CBPF-NF-017/96

"The Moment Method for the Seismic Inverse Problem"*

by

Luiz C.L. Botelho

Centro Brasileiro de Pesquisas Físicas - CBPF

Rua Dr. Xavier Sigaud, 150

22290-180 – Rio de Janeiro, RJ – Brasil

and

Departamento de Física

Universidade Federal do Pará

Campus Universitário do Guamá

66075-900 - Belém, Pará - Brasil

Abstract

We propose a new moment technique for the inverse problem of Wave Propagation in an inhomogeneous (half – space) medium in the acoustical Wave Field Born Approximation by using the Analytical Regularization Scheme of Bollini, Giambiagi and Dominguez.

Key-words: Seismic; Inverse Problem.

---

*IN MEMORIAM OF PROFESSOR JUAN JOSÉ GIAMBIAGI – CBPF

(DECEASED IN 1996, JANUARY)

One of the most chalenging problem in the inverse methods for the Geophysical problem of imaging and inhomogeneous medium described by a depth – variable refraction index $M(z)(0 \leqq z < \infty)$ from the backscattered acoustical wave field in the Born – approximation is the problem of its ill – posedeness ([1]) – pag. 323). In this Brief Report I will propose a solution for this difficulty by considering the Analytical Regularization Scheme previously used in Quantum Field Theory ([2]) to reduce the imaging of the medium to the well-know problem of determining a function by knowing its associated moments ([3]) – theorem 15.26).

Let us start our analysis by considering the corresponding Fredholm integral equation relating the medium refraction index $(M(z)$ $(0 \leqq z < \infty)$ to the Born approximated backscattered acoustical Wave Field $U_S(\xi, \xi; \omega)$ on the medium Boundary $(\xi = (x, y) \, \varepsilon R^2, -\infty < \omega < +\infty$ and written in the frequency domain

$$U_S(\xi, \varepsilon; \omega) = (\omega)^2 \cdot \int_{-\infty}^{+\infty} d^2\eta \int_0^{\infty} dz \; M(z) \cdot \exp\left[2i\omega\sqrt{(\eta - \xi)^2 + z^2}\right] \left((\eta - \xi)^2 + z^2\right)^{-1} \tag{1}$$

Unfortunately, the direct solution of the Fredholm Integral Equation eq. (1) to evaluate $M(z)$ is an ill – conditioned mathematical problem since its kernel is not square integrable and thus, defining a non – compact operator in the Square Integrable functions which by its turn make this operator loses its inverse continuity in this function space.

In order to overcome such mathematical problem, I follow a usual procedure borrowed from similar distribution multiplication problem in Quantum Field Theory ([2]). I propose to consider an analitically continued kernel in eq. (1) where the problem of ill – posedeness is absent and at the end of the calculation, I make an analytical continuation of the result to the Physical Probem as first proposed by Bollini, Giambiagi and Dominguez in 1964. Let us, then, study the analitically continued problem

$$U_S^{\alpha}(\xi, \xi; \omega) = (\omega)^2 \cdot \int_{-\infty}^{+\infty} d^2\eta \int_0^{\infty} dz \; M(z) \cdot \exp\left[2i\omega\sqrt{(\eta - \xi)^2 + z^2}\right] \left((\eta - \xi)^2 + z^2\right)^{-1} \tag{2}$$

where α is chosen in such way that it guarantee its square integrability. In this region of α values, it is possible to interchange the $\eta - z$ integration order by the use Fubbini –

Tomeli Theorem and evaluate explicitly the η – integration for $z > 0$ ([2]).

$$-\frac{U_S^{(\alpha)}(\xi,\xi,\omega)}{\omega^2} = \int_0^\infty dz\ m(z)\cdot\left[\sum_{m=0}^\infty \frac{(2i)^m\cdot(\omega)^3}{m!}\times\right.$$
$$\left.\left(\pi\frac{\Gamma\left(\frac{\alpha-m}{2}-1\right)}{\Gamma\left(\frac{\alpha-m}{2}\right)}\cdot(z^2)1-\frac{(\alpha-m)}{2}\right)\right] \tag{3}$$

where we have used the result

$$\int_{-\infty}^{+\infty} d^2p\left(p^2+\beta^2\right)^{\frac{m-\alpha}{2}} = \pi\ \frac{\Gamma\left(\frac{\alpha-m-2}{2}\right)}{\Gamma\left(\frac{\alpha-m}{2}\right)}\ (\beta^2)^{\frac{\alpha-m-2}{2}} \tag{4}$$

By considering now the existence of power expansion associated to the analitically continued Back-Scattered Acoustic Wave Field (which is mathematically correct assumption at least for large time)

$$-\frac{U_S^{(\alpha)}(\xi,\xi;\omega)}{\omega^2} = \sum_{m=0}^\infty \frac{U_m^{(\alpha)}\cdot\omega^m}{m!} \tag{5}$$

and comparing with the power expansion of Eq. (3), we get the moment problem relationship

$$U_m^{(\alpha)} = \frac{(2i)^m}{\Gamma\left(\frac{\alpha-m}{2}\right)}\cdot\pi\cdot\Gamma\left(\frac{\alpha-m-2}{2}\right)\cdot\int_0^\infty dz\cdot m(z)\cdot z^{m+2-\alpha} \tag{6}$$

At this point we implement the Physical Limit of $\alpha\to 2$ (see eq. (1) and taking into account that $U_0^{(\alpha-2)}$ has an infinite piece

$$U_0^{(\alpha-2)} = \int_0^\infty dz\cdot m(z)\cdot\lim_{\alpha\to 2}\left(\frac{2}{\alpha-2}\right)\cdot z^{2-\alpha} = -2\left(\int_0^\infty dz\ m(z)\right)\lim_{\alpha\to 2}\left(\frac{1}{\alpha-2}\right)$$
$$-\ 2\int_0^\infty dz\cdot m(z)\ lg\ z \tag{7}$$

The infinite piece is disregarded in our approach by suppossing that the "medium área" $\int_0^\infty m(z)dz$ vanishes identically. Otherwise eq. (7) must be understood as a finite-part prescription as in Quantum Field Theory studies ([2]).

By grouping togheter eq. (6) – eq. (7) we reduce the problem of imaging the refraction index $m(z)$ to the solution of the well-know Moment Problem of determining a function from its moment of order m

$$U_{(m>0)}^{(2)} = (2i)^m\pi\int_0^\infty dz\cdot m(z)\cdot z^m \tag{8.a}$$
$$U_{(0)}^{(2)} = -2\cdot\int_0^\infty dz\ m(z)lyz\ dz \tag{8.b}$$

In the case that $m(z)$ has a Fourier Transform which is an analytical function around the origin in the Fourier Domain (for instance when $m(z)$ has compact support) I can solve formally eq. (8.a). The validity of this claim is a result of the equation below

$$\tilde{m}(k) = \frac{1}{2\pi} \int_0^\infty dz \cdot e^{ikz} m(z) = \sum_{k=0}^{\infty} \frac{m^{(m)}}{m!} \; k^m \tag{9.a}$$

with

$$\begin{aligned} U^{(2)}_{(m>0)} &= (2i)^m \pi \int_0^\infty dz \cdot z^m \left(\int_{-\infty}^{+\infty} dx \; e^{ikz} \tilde{m}(k) \right) \\ &= (2i)^m \pi \cdot \frac{1}{(i)^m} \cdot \left(\frac{d^m \tilde{m}(k)}{dk^m} \right)_{k=0} \end{aligned} \tag{9.b}$$

As a consequence, I have the final result connecting the coefficients of eq, (9.a) and eq. (5).

$$U^{(2)}_{(m>0)} = 2^m \cdot \pi \cdot m_{(m)} \tag{10}$$

Finally, I remark that the above mathematical operations hold true at least if $|m_{(m)}| \leqq 2^{-m} \cdot C_m$, where C_m are the Taylor coefficients of an analytical real function with the property of $\Sigma \; C_m^2 < \infty$.

This research was supported by CNPq, Brazil.

References

[1] – Norman Bleistein, "Mathematical Methods for Wave Phenomena" – Computer Science and Applied Mathematics, Academic Press, 1994.

– Luiz C.L. Botelho and R. Vilhena, Phys. Rev. E, vol. 49, nº 2, R 1003 – R 1004 (1994).

[2] – Bollini, C.G., Giambiagi, J.J. and G. Dominguez, Il Nuovo Cimento 31, 550 (1964).

[3] – Walter Rudim, "Real and Complex Analysis", Second Edition, Tata Mc. Graw-Hill Publishing Co. Ltd., India.