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Some studies on functional integrals representations for fluid motion with random conditions

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Abstract

We write functional integral representations for the probability distribution over the statistical ensemble of Newtonian fluid velocity fields satisfying: a) A white noise initial condition; b) Beltrami fluxes with appropriate random stirrings; c) The one-dimensional Burger-Beltrami equation with general gaussian stirring.

1 Introduction

The main task in the statistical approach to random fluid dynamics ([1]) is solve the set of infinite hierarchy equations for the random fluid velocity correlation functions. One important scheme to solve these equations consists in considering directly for the appropriate fluxe equation the random conditions generating the fluxe stochasticity in the hope that the fluid turbulence is appropriated described in this statistical approach at least as an effective analytical theory.

Our aim in this paper is to present (formal) functional integral representations for the Navier-Stokes equation in the following random conditions:

1) A pure white-noise initial fluid velocity condition

2) A Soluble Beltrami fluxe with appropriate gaussian random stirrings

3) The Burger-Beltrami one-dimensional equation with a general gaussian random stirring.

Finally in a somewhat long appendix we show via path-integral techniques the appearance of vortex phase factors as important object in the advection physics of scalars on fluid fluxes.

2 The Functional Integral

Let us start this section by writing the Navier-Stokes equation for the velocity field of an incompressible fluid in the presence of a non-random external force $F_i(x,\tau)$ with a Gaussian (ultra-local) Random initial condition

$$\frac{\partial}{\partial\tau}\, v_i - \nu\Delta_x v_i + \left(v_k \frac{\partial}{\partial x_k} v^i\right)^{Tr} = F_i \tag{1.a}$$

$$v_i(x,0) = \varphi_i(x) \tag{1.b}$$

$$\langle \varphi_{i1}(x_1)\varphi_{i2}(x_2)\rangle = \lambda\delta^{(3)}(x_1 - x_2) \tag{1.c}$$

Let us remark that we have eliminated the pressure term $-\frac{1}{\rho}\vec{\Delta}\cdot\vec{p}$ by using the incompressibility condition $(\vec{\Delta v}) \equiv 0$ which, in turn, lead us to consider only the transverse part of the force and non-inertial field terms in Navier-Stokes equation. The transverse part of a generical vector field $\vec{W}(x,\tau)$ is defined by the relationship

$$\vec{W}^{Tr}(x,\xi) = \vec{W}(x,\xi) + \frac{1}{4\pi}\vec{\Delta}_x^{-1}\left(\int_{R^3} dy \frac{(\vec{\Delta}_y\vec{W})(y,\xi)}{|x-y|}\right)$$

$$\vec{W}(x,\xi) = \{\vec{W}_i(x,\xi)\ ;\ i=1,2,3\} \tag{2}$$

Our task, now, is to compute the φ-averge of the N-point fluide velocity fields eq. (1.a), for arbitrary space-time points, by means of a functional integral representation for the characteristic functional of the random fluid velocity fields $Z[J_i(x,\xi)]$; namely [3]

$$\langle V_{i1}(x_1,\xi_1),[\varphi])\cdots V_{iN}(x_N,\xi_N,[\varphi])\rangle_\varphi$$

$$= (-1)^N \frac{\delta^{(N)}}{\delta J_{i1}(x_1,\xi_1)\cdots\delta J_{iN}(x_N,\xi_N)}\ Z|J_i(x,\xi)|\Big|_{J_i(x,\xi)=0} \tag{3.a}$$

where

$$Z[J_i(x,\xi)] = \int_M d\mu[V_i]\times exp\left(-\int_{R^3} dx\int_0^\infty d\xi (V_i\cdot J_i)(x,\xi)\right) \tag{3.b}$$

The functional measure $d\mu[V_i]$ in eq. (3.b) is defined over the functional space M of all possible realizations of the random fluid motion defined by eq. (1). An explicit (formal) expression for the above functional measure should be given by the product of the usual Feynman measure weighted by a certain functional $S[V_i]$ to be determined,

$$d\mu[V_i] = D^F[V^i]exp(S[V^i]) \tag{4.a}$$

$$D^F[V^i] = \prod_{\substack{x\in R^3\\ 0<\xi<\infty\\ i=1,2,3}} (dV_i(x,\xi)) \tag{4.b}$$

In order to determine the Weight Functional $S[V_i]$ we first rewrite the Navier-Stokes Equation as a pure integral equation which has an explicit term taking into account the initial condition [4]

$$A_i[\tilde{v}] = B_i[\varphi] \tag{5}$$

with

$$A_i[v] = V_i(x,\xi) - \int_0^\infty ds \int_{R^3} dy \mathcal{O}_{ijk}(x-y,\xi-s) \times (V_j V_k)(y,s)$$
$$- \int_0^\infty ds \int_{R^3} dy H_{(1)}(x-y),\xi-s)F_i(y,s) \qquad (6.a)$$
$$B_i[\varphi] = \int_{R^3} dy H_{(0)}(x-y,\xi)\varphi_i(y) \qquad (6.b)$$

Here, the Kernels $\mathcal{O}_{ijk}, H_{(1)}, H_{(0)}$ are given respectively by

$$\mathcal{O}_{ik\ell}(z,\xi) = -\frac{1}{2}\left(\frac{\partial}{\partial Z_t}\overline{\mathcal{O}}_{ik} + \frac{\partial \overline{\mathcal{O}}_{i\ell}}{\partial Z_k}\right)(z,\xi) \qquad (7.a)$$
$$\overline{\mathcal{O}}_{pq}(z,\xi) = \delta_{pq}\theta(\xi)H_{(0)}(|z|,\xi) + \frac{\partial^2}{\partial z_p \partial z_q}\left(\frac{2\nu\xi}{|z|}\int_0^{|z|} H_0(|z'|,\xi)dz'\right) \qquad (7.b)$$
$$H_{(0)}(|z|,\xi) = \frac{1}{(4\pi\nu\xi)^{3/2}}\ exp\left(-\frac{|z|^2}{4\pi\nu\xi}\right) \qquad (7.c)$$
$$H_{(1)}(|z|,\xi) = \theta(\xi)H_{(0)}(|z|,\xi) \qquad (7.d)$$

Let us now introduce the following functional representation for the generating functional $Z[J_i(x,\xi)]$ [4]

$$Z[J_i(x,\xi)] = \int D^F[V^i]\langle\delta^{(F)}(V^i - \tilde{V}^i[\varphi])\rangle_\varphi \times exp\left(-\int_{R^3} dx \int_0^\infty \ d\xi (V^i \cdot J^i)(x,\xi)\right) \quad (8)$$

where $\delta^{(F)}(\cdot)$ denotes the delta – functional integral representation defined by the rule

$$\int_M D^F[V_i]\delta^{(F)}(V_i - A_i)\Sigma(V_i) = \Sigma(A_i) \qquad (9)$$

with $\Sigma(V_i)$ being an arbitrary functional defined on M.

By writing the φ-average in eq. (9) by means of a Gaussian functional integral in $\varphi(x,\xi)$, we obtain the following functional integral representation for the weight $S[V^i]$

$$exp(-S[V^i]) = \int D^F[\varphi^i] exp\left[-\frac{1}{2\lambda}\int_{R^3} dx(\varphi^i \cdot \varphi^i)(x)\right]$$
$$\times \left\{\int D^F[K^i] exp\ i \int_{R^3} dx \int_0^\infty d\xi K_i \cdot (A^i[v] - B^i|\varphi|)\right\} \qquad (10)$$

where we have used the Fourier Functional Integral representation for the Delta-functional in eq. (9)

$$\delta^{(F)}(V^i - V^i[\varphi]) = \delta^{(F)}(A_i[v] - B_i[\varphi]) = DET_F\left(\frac{\delta}{\delta V_i}A_k[v]\right) \times$$
$$\int D^F[K^i] exp\left(i\int_{R^3} dx \int_0^\infty d\xi K_i(A^i[v] - B^i[\varphi])(x,\xi)\right) \qquad (11)$$

It is worth remark that the functional determinant in eq. (11) is unity as a straightforward consequence of the fact that the Green function of the operator $\partial/\partial\xi$ is the step function.

We, then, face the problem of evaluating the φ and K functional integrals in eq. (11).

The φ-functional integral is of Gaussian type and easily evaluated

$$\begin{aligned}
&\int D^F[\varphi^i] exp\left(-\frac{1}{2\lambda}\int_{R^3} dx(\varphi_i\cdot\varphi_i)(x)\right)\times \\
&exp(i\int_{R^3} dx\int_0^\infty d\varepsilon(K_i\cdot B^{i,*}[\varphi])(x,\varepsilon)) \\
&=\exp\left\{-\frac{\lambda}{2}\int_{R^3}dx_1\int_{R^3}dx_2\int_0^\infty d\xi_1\int_0^\infty d\xi_2 K^i(x_1,\xi_1)\times\delta^{(3)}(x_1-x_2)\right. \\
&\left. C(x_1,\xi_1;x_2,\xi_2)K^i(x_2,\xi_2)\right\} \qquad (12)
\end{aligned}$$

where the Kernel $C(x_1,\xi_1;x_2,\xi_2)$ is given by

$$C(x_1,\xi_1;x_2,\xi_2)=\int_{R^3} dz H_{(0)}(x_1-z,\xi_1)H_{(0)}(z-x_2,-\xi_2) \qquad (13)$$

and is the (formal) Green function of the self-adjoint extension of the square $B^{i*}B_i$ diffusion operator

$$(B^i)^*B_i=\left(-\frac{\partial}{\partial\xi_1}-\nu\Delta_{x_1}\right)\left(\frac{\partial}{\partial\xi_1}-\nu\Delta_{x_1}\right)C(x_1,\xi_1;x_2,\xi_2) \qquad (14)$$

with the (well-posed) initial and boundary conditions

$$\lim_{\xi_1\to 0^+} C(x_1,\xi_1;x_2,\xi_2)=\delta^{(3)}(x_1-x_2) \qquad (15)$$

Its explicit expression in K-momentum space is given by (see ref. 6)

$$\tilde{C}(k;\xi_1,\xi_2)=-\frac{1}{\nu k^2}[e^{-\nu k^2|\xi_1-\xi_2|}-e^{-\nu k^2(\xi_1+\xi_2)}] \qquad (16)$$

As a consequence of eq. (12) we have represented the weight $S[v^i]$ by a Gaussian functional integral in the $K_i(x,\xi)$ field

$$\begin{aligned}
&exp(-S[v^i])=\int D^F[K^i] exp\left[-\frac{\lambda}{2}\int_{R^3}dx_1dx_2\right. \\
&\left.\int_0^\infty d\xi_1 d\xi_2\left(K^i(x_1,\xi_1)C(x_1,\xi_1;x_2,\xi_2)K_i(x_2,\xi_2)\right)\right] \\
&exp\left(i\int_{R^3}dx\int_{R^3}dx\int_0^\infty d\xi(K^iA_i[v])(x,\xi)\right) \qquad (17)
\end{aligned}$$

By evaluating eq. (17) we, thus, obtain the result

$$exp(-S[v^i]) = exp\left(-\frac{1}{2\lambda}\int_{R^3} dx_1 dx_2 \int_0^\infty d\xi_1 d\xi_2 \right.$$
$$\left. A^i[v](x_1,\xi_1)C^{-1}(x_1,\xi_1;x_2,\xi_2)A_i[v](x_2,\xi_2)\right) \tag{18}$$

By noting that (see ref. 4)

$$B_i^{-1}[A[v]] = \left(\frac{\partial}{\partial\xi} - \nu\Delta_k\right) A^i[v] = \left(\frac{\partial}{\partial\xi} - \nu\Delta_x\right) v^i + \left(v_k \frac{\partial}{\partial x_k} v^i\right)^{Tr} - F_i^{Tr} \tag{19}$$

we finally obtain the expression for the weight $S[v^i]$

$$S[v^i] = \frac{1}{2\lambda}\int_{R^3} dx \int_0^\infty d\xi d\xi' \left[\left(\frac{\partial}{\partial\xi} - \nu\Delta_x\right) A^i[v]\right]^* (x,\xi) \times$$
$$\left[\left(\frac{\partial}{\partial\xi'} - \nu\Delta_x\right) A_i[v]\right](x,\xi')$$
$$= \frac{1}{2\lambda}\int_{R^3} dx \int_0^\infty d\xi \int_0^\infty d\xi' \left[\left(\frac{\partial}{\partial\xi} - \nu\Delta_x\right) v^i + \right.$$
$$\left.\left(v_k \frac{\partial}{\partial x_k} v^i\right)^{Tr} + F_i^{Tr}\right](x,\xi) \times \left[\left(\frac{\partial}{\partial\xi'} - \nu\Delta_x\right) v^i + (v_k \frac{\partial}{\partial x_k} v^i)^{Tr} + F_i^{Tr}(x,\xi')\right] \tag{20}$$

we obtain, thus, our proposed functional integral representation for eq. (1)

$$Z[J_i(x,\xi)] = \int D^F[v^i] exp(-S[v^i] \; exp\left(-\int_{R^3} dx \int_0^\infty d\xi (J_i \cdot V_i)(x,\xi)\right) \tag{21}$$

The above written functional integral is the main result of this section.

A perturbative analysis for eq. (22) may be implemented by using the free propagator eq. (17) in the context of a background field decomposition $V_i = \bar{V}_i + \beta V_i^q$ where $\bar{V}_i$ satisfies the non-Random Navier Stokes equation

$$\frac{\partial}{\partial\xi}\bar{V}_i = \nu\Delta_x \bar{V}^i - \left(\bar{V}_k \frac{\partial}{\partial x_k}\bar{V}^i\right)^{Tr} + F^{Tr} \tag{22.a}$$

with

$$\bar{V}^i(x,0) \equiv 0 \tag{22.b}$$

with β being a coupling constant ($\beta << 1$). It is worth remaking that the cross term

$$\int_{R^3} dx \int_0^\infty d\xi (\partial_i v^i \Delta_x v_i)(x,\xi) \tag{23.a}$$

vanishes in $S[v^i]$ as a result of the boundary condition

$$v_i(x,0) = v_i(x,\infty) = 0 \tag{23.b}$$

for the pure random diffusion free propagator (eq. (17)).

Finally, we point out that our proposed functional integral eq. (21) differs from that proposed in ref. 7.

3 An Exact Soluble Path Integral Model for Stochastic Beltrami Fluxes and its String Properties

Our aim in this section is to present in our framework an exactly soluble path integral model for stochastic hydrodynamic motions defined here to be random regime of the *physical Navier-Stokes* eq. (1.a) equation in the incompressible case dominated by generalized Beltrami fluxes defined by the condition rot $\boldsymbol{v} = \lambda \boldsymbol{v}$ with λ a positive parameter.

Let us, thus, start with the usual Navier-Stokes equation, eq. (1a)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \left(\frac{1}{2}\text{grad}(\boldsymbol{v}^2) - (\boldsymbol{v} \times \text{rot } \boldsymbol{v})\right) = -\frac{\text{grad}P}{\rho} + \nu\Delta\boldsymbol{v} + \boldsymbol{F}^{ext} \tag{24}$$

where, the random stirring force is such that its satisfies the following spatially nonlocal Gaussian statistics in our reduced model for turbulence, i.e.,

$$\langle (F^{ext})_i(\boldsymbol{r},t)(F^{ext})_j(\boldsymbol{r}',t')\rangle = \lambda^2\delta_{ij}((\Delta^{-1})\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') \tag{25}$$

where Δ_r^{-1} denotes the Laplacean Green function.

At this point we take curl of Eq. (24) and consider the already mentioned Beltrami flux condition and its direct consequence, namely:

$$\lambda^2\boldsymbol{v} = \text{rot}(\text{rot } v) = \text{grad}(div\ \boldsymbol{v}) - \Delta\boldsymbol{v} = -\Delta\boldsymbol{v}\ , \tag{26.a}$$

$$\boldsymbol{v} \times \text{rot } \boldsymbol{v} = \boldsymbol{v} \times (\lambda\boldsymbol{v}) = 0\ , \tag{26.b}$$

in order to replace the Navier-Stokes equation, Eq. (24) by the exactly soluble Langevin equation for the fluid flux stirred by the external force $\boldsymbol{\Omega}^{ext} = \text{rot}(\boldsymbol{F}^{ext})$ in our proposed model of Navier-Stokes turbulence dominated by generalized Beltrami fluxes

$$\frac{\partial\boldsymbol{v}(\boldsymbol{r},t)}{\partial t} = -\nu\lambda^2\boldsymbol{v}(\boldsymbol{r},t) + \frac{1}{\lambda}\boldsymbol{\Omega}^{ext}(\boldsymbol{r},t)\ . \tag{27}$$

The new external stirring $\boldsymbol{\Omega}^{ext} = \mathrm{rot}(\boldsymbol{F}^{ext})(\boldsymbol{r},t)$ satisfies a Gaussian process with the following two-point correlation function

$$
\begin{aligned}
&\langle \Omega_\ell^{ext}(\boldsymbol{r},t)\Omega_{\ell'}^{ext}(\boldsymbol{r}',t')\rangle = (\varepsilon^{\ell jk}\partial_j^{(r)})(\varepsilon^{\ell' j' k'}\partial_{j'}^{(r')}\langle F_k^{ext}(\boldsymbol{r},t)F_{k'}^{ext}(\boldsymbol{r}',t')\rangle \\
&= \lambda^2(\delta^{\ell\ell'}\delta^{\ell j j'} - \delta^{\ell j'}\delta^{\ell' j})\partial_j^{(r)}\partial_{j'}(\Delta_r^{-1}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}') \times \delta(t-t') \\
&= \lambda^2\delta^{\ell\ell'}\delta^{(3)}(r-r')\delta(t-t') - \lambda^2\partial_\ell^{(r)}\partial_{\ell'}^{(r')}(\Delta_r^{-1}\delta(\boldsymbol{r}-\boldsymbol{r}'))\delta(t-t') \ .
\end{aligned}
\tag{28}
$$

It is obvious that Eq. (28) satisfies the incompressibility condition necessary for the incompressibility consistency of our Brownian-Langevin fluid Eq. (28) and its stochastic version below.

It is important to remark that the formal wave vectors of the Beltrami hydrodynamical motions have eddies of a fixed scale $|\boldsymbol{k}| = \gamma$ in our reduced model. As a consequence of this fact, we assume implicitly the same wave vector constraint in our random strings Eqs. (25) and (28).

Proceeding as in section 2 it leads to the exactly generating path integral for our Brownian reduced model, where we have used the incompressibility constraint $\partial_i^{(r)}v^i(\boldsymbol{r},t) = 0$ to see that the spatially nonlocal piece of Eq. (28) does not contributes to the final path integral weight Eq. (29)

$$
\begin{aligned}
&Z[\boldsymbol{j}(\boldsymbol{r},t)] = \int D[\boldsymbol{v}(\boldsymbol{r},t)]exp\left(i\int_{-\infty}^{+\infty} d^3\boldsymbol{r}\int_0^\infty dt(\boldsymbol{j}\cdot\boldsymbol{v})(\boldsymbol{r},t)\right) \\
&\times det\left[\frac{\partial}{\partial t} - \nu\lambda^2\right]\delta^{(F)}(div\ \boldsymbol{v}\ exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^3\boldsymbol{r}d^3\boldsymbol{r}'\int_0^{+\infty} dtdt'\right. \\
&\times\left(\frac{\partial v_i}{\partial t} + \nu\lambda^2 v_i\right)(\boldsymbol{r}-\boldsymbol{r}')\delta^{ii'}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') \\
&\left.-\partial_i^{(r)}\partial_{i'}^{(r)}(\Delta_r^{-1}\delta(\boldsymbol{r}-\boldsymbol{r}'))\delta(t-t')]\left(\frac{\partial v_{i'}}{\partial t} + \nu\lambda^2 v_{i'}\right)(\boldsymbol{r}-t')\right\} \\
&= \int D[(\boldsymbol{v}(\boldsymbol{r},t)]exp\left(i\int_{-\infty}^{+\infty} d^3\boldsymbol{r}\int_0^\infty dt(\boldsymbol{j}\cdot\boldsymbol{v})(\boldsymbol{r},t)\right) \\
&\times exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^3\boldsymbol{r}\int_0^{+\infty} dt\left(\frac{\partial\boldsymbol{v}}{\partial t} + \nu\lambda^2\boldsymbol{v}\right)^2(\boldsymbol{r},t)\right\} \ .
\end{aligned}
\tag{29}
$$

At this point it is worth to compare the exactly soluble path integral above written (note the fixed wave vector $|\boldsymbol{k}| = \gamma$ imposed implicitly on Eq. (29)) with that one associated to the complete Navier-Stokes equation for ultra-local random external stirring with strength D namely: $\langle F_i(\boldsymbol{r},t)F_j(\boldsymbol{r}',t')\rangle = D\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')\delta_{ij}$ and full range scale $0 \leq |\boldsymbol{k}| < \infty$ (see section 2)

$$
\begin{aligned}
&Z[\boldsymbol{j}(\boldsymbol{r},t)] = \\
&= \int D[\boldsymbol{v}(\boldsymbol{r},t)] det \left[\left(\frac{\partial}{\partial t} - \nu\Delta \right) \delta_{\ell k} + \sqrt{D}\frac{\delta}{\delta v_\ell}((\boldsymbol{v}\cdot\boldsymbol{\Delta})v)_k \right] \\
&\times \left\{ -\frac{1}{2}\int_{-\infty}^{+\infty} d^3r \int_0^{+\infty} dt \left(\frac{\partial}{\partial t} - \nu\Delta\boldsymbol{v} + \sqrt{D}(\boldsymbol{v}\cdot\boldsymbol{\nabla})v + \frac{\text{grad}P}{\rho} \right)^2 (\boldsymbol{r},t) \right\} \\
&\times exp\left\{ i\sqrt{D}\int_{-\infty}^{+\infty} d^3r \int_0^{+\infty} (\boldsymbol{j}\cdot\boldsymbol{v})(\boldsymbol{v},t) \right\} \ . \qquad (30.a)
\end{aligned}
$$

Let us remark that it is possible to eliminate the pressure term $-(1/\rho)\text{grad}P$ in this path-integral framework by using the incompressibility condition $div(\boldsymbol{v}) = 0$, which, by its turn leads one to consider only the transverse part of the external force and of the nonlinear term in the effective action in Eq. (30a) (see section 2)

$$
\begin{aligned}
&Z[\boldsymbol{j}(\boldsymbol{r},t)] = \int D^F[\boldsymbol{v}(\boldsymbol{r},t)] \\
&\times exp\left\{ -\frac{1}{2}\int_{-\infty}^{-\infty} d^3r \int_0^{+\infty} dt \left(\frac{\partial}{\partial t}\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \sqrt{D}((\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v})^{tr})^2 \right) \right\} \ . \qquad (30.b)
\end{aligned}
$$

Here the transverse part of a generic vector field is defined by the expression (see eq. (2))

$$
(\boldsymbol{W}(\boldsymbol{r},t))^{tr} = \boldsymbol{W}(\boldsymbol{r},t) - \frac{1}{4\pi}\text{grad}_r(\Delta^{-1}(div\ \boldsymbol{W})) \ . \qquad (31)
$$

Note that now one should postulate the nonlocal two-point correlation function in order to get Eq. (30b) $\langle F_i^{tr}(\boldsymbol{r},t)F_j^{tr}((\boldsymbol{r}',t'))\rangle = \mathcal{D}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')\delta_{ij}$.

It is worth remark that Eqs. (3a)-(5b), applied to the Burger equation leads to a different path integral than that proposed in ref. [8] since in the path-integral framework the viscosity is not a perturbative parameter which, in our case, is $\sqrt{D}$. Besides, the propagator in the free case for *the time parameter in the range* $0 \leq t \leq \infty$ is given (see eq. (16))

$$
\left(\left(\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} \cdot \left(-\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} \right)(k,t,t') = -\frac{1}{\nu k^2}\left[e^{-\nu k^2|t-t'|} - e^{-\nu k^2|t+t'|} \right] \qquad (32)
$$

and differing from the Dominicis-Martin propagator suitable for the range $-\infty \leq t \leq \infty$ ([8])

$$
\left(\left(\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} \cdot \left(-\frac{\partial}{\partial t} - \nu\Delta \right)^{-1} \right)(k,t,t') = \int_{-\infty}^{+\infty} dw(e^{iw(t-t')})\frac{1}{w^2+\nu^2|\boldsymbol{k}|^4} \ . \qquad (33)
$$

Let us now evaluate the vortex phase factor defined by a fixed-time spatial loop $\ell = \{\ell(\sigma), a \leq \sigma \leq b\}$ in our exactly soluble model Eq. (27) in order to see the connection with strings (random surfaces) ([8]) (see Appendix A for the relevance of theses non-local objects for advection phenomema)

$$\begin{aligned} &\left\langle exp\left(i\oint \boldsymbol{v}(\ell(\sigma),t)d\ell(\sigma)\right)\right\rangle_{\boldsymbol{v}} \\ &\equiv \int D^F[\boldsymbol{v}(\boldsymbol{r},t)]exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^3\boldsymbol{r}\int_0^{+\infty} dt\left(\frac{\partial \boldsymbol{v}}{\partial t}+\nu\lambda^2\boldsymbol{v}\right)^2(\boldsymbol{r},t)\right\} \\ &\times exp(i\oint \boldsymbol{v}(\ell(\sigma),t)d\ell(\sigma))\ . \end{aligned} \tag{34}$$

Since the flux is of a Beltrami type in our soluble model Eq. (29), we propose to rewrite the circulation phase factor as a sum over all surfaces bounding the fixed loop ℓ by making use of Stokes theorem and by taking into account again the Beltrami condition, i.e.,

$$\begin{aligned} \left\langle e^{i\oint_c \boldsymbol{v}\cdot d\ell}\right\rangle &= \int \mathcal{D}^F[\boldsymbol{v}(\boldsymbol{r},t)]exp\left\{-\frac{1}{2}\int_{\infty}^{+\infty} d^3r\int_0^{+\infty} dt\left[\boldsymbol{v}\left(-\frac{\partial}{\partial t^2}+\nu\lambda^2\right)\boldsymbol{v}\right]\right. \\ &\times\left.\left(\sum_S exp\left(i\lambda\int\int_S \boldsymbol{v}(x,t)\cdot d\boldsymbol{A}(x)\right)\right)\right\}\ . \end{aligned} \tag{35}$$

By observing now that the two-point correlation of our Brownian-Beltrami turbulent flux is exactly given by

$$\langle v_i(\boldsymbol{r},t)v_j(\boldsymbol{r}',t')\rangle_v = \int_{|\boldsymbol{k}|=\lambda} e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r})}\ \frac{e^{-\nu\lambda^2|t-t'|}}{\nu\lambda^2}\ \theta(t-t')\delta_{ij}\ , \tag{36}$$

with $t, t'; \in [0,\infty]$ and $\theta(0) = 1/2$ in this initial value problem, we can easily evaluate the average Eq. (29) and producing a strongly coupled area dependent functional for the spatial vortex phase factor, in our proposed turbulent flux regime

$$W[\ell,\boldsymbol{v}] \equiv \left\langle e^{i\oint_\ell \boldsymbol{v}(\ell,t)\cdot d\ell}\right\rangle = \sum_{\{S\}}\ exp\left\{-\frac{\lambda}{\nu}\int\int_S d\boldsymbol{A}(y)\frac{sin(\lambda|x-y|)}{|x-y|}d\boldsymbol{A}(y)\right\}\ . \tag{37}$$

Just for completeness of our study and in order to generalize the Beltrami flux turbulence analysis represented in the main text, for the physical case of the complete wave vector range $0 \leq |\boldsymbol{k}| < \infty$ in our turbulent path integral soluble model studies, we propose to consider a kind of generalized Beltrami condition to overcome this possible drawback of our turbulence modeling, namely:

$$\text{rot}\ \boldsymbol{v}(\boldsymbol{r},t) = \lambda(\boldsymbol{r})\boldsymbol{v}|\boldsymbol{r},t| \tag{38}$$

where $\lambda(\boldsymbol{r})$ is a positive function varying in the space and to be determined from a phenomenological point of the view. Note that the Fourier transformed (wave-vector) condition takes now the general form

$$|\boldsymbol{k}|\cdot|\tilde{\boldsymbol{v}}(\boldsymbol{k},t)| = \int_{R^3} d^3\boldsymbol{p}|\tilde{\lambda}(\boldsymbol{p}-\boldsymbol{q})|\cdot|\tilde{\boldsymbol{v}}(\boldsymbol{p},t)| \tag{39}$$

which, by its turn, leads to the full range scale $0 < |\boldsymbol{k}| < \infty$ for the eddies hydrodynamical motions undeer study. By supposing that the "vortical" stirring Eq. (28) is a pure white noise process with strength D,

$$\langle\Omega_\ell^{ext}(\boldsymbol{r},t)\Omega_{\ell'}^{ext}(\boldsymbol{r},t')\rangle = D\cdot\delta^{\ell\ell'}\delta^{3)}(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')\ . \tag{40}$$

It is a straightforward deduction by following our procedures as exposed in the text to arrive at an analogous Gaussian path integral for the Generalized Beltrami random hydrodynamical defined by Eq. (38). The generalized effective motion equation is given, in this new situation, by

$$\begin{aligned}&\left[\left(\frac{\partial}{\partial t}-\nu\left(\frac{\Delta\lambda}{\lambda}\right)(\boldsymbol{r})-\frac{\nu}{\lambda(\boldsymbol{r})}\frac{\partial\lambda(\boldsymbol{r})}{\partial x_e}\frac{\partial}{\partial x_e}+\nu\lambda^2(\boldsymbol{r})\right)\delta^{ik}+\right.\\&\left.+\nu\left(\varepsilon^{ijk}\ \frac{\partial\lambda(\boldsymbol{r})}{\partial x_j}\right)\right]v_k(\boldsymbol{r},t)=\Omega_i^{ext}(\boldsymbol{r},t)\ .\end{aligned} \tag{41}$$

The Gaussian path-integral, thus, is exactly written below

$$\begin{aligned}Z[j_i(\boldsymbol{r},t)] &= \int\prod_{i=1}^{3}D^F[v_i(\boldsymbol{r},t)]exp\left(i\int_{\infty}^{+\infty}d^3\boldsymbol{r}\int_0^\infty dt(j^iv_i)(\boldsymbol{r},t)\right)\\&\times exp\left[-\frac{1}{2D}\int_{-\infty}^{+\infty}d^3\boldsymbol{r}\int_0^\infty dtv_k(\boldsymbol{r},t)(M^{*ki}\cdot M^{is})v_s(\boldsymbol{r},t)\right].\end{aligned} \tag{42}$$

Here, the differential operators entering in the kinetic term of the turbulent path integral are

$$M^{*ki} = \left(-\frac{\partial}{\partial t}+\frac{\nu}{\lambda(\boldsymbol{r})}\frac{\partial\lambda(\boldsymbol{r})}{\partial x_e}\frac{\partial}{\partial x^e}+\frac{\nu}{\lambda(\boldsymbol{r})}\cdot\Delta\lambda\boldsymbol{r}-v\ \frac{\Delta\lambda(\boldsymbol{r})}{\lambda(\boldsymbol{r})}+\nu\lambda^2(\boldsymbol{r})\right)\delta^{ki}+\nu\varepsilon^{kji}\ \frac{\partial\lambda(\boldsymbol{r})}{\partial x_j} \tag{43}$$

and

$$M^{is} = \left(+ \frac{\partial}{\partial t} - \frac{\nu}{\lambda(\boldsymbol{r})} \frac{\partial \lambda(\boldsymbol{r})}{\partial x_e} \frac{\partial}{\partial x^e} + \nu \lambda^2(\boldsymbol{r}) - \nu \frac{\Delta \lambda(\boldsymbol{r})}{\lambda(\boldsymbol{r})} \right) \delta^{is} + \nu \varepsilon^{ijs} \, \frac{\partial \lambda(\boldsymbol{r})}{\partial x_j} \; . \tag{44}$$

It is worth pointing out that the exact evaluation of the variance in Eq. (42) depends on the *exact* form of our rotation $\lambda(\boldsymbol{r})$ defining the Beltrami condition (38).

The vortex phase factor Eq. (34), takes now a form closely related to the pure self-avoiding string theory in the case of a slowly varying function $|\text{grad}\ \lambda(\boldsymbol{r})| \ll \lambda(\boldsymbol{r})$ and $\lambda(\boldsymbol{r}) \sim 1$ (a very slowly $\boldsymbol{r}$-varying function: for instance as $\lambda(\boldsymbol{r}) = \lambda_0\ exp(-10^{-5}|\boldsymbol{r}|^2)$)

$$\langle e^{i \oint \boldsymbol{v}(\ell,t) d\ell} \rangle = \sum_{\{S\}} exp \left\{ -\frac{1}{\nu} \int_S \int_S d\boldsymbol{A}(x) \cdot \delta^{(3)}(x-y) \cdot d\boldsymbol{A}(y) \right\} \sim exp \left(-\frac{1}{\nu} \text{area}(S) \right) \; . \tag{45}$$

Now, if we follows Refs. [9] it is an easy task to deduce that the above written *time-fixed* vortex phase factor satisfies the famous loop wave equation for Abelian Q.C.D. at very low energy and a large number of colors. It may be written in the geometrical (infinitely differentiable loops $\ell(\sigma)$) as the following

$$\partial_\mu^x \, \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \left(\langle e^{i \oint \boldsymbol{v}(\ell,t) d\ell} \rangle \right) = \frac{1}{\nu} \oint d\boldsymbol{y} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \langle e^{i \oint \boldsymbol{v}(\ell,t) d\ell} \rangle \; . \tag{46}$$

The above obtained results rise hopes again that a string theory may be relevant to understand turbulence modeled as an amalgamation of “rough” roll up of random stirred fluid motions.

4 A Complex Trajectory Path-Integral Representation for the Burger-Beltrami Fluid Fluxe

The Hopf wave equation for turbulence is a master functional compressing the infinite hierarchy fluid velocity correlation functions in a single functional differential equation ([3]).

Our aim in this section is to a certain extent complete the previous path integral studies by presenting a complex *trajectory* path integral representation for a reduced model simulating “Burger turbulence” by considering directly the “experimental observable”

N-point grid velocity observable as a fundamental object of the proposed dynamically reduced Burger-Beltrami turbulent flux model below defined.

Let us start with the dynamical equation defining our one-dimensional Brownian like fluid flux

$$\begin{aligned} &\frac{\partial v(x,t)}{\partial t} + \left(v\ \frac{\partial v}{\partial x}\right)(x,t) = -(\nu\lambda^2)v(x,t) + f(x,t) \\ &v(x,0) = g(x) \end{aligned} \tag{47}$$

where we have replaced the usual fluid viscosity term $\nu d^2 v(x,t)/dx^2$ by the pure damping term $-\nu\lambda^2 v(x,t)$ (the reader should compare our proposed Brownian like flux with that of the Navier-Stokes-Beltrami studied in section 3).

One of the most important observable object in fluid turbulence is the fixed velocities measurements at the grid points (x_ℓ) and *at a common observation time t*

$$\langle \prod_{\ell=1}^{N} \delta(v(x_\ell;t) - v_\ell) \rangle \tag{48}$$

where the average $\langle\ \rangle$ is defined by the random stirring satisfying the gaussian statistics ([10])

$$\langle f(x,t) f(x',t') \rangle = k(x-x')\delta(t-t') \tag{49}$$

In momentum space, the observable eq. (48) is given by the following (grid dependent) characteristic functional (the Hopf wave functional restricted on the N-point grid)

$$\psi((x_1,\cdots,x_N);(p_1,\cdots,p_N);t) = \left\langle exp\left(i\sum_{\ell=1}^{N} p_\ell v(x_\ell,t)\right)\right\rangle \tag{50}$$

The Hopf wave equation associated to our model eq. (48) is given in a closed form by applying straightforward the methods of ref. [10]

$$\begin{aligned} &-i\frac{\partial}{\partial t}\psi((x_1,\cdots,x_N);(p_1,\cdots,p_N);t) \\ &\left\{\sum_{\ell=1}^{N}\left[p_\ell\ \frac{\partial}{\partial p_\ell}\left(\frac{1}{p_\ell}\frac{\partial}{\partial x_\ell}\right) - (\nu\lambda^2)p_\ell\frac{\partial}{\partial p_\ell}\right]\right. \\ &\left. + \sum_{\ell=1,\ell'=1}^{N}(k(x_\ell - x_{\ell'})p_\ell p_{\ell'}\right\}\psi((x_1,\cdots,x_N);(p_1,\cdots,p_N);t) \end{aligned} \tag{51}$$

Note that we must added the deterministic initial date condition to eq. (51)

$$\psi(x_1, \cdots, x_N); (p_1, \cdots, p_N); t \to 0^+) = exp\left(i\sum_{\ell=1}^{N} p_\ell g(x_\ell)\right) \tag{52}$$

Let us remark that in the physical grid on $R^3 = \{x_k^{(a)}; a = 1,2,3; k = 1, \cdots, N\}$, eq. (51) naturally reads

$$-i\frac{\partial}{\partial t}((x_1^{(a)}, \cdots, x_N^{(a)}); (p_1^{(a)}, \cdots, p_N^{(a)}); t)$$
$$\sum_{\ell=1}^{N}\sum_{a=1}^{3}\left[p_\ell^{(a)}\frac{\partial}{\partial p_\ell^{(a)}}\left(\frac{1}{p_\ell^{(a)}}\frac{\partial}{\partial x_\ell^{(a)}}\right) - \nu\lambda^2 p_\ell^{(a)}\frac{\partial}{\partial p_\ell^{(a)}}\right] + \psi((x_1, \cdots, x_N^{(a)}); (p_1^{(a)}, \cdots, p_N^{(a)}); t) +$$
$$\left[\sum_{\ell=1,\ell'=1}^{N} K_{ab}(x_\ell^{(a)} - x_{\ell'}^{(a)})p_\ell^{(a)}p_\ell^{(b)}\right]\psi((x_1^{(a)}, \cdots, x_N^{(a)}); (p_1^{(a)}, \cdots p_N^{(a)}); t) \tag{53}$$

Hereafter as said before we will present our study of eq. (53) for the one dimensional case eq. (51). By introducing the mixed coordinates defined by the transformation law.

$$p_j + x_j = u_j \quad ; \quad p_j - x_j = v_j. \tag{54}$$

The turbulent wave equation eq. (51) takes the more invariant form similar to a many-particle Schrödinger equation in Quantum Mechanics.

$$-i\frac{\partial}{\partial t}(\psi(u_1, \cdots, u_N); (v_1, \cdots v_N); t)$$
$$= \sum_{\ell=1}^{N}\frac{1}{4}\left[\frac{\partial^2}{\partial^2 u_\ell} - \frac{\partial^2}{\partial^2 v_\ell} - \left(\frac{2}{u_\ell + v_\ell}\right)\left(\frac{\partial}{\partial u_\ell} - \frac{\partial}{\partial v_\ell}\right) - \frac{\nu\lambda^2}{2}(u_\ell + v_\ell)\left(\frac{\partial}{\partial u_\ell} + \frac{\partial}{\partial v_\ell}\right)\right]$$
$$\psi((u_1, \cdots, u_N); (v_1, \cdots, v_N); t) + \frac{1}{4}\left[\sum_{\ell=1;\ell^1=1}^{N}(u_\ell + v_\ell)(u_{\ell'} + v_{\ell'})K\left(\frac{u_\ell - u_{\ell'} + (v_{\ell'} - v_\ell)}{2}\right)\right]$$
$$\psi((u_1, \cdots, u_N); (p_1, \cdots, p_N); t) \tag{55}$$

and

$$\psi((u_1, \cdots, u_N); (v_1, \cdots v_N); 0) = exp\left(i\sum_{\ell=1}^{N}\left(\frac{u_\ell + v_\ell}{2}\right)g\left(\frac{u_\ell - v_\ell}{2}\right)\right) \tag{56}$$

The above written closed partial differential equation is the basic result of this section. At this point we can implemente perturbative calculations for our turbulent wave equation by considering a physical slowly varying (even function) correlation function of the form

$$K(x) \approx K(0) - \frac{\kappa_0}{2}x^2 \quad ; \quad |x| << \left(\frac{K(0)}{\kappa_0}\right)^{1/2} \equiv L$$
$$0 \quad ; \quad |x| >> L \tag{57}$$

Which by its turn leads to the Harmonic and quartic anharmonic potential bellow written

$$\sum_{\ell=1;\ell'=1}^{N} \{K(0)(u_\ell u_{\ell'} + v_\ell v_{\ell'} + u_\ell v_{\ell'} + v_\ell u_\ell) \\ -\frac{1}{8}\kappa_0(u_\ell + v_\ell)(u_{\ell'} + v_{\ell'}[u_\ell^2 + u_{\ell'}^2 + v_{\ell'}^2 + v_\ell^2 - 2u_\ell u_{\ell'} - 2v_\ell v_{\ell'}]\Big\} \quad (58)$$

In the important case of the single fluid velocity average, our turbulent wave equation takes the following form, after making an analitic continuation $v \to iv$; namely

$$i\frac{\partial}{\partial t}\psi(u,v;t) = (\mathcal{L}_0 + \mathcal{L}_1)\psi(u,v;t) \quad (59)$$

With the initial condition

$$\psi(u,v;t \to 0^+) = exp\left[i\left(\frac{u+iv}{2}\right)g\left(\frac{u-iv}{2}\right)\right] \quad (60)$$

Here the Kinetic and perturbation terms are:

$$\mathcal{L}_0 = -\frac{1}{4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{k(0)}{4}(u^2 - v^2) \\ \mathcal{L}_1 = \frac{2}{u+iv}\left(\frac{\partial}{\partial U} - \frac{1}{i}\frac{\partial}{\partial v}\right) - \nu\lambda^2(u+iv)\left(\frac{\partial}{\partial u} + \frac{1}{i}\frac{\partial}{\partial v}\right) \quad (61)$$

The Harmonic oscilator propagator of the kinetic term eq. (61) is determined in a straightforward way and a Feynman diagramatic analysis may be easily implemented for $\nu << 1$ by the same perturbative procedure used in quantum mechanical problems. Similar remarks hold true in the general case eq. (55).

It is worth point out that analogous results are easily obtained in the physical case of turbulent Beltrami flux in the three dimensional case.

Let us comment the case of general turbulent flux. In this case, although being impossible to write a closed partial differential equation as we did in this section 4, we can develop approximate schemes to solve the full functional Hopf equation by approximating the fluid shear stress tensor by finite differences, namely:

$$\left\{\nu\frac{d^2 v(x_j;t)}{dx_j^2} \approx \frac{\nu}{\Delta}(-2v(x_j,t) + v(x_{j+1};t) + v(x_{j-1},t))\right\} \quad (62)$$

With the grid spacing $\Delta = |x_{j+1} - x_j|$.

Let us now write a trajectory functional integral representation for the initial-value problem eq. (55) after taking into account the analytic continuation $v_\ell \to iv_\ell$ there.

As a first step to achieve our goal, we write the associated Green functional of eq. (55) in an operator form (the Feynman-Dirac propagation) for the free case $k \equiv 0$

$$\overline{G}[(u_\ell, u_\ell); (u'_\ell, v'_\ell; t)] = \langle (v_\ell, v_\ell)| \\ exp\left(it \left[\sum_{\ell=1}^{N} \frac{1}{4}\Delta_{(v_\ell,v_\ell)} - \left(\frac{2}{u_\ell + iv_\ell} \right) \left(\frac{\partial}{\partial v_\ell} - \frac{1}{i}\frac{\partial}{\partial v_\ell} \right) \right.\right. \\ \left.\left. -\frac{\nu\lambda^2}{2}(u_\ell + iv_\ell) \left(\frac{\partial}{\partial u_\ell} + \frac{1}{i}\frac{\partial}{\partial v_\ell} \right) \right] \right) |(u'_\ell, v'_\ell)\rangle \tag{63}$$

As in the usual Feynman analysis we write eq. (63) as an infinite product of short-time t-propagation and consider the standard short-time expansion

$$\lim_{s\to 0^+} \langle (u_\ell^{(I)}, v_\ell^{(I)})| exp(isH) |(v_\ell^{(I-1)}, u_\ell^{(I-1)}\rangle = \lim_{s\to 0^+} \int d^N p_I d^N q_I \times \\ exp\left\{ is \left[\frac{p_I^2 + q_I^2}{4} - \left(\frac{2}{u_\ell^{(I)} + iv_\ell^{(I)}} \right) \left(ip_I^{(\ell)} - q_I^{(\ell)} \right) \right] \right\} - \\ -\frac{\nu\lambda^2}{2}(u_\ell^{(I)} + iv_\ell^{(I)}) \left(ip_I^{(\ell)} - q_I^{(\ell)} \right) - \\ exp\left\{ \sum_{\ell=1}^{N} \left[ip_I^{(\ell)}(u_I^{(\ell)} - u_{I-1}^{(\ell)}) \right] \right\} \times exp\left\{ \sum_{\ell=1}^{N} \left[ig_I^{(\ell)}(v_I^{\ell} - v_{I-1}^{\ell-1}) \right] \right\} \tag{64}$$

where H denotes the second order differential operator inside the brackets of eq. (64).

If we substitute eq. (65) into the short-time product expansion of eq. (64), namelly

$$\langle u_\ell, v_\ell | \ell^{itH} | u'_{\ell'}, v'_\ell \rangle = \prod_{I=1}^{M} \int_{-\infty}^{+\infty} du_I dv_I \langle (u_\ell^I, v_\ell^I)| exp\left(i \left(\frac{t}{M} \right) H \right) |(u_\ell^{I-1}, v_\ell^{I-1})\rangle \tag{65}$$

and evaluate the (p_I, q_I)-momenta functional integrals (see ref. [11] for a detailed exposition), we get our searched trajectory path integral representation for the Green-function of eq. (55)-(56) in the free case $k \equiv 0$

$$\overline{G}\,[(u_\ell, v_\ell);((u'_\ell, v'_\ell);t] =$$

$$\int_{\substack{\overline{U}_\ell(0)=u_\ell \\ \overline{U}_\ell(t)=u_\ell}} \int_{\substack{\overline{v}_\ell(0)=v_{\ell'} \\ \overline{v}_\ell(t)=v_{\ell'}}} exp\left\{\frac{i}{4}\int_0^t d\sigma\left[\left(\frac{d}{\partial\sigma}\overline{U}(\sigma)\right)^2\right.\right.$$

$$-2\sum_{\ell=1}^{N}\left(\frac{d}{\partial\sigma}\overline{U}^{(\ell)}(\sigma)\right)\left(-\frac{2}{\overline{U}_\ell(\sigma)+i\overline{V}_\ell(\sigma)}-(\nu\lambda^2)(\overline{U}_\ell(\sigma)+i\overline{V}_\ell(\sigma))\right)$$

$$\left.\left.+\sum_{\ell=1}^{N}\left(-\frac{2}{\overline{U}_\ell(\sigma)+i\overline{V}_\ell(\sigma)}-(\nu\lambda^2)(\overline{U}_\ell(\sigma)+i\overline{V}_\ell(\sigma))\right)^2\right]\right\}$$

$$exp\left\{\frac{i}{4}\int_0^t d\sigma\left[\left(\frac{\partial}{\partial\sigma}\overline{V}_\ell(\sigma)\right)^2-2\sum_{\ell=1}^{N}\left(\frac{\partial}{\partial\sigma}\overline{V}_\ell(\sigma)\right)\right]\times\right.$$

$$\left(-\frac{2\cdot i}{V_\ell(\sigma)+i\overline{V}_\ell(\sigma)}+i(\nu\lambda^2)(\overline{V}_\ell(\sigma)+i\overline{V}_\ell(\sigma))\right)^2$$

$$\left.\sum_{\ell=1}^{N}\left(-\frac{2i}{\overline{U}_\ell(\sigma)+i\overline{V}_\ell(\sigma)}+i(\nu\lambda^2)(\overline{U}_\ell(\sigma)+iV_I^{(\ell)}(\sigma))\right)\right\} \tag{66}$$

Note that the discrete index $I = 1, \cdots, M$ has became the continuous time parameter σ ranging in $[0, t]$.

The general $k \equiv 0$ case is straightforwardly obtained from eq. (66) by only considering the additional weight

$$exp\left\{i\left[\sum_{\ell=1,\ell'=1}^{N}(\overline{U}^{\ell}(\sigma)+i\overline{V}^{\ell}(\sigma))(\overline{U}^{\ell'}(\sigma)+i\overline{V}^{\ell'}(\sigma))\times\right.\right.$$

$$\left.\left.K\left(\frac{(\overline{U}^{\ell}(\sigma)-\overline{U}^{\ell'}(\sigma))+i(\overline{V}^{\ell'}(\sigma)-\overline{V}^{\ell}(\sigma))}{2}\right)\right]\right\} \tag{67}$$

It is obvious from the above written N-body (complex valued!) trajectory path integrals representations that any analytical analysis will be somewhat combersome. However, its numerical (Monte-Carlo and F.F.T alghoritmos) studies may be usefull to implement approximated evaluations on applied problems.

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5 Appendix A – The Vortex Phase Factor in Scalar Advected Diffusion

In this somewhat pedagogical appendix, we show the appearance of vortex phase factors in hydrodymaic advected diffusion as used in section 3 of this paper and similar in it structure to the loop space approach used on Quantum Chromodynamics ([9]). Let us, thus, consider the motion equation for a scalar field $\phi(x,t)$ advected by a incompressible fluid with velocitty $\vec{v}(x,t)$, namely

$$\frac{\partial\phi(x,t)}{\partial t} = (D(t)D_0)\,\Delta\phi(x,t) - \left([\vec{v}\cdot\vec{\Delta}]\phi\right)(x,t) + j(x,t)\phi(x,t) \tag{68}$$

with the initial value condition

$$\phi(x, t \to 0^+) = f(x) \tag{69}$$

Here $D(t)$ is a time dependente molecular diffusion constant. $j(x,t)$ is an external source field and D_0 a reference value for the scalar diffusion constant.

As a first step to analyze eq. (68), let us consider the following time variable change

$$\tau = \int_0^t D(s)ds \tag{70}$$

$$\phi(x,\tau) \equiv \phi(x,t(\tau)) \tag{71}$$

$$j(x,\tau) \equiv \tau(x,t(\tau))\big/D((t)(\tau)) \tag{72}$$

$$\vec{\alpha}(x,\tau) \equiv \vec{v}(x,t(\tau))\big/t(\tau) \tag{73}$$

We obtain, thus, the more amenable form for eq. (69) with a constant molecular diffusion constant in this new time scale τ.

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = D_0\Delta\phi(x,\tau) - \left([\vec{a}\cdot\vec{\nabla}]\phi\right)(x,\tau) + j(x,\tau)\phi(x,\tau) \tag{74}$$

$$\lim_{\tau\to 0^+}\phi(x,\tau) = f(x) \tag{75}$$

At this point let us remark that in the simple case of eq. (68) with $\vec{v}(\vec{x},t) = 0$ and $j(\vec{x},t) = 0$, the problem Green function is easily given by

$$G_{dif}\left((x,t');(x,t')\right) = \left(D_0\int_{t'}^{t} D(s)ds\right)^{-\frac{3}{2}} \times \exp\left\{-\frac{[(\vec{x}-\vec{x}\,')^2]}{D_0\left(\int_{t'}^{t} D(s)ds\right)}\right\} \tag{76}$$

and leading thus to the quadratic mean desviation

$$\langle(\vec{x})^2\rangle = \frac{3}{2}\, D_0 \int_0^t D(s)ds \tag{77}$$

which, by its turn, leads to a super-difuse behavior if $D(s) \sim s^\alpha$ for $\alpha > 0$

Usual perturbative calculations may be formally implemented by considering the zeroth-order Green function as given by eq. (76).

Let us write a (non-perturbative) path-integral representation for the Green function $G\left((x,\tau):(x',\tau')\right)$ – eq. (74)-eq. (75). In order to implement such analysis we compare it with the analogous problem in Quantum Mechanics of a particle interacting with an eletromagnetic field $\vec{A}$ and a scalar potential V. The Schrödinger equation for this quantum mechanical problem in Landau gauge $\vec{\nabla}\vec{A} = 0$ is given by

$$i\hbar\,\frac{\partial\psi(\vec{x},\tau)}{\partial\tau} = \left\{-\frac{\hbar^2}{2m}\,\Delta\psi + \frac{ie\hbar}{mc}\,(\vec{A}\cdot\vec{\nabla}\psi) + \frac{ie\hbar}{2mc^2}\,(\vec{\nabla}\cdot\vec{A})\psi + \left(\frac{e^2}{2mc^2}\,(\vec{A})^2 + V\right)\psi\right\}(\vec{x},\tau) \tag{78}$$

It is well-known that the Green function associated to a initial value problem is given by the following (formal) Feynman path-integral

$$\begin{aligned}\tilde{G}\left[(x,\tau);(x',\tau')\right] \;=\; & \int_{\vec{r}(\tau')=x';\vec{r}(\tau)=x} D^F[\vec{r}(\sigma)]\,\times \\ & \exp\left\{\frac{i}{\hbar}\int_{\tau'}^{\tau} d\sigma\left[\frac{1}{2}\,m\left(\frac{\vec{r}(\sigma)}{d\sigma}\right)^2 + ie\vec{A}\left(\vec{r}(\sigma),\sigma\right) - V\left(\vec{r}(\sigma),\sigma\right)\right]\right\}\end{aligned} \tag{79}$$

It is straightforward to note that if one makes the following identification on eq. (79)

$$\begin{aligned}\hbar &= -i \quad ; \quad \vec{A} = -\vec{v} \quad ; \quad V = -\frac{1}{4D_0}\,(\vec{a})^2 + j \quad ; \\ m &= \frac{1}{2D_0} \quad ; \quad \frac{e}{c} = \frac{1}{2D_0} \quad ; \quad c = 1\end{aligned} \tag{80}$$

one can see that the Schrödinger equation (78) reduces to our scalar advected equation (76).

As a direct consequence of the above made remark, we obtain the result anounced on the begining of our study. Namely, the Green function $G[(x,\tau),(x',\tau')]$ is given explicitly with a closed form by the following (now well-defined) Wiener path-integral weighted with the vortex phase factor used in the studies presented in section 3

$$G_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau')\right]=\int_{\vec{Z}(\tau)=\vec{x};\vec{Z}(\tau')=\vec{x}'}D^F\left[\vec{Z}(\sigma)\right]$$

$$\times\exp\left\{-\frac{1}{4D_0}\left(\int_{\tau'}^{\tau}d\sigma\left[\frac{d\vec{Z}}{d\sigma}-\vec{a}\left(\vec{z}(\sigma),\sigma\right)\right]^2\right)\right\}\times\exp\left\{-\int_{\tau'}^{\tau}d\sigma j(\vec{z}(\sigma),\sigma)\right\}$$

$$\equiv\int_{\vec{Z}(\tau)=\vec{x},Z(\tau')=\vec{\tau}'}d_{\mu}^{Wiener}[z(\sigma)]\mathbf{exp}\left\{-\frac{\mathbf{1}}{\mathbf{4D_0}}\int_{\tau'}^{\tau}\boldsymbol{\vec{a}(\vec{Z}(\sigma),\sigma)}\ \frac{\boldsymbol{d\vec{Z}}}{\boldsymbol{d\sigma}}\ \boldsymbol{(\sigma)}\right\}\times$$

$$\exp\left\{-\int_{\tau'}^{\tau}\left(\frac{\vec{a}^{\,2}}{4D_0}+\vec{j}\right)\left(\vec{Z}(\sigma),\sigma\right)\right\}\tag{81}$$

in the other words

$$\phi(x,\tau)=\int_0^{\tau}d\tau'\int dx'G\left[(x,\tau);(x,\tau')\right]\phi(x,\tau')\tag{82}$$

At this point let us remark that in the pratical important case of large-scale transport where one can set $D_0\equiv 0$ on eq. (68) (with $j(x,\tau)\equiv 0$ for simplicity), an exactly expression for the first-order resulting equation

$$\frac{\partial\phi(x,\tau)}{\partial\tau}+\left[\vec{\nabla}\cdot(\vec{a}\phi)\right](x,\tau)=0\tag{83}$$

is exactly obtained by considering the limit $D_0\to 0$ on eq. (81) and producing the result

$$\bar{G}_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau')\right]=\delta^{(3)}\left[\vec{x}-\vec{Z}_{(x',\tau')}(\tau)\right]\tag{84}$$

where $\vec{Z}_{(x',\tau')}(\tau)$ satisfies the Saddle-point (minimum) of the positive path-integral weight, namely: $\vec{Z}_{(x',\tau')}(\tau)\equiv\vec{Z}(\sigma)\Big|_{\sigma=\tau}$, here $\vec{Z}(\sigma)$ satisfies the Liouville boundary value problem

$$\frac{d\vec{Z}(\sigma)}{d\sigma}=\vec{a}\left(\vec{Z}(\sigma),\sigma\right)\tag{85}$$

with

$$\vec{Z}(\tau')=\vec{x}\qquad\qquad\vec{Z}(\tau)=\vec{x}\tag{86}$$

Next D_0 corrections are implemented on the path-integral eq. (81) by similar procedures used in the Feymann path-integral theory. We, thus, consider the following background decomposition of the path manifold on eq. (81)

$$\vec{Z}(\sigma)=\vec{Z}_{(x',\tau')}(\sigma)+\sqrt{D}\ \vec{Y}(\sigma)\tag{87}$$

with the "fractal" path $\vec{Y}(\sigma)$ such that

$$\vec{Y}(\tau') = \vec{Y}(\tau) = \vec{0} \tag{88}$$

As a consequence eq. (87) – eq. (88), we get the next $\sqrt{D}$-correction for the diffusion Green function eq. (81)

$$\begin{aligned} G_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau'\right] &\sim \tilde{G}_{dif}\left[(\vec{x},\tau);(\vec{x}',\tau')\right] \times \\ & \det{}_F^{-\frac{1}{2}}\left\{-\frac{d^2}{d^2\sigma}\delta_{AB} + \left[(\partial_A a_s)(\partial_B a_s)\right]\left(\vec{Z}_{x',\tau'}(\sigma)\right)\right. \\ &- \left. 2\left[\partial_A a_B\right]\left(\vec{Z}_{x',\tau'}(\sigma)\right) \times \frac{d}{d\sigma}\right\} + 0(D_0) \end{aligned} \tag{89}$$

where $A, B = 1, 2, 3$ denote the vectorial indixes on R^3 $(\vec{a}(\vec{x}) \equiv (a_A)(x_B))$ and the functional determinant associated to the fluctuation operator at the one-loop order should be evaluated with Dirichlet boundary conditions defined by eq. (88). Exactly evaluation of the above cited functional determined needs the closed form of the transport fluid fluxe $\vec{a}(\vec{x})$.

Let us exemplify this last point for the two-dimensional vortex configuration with *constant vorticity* $(\vec{x} = (x, y))$

$$\vec{a}(\vec{x}, t) = \left(-\frac{1}{2}\, wy\ , \frac{1}{2}\, wx\right) \tag{90}$$

In this case the classical trajectory equations eq. (85) – eq. (86) are given exactly by

$$z_1(\sigma) = A_1 sen\left(-\frac{w}{2}\,\sigma + \rho_1\right) \tag{91}$$

$$z_2(\sigma) = A_2 sen\left(-\frac{w}{2}\,\sigma + \rho_2\right) \tag{92}$$

where the integration constants $(A_1, A_2, \rho_1, \rho_2)$ must be choosen in order to satisfy the boundary conditions eq. (86). Namely,

$$A_1 sen\left(\frac{w}{2}\,\tau' + \rho_1\right) = x_1' \tag{93}$$

$$A_1 sen\left(\frac{w}{2}\,\tau + \rho_1\right) = x_1 \tag{94}$$

$$A_2 sen\left(\frac{w}{2}\,\tau' + \rho_2\right) = y_1' \tag{95}$$

$$A_2 sen\left(\frac{w}{2}\,\tau' + \rho_2\right) = y_1 \tag{96}$$

The functional determinant on eq. (89) is easily evaluated by the usual path-integral techniques applied to the problem of a particle can the presence of a harmonic oscillator and a constant magnetic field

$$det_F^{-\frac{1}{2}}\left\{\left[\begin{array}{ll} -\frac{d^2}{dv^2}+\frac{1}{4}\,w^2 & -w\,\frac{d}{dv} \\ w\,\frac{d}{dv} & -\frac{d^2}{dv^2}+\frac{1}{4}\,w^2 \end{array}\right]\right\} = \frac{w}{4\pi(\tau-\tau')\cdot sen\left(\frac{w(\tau-\tau')}{2}\right)} \tag{97}$$

As a last point worth remarking let us consider the Boltzman-Vlasov advected *damped* equation on R^6 wiht an external stirring $f(\vec{x},t)$

$$\frac{\partial N(\vec{x},t)}{\partial t} = -\nu N(\vec{x},t) - \left([\vec{V}\cdot\vec{\nabla}]N\right)(\vec{x},t) + f(\vec{x},t) \tag{98}$$

with the inital condition

$$\lim_{t\to 0^+} N(\vec{x},t) = f(\vec{x}) \tag{99}$$

By following the above exposed study, it is straighforward to write the solution of eq. (98) as the sum of the homogneous case with non zero initial condition added with that of the non-homogenous case but now with zero initial condition, namely

$$\begin{aligned} N(\vec{x},t) &= e^{-\nu t}\int d^6\vec{x}\,'\delta^{(6)}\left[(\vec{x}-\vec{Z}_{x',0}(t))\right]g(x') \\ &+ \int_0^t dt' e^{\nu(t'-t)}\int d^6\vec{x}\,'\delta^{(6)}\left[\vec{x}-\vec{Z}_{x',t'}(t))\right]f(\vec{x}\,',t') \end{aligned} \tag{100}$$

Here $\vec{Z}_{x',t'}(t)\equiv\vec{Z}_{x',t'}(\sigma)\Big|_{\sigma=t}$ satisfies the equations (85) – (86)).