}_{-}}{D^{1/4}\left|\lambda_- + p_\varphi\right|^{1/2}},\ n^{\mu},\ z^{\mu},\ \frac{\xi^{\mu}_{+}}{D^{1/4}\left|\lambda_+ + p_\varphi\right|^{1/2}} \right\}. \tag{4.16}$$

Then, the corresponding EMT also takes the form of Eq.(4.12) but now with [‖]

$$\rho_{(0)} \;\; = \;\; -\frac{(\lambda_- + p_\varphi)}{2q^2}\left\{D^{1/2}p_\varphi + \left[p_\varphi(\rho + p_\varphi) - 2q^2\right]\right\}$$

[‖]Note the typos in Eq.(42) of [14]. After the corrections, they should be given by the first parts of Eq.(4.17) in each of their corresponding expressions.

$$
\begin{aligned}
&= \frac{1}{2}\left[(\rho - p_\varphi) - D^{1/2}\right] = \frac{\sigma(1-2\sigma)}{8\pi R_0} - \frac{\sqrt{D}}{2}, \\
p_{(3)} &= -\frac{(\lambda_+ + p_\varphi)}{2q^2}\left\{D^{1/2}p_\varphi - \left[p_\varphi(\rho + p_\varphi) - 2q^2\right]\right\} \\
&= \frac{1}{2}\left[(p_\varphi - \rho) - D^{1/2}\right] = -\frac{\sigma(1-2\sigma)}{8\pi R_0} - \frac{\sqrt{D}}{2}, \\
p_{(2)} &= p_z = \frac{\sigma(1-2\sigma)}{4\pi R_0}, \quad (\lambda_+ + p_\varphi < 0).
\end{aligned} \tag{4.17}
$$

It is not difficult to show that in this case none of the three energy conditions is satisfied. Thus, it is concluded that in the present case there does not exist physically acceptable rotating thin shell such that the conditions (4.6) and (4.15) are satisfied.

4.2 $D = 0$

In this case from Eq.(4.5) we find that

$$
J(r_0) = \frac{1}{2}\left(\frac{\Sigma}{1-4\sigma} + \frac{1-4\sigma}{\Sigma}\right). \tag{4.18}
$$

Substituting the above expression into Eq.(3.18), we have

$$
\begin{aligned}
\rho &= \frac{\sigma}{8\pi\Sigma R_0}\left[2 - 5\sigma + 4\sigma^2(1-\sigma)\right], \\
p_z &= \frac{\sigma(1-2\sigma)}{4\pi R_0}, \quad p_\varphi = \frac{3\sigma^2(1-2\sigma)^2}{8\pi\Sigma R_0}, \\
q &= -\frac{|\sigma(1-\sigma)(1-4\sigma^2)|}{8\pi\Sigma R_0},
\end{aligned} \tag{4.19}
$$

from which we can see that when $\sigma = 0$ the shell disappears, and when $\sigma = -1/2$ we have $q = 0$. In the latter case, we obtain

$$
\rho = -p_\varphi = \frac{1}{2}p_z = -\frac{1}{8\pi R_0}, \quad q = 0, \quad \left(\sigma = -\frac{1}{2}\right). \tag{4.20}
$$

Clearly, in this case none of the three energy conditions is satisfied. On the other hand, impossing (3.9) we find that σ must satisfy the condition $\sigma < 1/4$, since now we have $a > 0$. Therefore, in this subsection we need to consider only the subcases, $\sigma < -1/2$, $-1/2 < \sigma < 0$ and $0 < \sigma < 1/4$. From Eq.(3.17) we can see that the case $q < 0$ can be obtained from the case $q > 0$ by replacing φ_μ by $-\varphi_\mu$. This is physically equivalent to a counter-rotation. Obviously, the energy conditions should not depend on such rotations. Thus, without loss of generality, in the following we shall drop the negative sign of q given in Eq.(4.19). After this replacement is done, we find that

$$
q = \begin{cases} \frac{1}{2}(\rho + p_\varphi) > 0, & 0 < \sigma < \frac{1}{4}, \\ -\frac{1}{2}(\rho + p_\varphi) > 0, & -\frac{1}{2} < \sigma < 0, \\ \frac{1}{2}(\rho + p_\varphi) > 0, & \sigma < -\frac{1}{2}. \end{cases} \tag{4.21}
$$

In each of the above subcases, it can be shown that the two roots $\lambda_\pm$ given by Eq.(4.3) degenerate into one. As shown in [14], this multiple root corresponds to two null independent eigenvectors,

$$\xi^\mu_\pm = \frac{u^\mu \pm \varphi^\mu}{\sqrt{2}}. \tag{4.22}$$

From these two null vectors we can construct two unit vectors, one is timelike and the other is spacelike, but these are exactly u^μ and φ^μ. Thus, in the basis

$$E^\mu_{(a)} = \{u^\mu,\ n^\mu,\ z^\mu,\ \varphi^\mu\}, \tag{4.23}$$

the surface EMT takes the form

$$\left[\tau_{(a)(b)}\right] = \begin{bmatrix} \rho & 0 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_z & 0 \\ q & 0 & 0 & p_\varphi \end{bmatrix}. \tag{4.24}$$

To further study the problem, let us consider the cases (1) $0 < \sigma < 1/4$ or $\sigma < -1/2$, and (2) $-1/2 < \sigma < 0$, separately. **Case A.2.1** $0 < \sigma < 1/4$ or $\sigma < -1/2$: In this case from Eq.(4.21) we can see that

$$q = \frac{1}{2}(\rho + p_\varphi) > 0, \tag{4.25}$$

and the corresponding surface EMT (4.24) can be written in the form

$$\left[\tau_{(a)(b)}\right] = q \begin{bmatrix} 1+\kappa & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_{(2)} & 0 \\ 1 & 0 & 0 & 1-\kappa \end{bmatrix}, \quad (\rho + p_\varphi > 0), \tag{4.26}$$

where

$$\kappa \equiv \frac{\rho - p_\varphi}{\rho + p_\varphi}, \quad p_{(2)} \equiv \frac{2p_z}{\rho + p_\varphi}. \tag{4.27}$$

Eq.(4.26) is exactly in the form of the type II fluid classified in [11]. Then, it can be shown that *the weak and strong energy conditions are satisfied for* $\sigma \in (0, 1/4)$, *while the dominant energy condition is violated for any value of* σ *within the above given range.* **Case A.2.2** $-1/2 < \sigma < 0$: In this case from Eq.(4.21) we find that

$$q = -\frac{1}{2}(\rho + p_\varphi) > 0, \tag{4.28}$$

and the corresponding surface EMT cannot be written in the form of Eq.(4.26). In order to study the energy conditions, let us consider an observer with its four-velocity given by

$$w^\mu = \alpha t^\mu + \beta n^\mu + \gamma z^\mu + \delta \varphi^\mu, \tag{4.29}$$

where α, β, γ and δ are arbitrary constants, subject to the condition,

$$w^{\mu}w_{\mu} = \alpha^2 - \beta^2 - \gamma^2 - \delta^2 \geq 0. \tag{4.30}$$

The weak energy condition requires that [11]

$$\tau_{\mu\nu}w^{\mu}w^{\nu} = \alpha^2\rho + \gamma^2 p_z + \delta^2 p_{\varphi} - 2\alpha\delta q \geq 0. \tag{4.31}$$

It can be shown that Eq.(4.31) is satisfied for any observer given by Eqs.(4.29) and (4.30) only when the conditions $\rho \geq 0$, $\rho + p_z \geq 0$, $\rho + p_{\varphi} - 2q \geq 0$ and $\rho + p_{\varphi} + 2q \geq 0$ are true. On the other hand, the strong energy condition holds when [11]

$$\begin{aligned} \left(\tau_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tau\right)w^{\mu}w^{\nu} &= \frac{1}{2}\Big[(\alpha^2+\beta^2+\gamma^2+\delta^2)\rho + (\alpha^2+\gamma^2-\beta^2-\delta^2)p_z \\ &\quad + (\alpha^2+\delta^2-\beta^2-\gamma^2)p_{\varphi} - 4\alpha\delta q\Big] \geq 0, \end{aligned} \tag{4.32}$$

which is equivalent to $\rho + p_z \geq 0$, $\rho + p_{\varphi} - 2q \geq 0$, $\rho + p_{\varphi} + 2q \geq 0$ and $\rho + p_z + p_{\varphi} \geq 0$, while the dominant energy condition requires that [11] $\rho \geq |p_z|$, $\rho \geq |p_{\varphi}|$, $\rho \geq |q|$. To summarize, for any given $\tau_{\mu\nu}$ of the form (4.24), the energy conditions are the following:

(a) *The Weak Energy Condition*:

$$i)\ \rho \geq 0, \quad ii)\ \rho + p_z \geq 0, \quad iii)\ \rho + p_{\varphi} + 2q \geq 0, \quad iv)\ \rho + p_{\varphi} - 2q \geq 0. \tag{4.33}$$

(b) *The Dominant Energy Condition*:

$$i)\ \ \rho \geq |p_z|, \quad ii)\ \ \rho \geq |p_{\varphi}|, \quad iii)\ \ \rho \geq |q|. \tag{4.34}$$

(c) *The Strong Energy Condition*:

$$i)\ \ \rho \geq 0, \quad ii)\ \ \rho+p_z \geq 0, \quad iii)\ \ \rho+p_{\varphi}-2q \geq 0, \quad iv)\ \ \rho+p_{\varphi}+2q \geq 0, \quad v)\ \ \rho+p_z+p_{\varphi} \geq 0. \tag{4.35}$$

Applying the above energy conditions to the surface EMT given by Eq.(4.21), we find that in the present case, $-1/2 < \sigma < 0$, none of the three energy conditions is satisfied.

Therefore, *when $a > 0$ and $D = 0$ the weak and strong energy conditions are satisfied when $\sigma \in (0, 1/4)$, and there does not exist any value of σ in the range $\sigma \in (-\infty, 1/4)$, for which the dominant energy condition is satisfied.*

4.3 $D < 0$

The condition $D < 0$ is equivalent to

$$J(r_0) > \frac{\Sigma^2 + (1-4\sigma)^2}{2(1-4\sigma)\Sigma}, \quad (D < 0). \tag{4.36}$$

From Eqs.(4.3) and (4.4), on the other hand, we can see that now the eigenvalues $\lambda_{\pm}$ are complex, and so do the two eigenvectors $\xi^{\mu}_{\pm}$. This means that in the present case the surface EMT cannot be diagonalized (by real similarity transformations), and Eq.(4.24) is already in its canonical form. As shown in the last subsection, for this form of EMT, the three energy conditions are those given, respectively, by Eqs.(4.33)-(4.35). In order to analize these conditions, let us first note that when $\sigma < 1/4$ the condition $\rho \geq 0$ is equivalent to

$$J(r_0) \leq \frac{1}{1-4\sigma}, \quad (\rho \geq 0). \tag{4.37}$$

Eqs.(4.36) and (4.37) have solution when $0 < \sigma < 1/4$. However, for this range of σ it can be shown that the condition $\rho + p_{\varphi} - 2q \geq 0$ requires

$$J(r_0) \leq \frac{\Sigma^2 + (1-4\sigma)^2}{2(1-4\sigma)\Sigma}, \quad (\rho + p_{\varphi} + q \geq 0), \tag{4.38}$$

which is inconsistent with Eq.(4.36). Combining the above analysis with Eqs.(4.33) and (4.35), we find that *in the present case the weak and strong energy conditions are violated for values of* σ *within the range* $0 < \sigma < 1/4$. On the other hand, it can be shown that now the dominant energy condition requires,

$$0 \leq \sigma < \frac{1}{4}, \qquad \frac{\Sigma^2 + (1-4\sigma)^2}{2(1-4\sigma)\Sigma} < J(r_0) \leq \frac{8\sigma^2 - 4\sigma + 1}{1-4\sigma}. \tag{4.39}$$

In review of all the above, it is concluded that *for the case* $a > 0$ *the Lewis vacuum solutions can be produced by physically acceptable rotating cylindrical thin shells for* $0 \leq \sigma < 1/4$. *Moreover, to this range of* σ *the radius of the shell has to be chosen such that the condition* $D > 0$ *is satisfied, in which the surface EMT can be diagonalized and given by Eq.(4.12).* A similar analysis shows that *for the case* $a < 0$ *the Lewis vacuum solutions can be produced by physically acceptable rotating cylindrical thin shells for* $1/4 < \sigma \leq 1/2$, *by properly choosing the radius of the shell so that the condition* $D > 0$ *is satisfied, for which the surface EMT can be diagonalized and given by Eq.(4.12).*

5 The vorticity of the shell and its energy per unit length

The four velocity of a comoving observer in the system of the chosen coordinates is given by

$$u^{\mu} = \frac{1}{\sqrt{g_{tt}}}\delta^{\mu}_{t}, \tag{5.1}$$

for which it can be shown that the vorticity tensor $\omega_{\alpha\beta}$ has only two non-vanishing components, given by

$$\begin{aligned} \omega_{\alpha\beta} &= u_{[\alpha;\beta]} + u_{[\alpha;\mu}u^{\mu}u_{\beta]} \\ &= \frac{g'_{tt}}{2\sqrt{g_{tt}}}\left(\delta^{t}_{\mu}\delta^{r}_{\nu} - \delta^{r}_{\mu}\delta^{t}_{\nu}\right) + \frac{g'_{t\varphi}}{2\sqrt{g_{tt}}}\left(\delta^{r}_{\mu}\delta^{\varphi}_{\nu} - \delta^{\varphi}_{\mu}\delta^{r}_{\nu}\right). \end{aligned} \tag{5.2}$$

Then, the vorticity vector ω_α takes the form

$$\omega^\alpha = \frac{\epsilon^{\alpha\beta\gamma\delta}}{2\sqrt{-g}} u_\beta \omega_{\gamma\delta} = \frac{1}{2\sqrt{-g}} \left({g'}_{t\varphi} - \frac{g_{t\varphi} {g'}_{tt}}{g_{tt}} \right) \delta^\alpha_z. \tag{5.3}$$

Calculating the above quantity at $r = r_0$ using the external metric (3.4), we find that

$$\omega^\alpha_+(r_0^+) = \frac{\gamma(4\sigma - 1) r_0^2}{(1 - \omega^2 r_0^2) R^{1+1/\Sigma}} \delta^\alpha_z, \tag{5.4}$$

while for the interior metric (3.1), we have

$$\omega^\alpha_-(r_0^-) = \frac{\omega}{1 - \omega^2 r_0^2} \delta^\alpha_z. \tag{5.5}$$

Clearly, now we have $\omega^\alpha_- \neq \omega^\alpha_+$. The reason is that the derivatives of the metric are discontinuous across the shell $r = r_0$.

On the other hand, considering Israel's definition [19] of energy density per unit length μ, from Eqs.(3.17) and (3.18) we find that,

$$\begin{aligned} \mu &= \int_0^\infty \int_0^{2\pi} (T^t_t - T^r_r - T^z_z - T^\phi_\phi) \sqrt{-g}\, dr d\phi \\ &= \frac{\sigma}{\Sigma} + (1 - 4\sigma) \frac{b\gamma}{2\Sigma}. \end{aligned} \tag{5.6}$$

The tangential velocity ωr_0 given by (3.11) for the case $a > 0$, up to first order, $O(b)$ and $O(\gamma)$, becomes

$$\omega r_0 \approx -\frac{b R_0^{1/\Sigma}}{r_0^2} - \frac{\gamma r_0^2}{R_0^{1/\Sigma}}. \tag{5.7}$$

The first and second terms in the right hand side of (5.7) correspond to the tangential velocity of the shell due to b, and γ, respectively. Then we can see that $b\gamma \sim$(tangential velocity of the shell)^2.

Hence, the second term in the right hand side of Eq.(5.6), due to rotation, can be associated to the kinetic energy of the shell.

6 Discussions and Conclusions

In this paper, we first studied the local and global properties of the stationary cylindrically symmetric general vacuum solutions (Lewis), and found that the condition for the non-existence of closed time-like curves outside the shell can be satisfied if $a > 0$, $\sigma \leq 1/4$ or $a < 0$, $\sigma \geq 1/4$. To further study the solutions, we also constructed rotating thin-shell-like sources, by assuming that the spacetime inside the shell is flat. It was shown that such constructed cylindrical shells can satisfy the three energy conditions, weak, dominant and strong, when $a > 0$, $0 \leq \sigma \leq 1/4$ or $a < 0$, $1/4 \leq \sigma \leq 1/2$. It was also found that in the latter cases the corresponding surface EMT can be diagonalized and takes the form of Eq.(4.12). Moreover, in the cases $a < 0$ the first junction condition does not allow the static limit $b = 0 = \gamma$ [cf. Eq.(3.9)]. The vorticity of the rotating shell and its mass per unit length were also calculated. When $\gamma = 0$, the vorticity of the shell calculated from outside, vanishes as can be seen from Eq.(5.4), while the energy per unit length as given by (5.6) is the same as that in the corresponding static case [20]. However the stationarity of the spacetime manifests itself through the dragging of a gyroscope at rest in the frame where (2.1) takes a diagonal form [21]. This situation is reminiscent of the behaviour of a gyroscope in the field of a charged magnetic dipole. In this latter case, even though the metric is static, dragging of inertial frames appears and is explained as due to the presence of a flow of electromagnetic energy in the angular direction [22]. In our case also, even if the vorticity of the shell vanishes, there is still a flow of energy (if $b \neq 0$) along ϕ^{μ}, which might be interpreted as the "source" of the dragging. Finally it is worth mentioning that we have ensured no CTC's and energy conditions being satisfied simultaneously in our models by restraining the range of σ to $0 \leq \sigma \leq 1/4$, (if $a > 0$). It remains to be proved if, and to what extent, this range can be safely extended. Since physically reasonable sources for the Levi-Civita spacetime have been found for $\sigma > 1/4$ (see [8], [9] and references therein), it could be conjectured that for sufficiently small values of b and γ, this is also possible for Lewis. However a bifurcation might be present , but we conclude without giving a definite answer to that question.

Acknowledgements

The financial assistance from UERJ (MFAdaS, AZW), CNPq (NOS, AZW) and M.C.T.,Spain, BFM2000-1322 (LH) is gratefully acknowledged.

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