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ON THE PHYSICAL INTERPRETATION OF COMPLEX POLES OF THE S-MATRIX - I.

by

G. Beck and H. M. Nussenzveig

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

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ON THE PHYSICAL INTERPRETATION OF COMPLEX POLES OF THE S-MATRIX - I.*

G. Beck and H. M. Nussenzveig**
Centro Brasileiro de Pesquisas Físicas

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SUMMARY: - To improve the usual treatment of the transient behaviour of continuous systems by the "method of complex eigenvalues", it is necessary to take into account the excitation conditions. This is done by considering initial-value problems. Three examples are investigated: (1) a harmonic oscillator coupled with a vibrating string; (2) the electromagnetic oscillations of a perfectly conducting spherical antenna; (3) the scattering of Schrödinger particles by a hard sphere. In each case the general solution of the initial-value problem is related to the "method of complex eigenvalues" by associating a propagator with each pole of the S-matrix. In this way, the difficulty of exponential growth, which occurs in the usual treatment, is eliminated, and the dependence of the decay law on the excitation is exhibited. For Schrödinger particles, the spreading of wave packets restricts the domain of validity of the exponential decay law. The origin of the "transient modes" which are associated with the poles of the S-matrix is discussed. It is shown that the antenna modes originate from the effect of inertial forces. The limitations on the physical interpretation in the case of short-lived modes are emphasized.

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1. - Introduction.

It is well known that the transient behaviour of a discrete system, such as an electric network with lumped parameters or a mechanical system with a finite number of degrees of freedom, is closely related to the "complex eigenfrequencies" of the system. Let us consider, for instance, an electric network with lumped parameters. Its response to a time-harmonic excitation may be described by giving a characteristic function of the network (steady-state admittance or impedance) as a function of the frequency. This function usually has an analytic continuation with poles in the "complex-frequency" plane. The poles are associated with the "free modes of oscillation" of the network. The response of the network to inhomogeneous initial conditions (given charges in the capacitors or currents in the inductors), in the absence of external driving functions, is a superposition of these modes, with amplitudes determined by the initial conditions. The poles also play an important role in the determination of the response of the network to an arbitrary excitation, including external driving functions [1].

The "method of complex eigenvalues" is also employed in the theory of transients in continuous systems, but the situation is far less satisfactory in this case. Perhaps the earliest example is Thomson's treatment [2] of the elctromagnetic oscillations of a perfectly conducting sphere. Thomson's "natural modes of oscillation" satisfy the requirement of containing only outgoing radiation. They

1. M.F. Gardner & J.L. Barnes: Transients in Linear Systems (New York, 1942).

2. J.J. Thomson: Proc. Lond. Math. Soc. (1), 15, 197 (1884).

are associated with the "complex eigenfrequencies" $\omega_n = \omega'_n - i\gamma_n$, $\gamma_n > 0$ $(n = 1, 2, \ldots)$, with a corresponding time factor $\exp(-i\omega_n t)$. According to Thomson, ω'_n represents the frequency, and γ_n the damping constant, associated with the nth natural mode. All the modes are strongly damped. The same method was applied by ABRAHAM [3] to a thin, perfectly conducting, prolate spheroid. The damping is much weaker in this case.

Thomson's method was criticized by LAMB [4], on the ground that the solutions are not bounded at infinity. In fact, Thomson's modes behave like $r^{-1} \exp\left[-i\,\omega_n(t - \frac{r}{c})\right]$ at large distances from the sphere, so that they increase exponentially for $r \to \infty$. This "exponential catastrophe" is a characteristic feature of such damped, purely outgoing waves, since the field at large distances was in the neighbourhood of the source at a correspondingly remote time. As was pointed out by Lamb, the difficulty is related to the unphysical assumption that the modes have been in existence for an indefinitely long time. It may be overcome by taking into account the excitation conditions, as was shown by Lamb in an example.

Some illustrations of this point in connections with Thomson's problem were given by LOVE [5]. He considered the case in which the initial field around the sphere is identical to an electrostatic multipole field. This gives rise to an outgoing disturbance with a

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3. M. Abraham: Ann. Physik 66, 435 (1898).
4. H. Lamb: Proc. Lond. Math. Soc. (1), 32, 208 (1900).
3. A.E.H. Love: Proc. Lond. Math. Soc. (2), 2, 88 (1904).

sharp front, which travels with the velocity of light. It was shown by Love that the field beyond the wave front remains undisturbed (so that there is no exponential catastrophe), whereas the field behind the wave front is a superposition of Thomson's modes with constant amplitudes. Thus, Love's paper shows the way to find a proper physical interpretation of Thomson's modes, for a particular type of excitation. However, it does not indicate how the results depend on the excitation.

The solution of Thomson's problem for an arbitrary initial field was given by HILL and GELBAUM [6], in the form of an expansion in stationary states. However, the connection between this form of the solution and Thomson's modes was not discussed.

The method of complex eigenvalues was introduced in quantum mechanics by GAMOW [7], in connection with the theory of alpha-decay. The "complex-energy wave functions" which correspond to Thomson's modes are associated with "decaying states". The exponential catastrophe is also found in this case. Methods for dealing with this difficulty have been suggested in many papers. The usual method [8,9,10,11,12] is to consider the decay of a wave packet which is initi-

6. E.L. Hill & B. Gelbaum: unpublished (1954). We are indebted to Professor W.B. Cheston for bringing this paper to our attention. We wish to thank Professor E.L. Hill for sending us a copy of the manuscript.

7. G. Gamow: Z. Phys., 51, 204 (1928).

8. O.K. Rice: Phys. Rev., 35, 1538 (1930).

9. H.B.G. Casimir: Physica (Haag), 1, 193 (1934).

10. G. Breit & F.L. Yost: Phys. Rev., 48, 205 (1935).

11. B. Breit: Handbuch der Physik, Bd. XLI/1 (Berlin 1959), p. 28.

12. A.M. Lane & R.G. Thomas: Rev. Mod. Phys., 30, 257 (1958), 343.

ally concentrated within the nucleus. For a Schrödinger particle, in contrast with the electromagnetic case, there is no limiting velocity, and an outgoing wave packet with a sharp front is impossible[13]. What one tries to show, then, is that the wave function is very small for $r \gg vt$, and differs very little from the "wave function of a decaying state" for $r \ll vt$, where v is a mean velocity associated with the emitted particle. This has been done, however, only for special choices of the wave packet, and the approximations are valid only for long-lived and widely separated "decaying states".

A general relation between the decay law and the energy spectrum of the initial state was given by KRYLOV and FOCK[14]. However, their definition of "decay law" cannot be accepted without restrictions.

A time-dependent theory of resonance reactions was given by MOSHINSKY[15], and applied by LOZANO to the problems of decay[16] and of scattering by a potential[17]. The connection between Moshinsky's approach and that of the present paper will be discussed later.

Complex eigenvalues have also been employed in the theory of emission of light[18] and in the theory of unstable elementary par-

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13. N. G. Van Kampen: Phys. Rev., 91, 1267 (1953).

14. N.S. Krylov & V.A. Fock: J. Exptl. Theoret. Phys. (U.S.S.R.), 17, 93 (1947). See also L.A. Khalfin: Soviet Physics (JETP), 6, 1053 (1958).

15. M. Moshinsky: Phys. Rev., 84, 525 (1951).

16. J.M. Lozano: Rev. Mex. Fis., 3, 63 (1954).

17. J.M. Lozano: Rev. Mex. Fis., 2, 155 (1953).

18. V.F. Weisskopf & E.P. Wigner: Z. Physik, 63, 54 (1930); 65, 18 (1930); W. Heitler: The Quantum Theory of Radiation, 3rd. ed. (Oxford, 1954).

ticles [19], but these problems will not be considered here.

In the present paper, we shall investigate three examples of transients in continuous systems. The excitation conditions will be taken into account by looking for the general solution of the initial-value problem. The treatment is based on an extension of the standard methods which are employed in the case of discrete systems. In this way, it is possible to obtain a rigorous foundation for the method of complex eigenvalues. The "complex eigenvalues" are the poles of the S-matrix. Thus, the main questions to be considered are: (a) What is the relation between the transient behaviour of the system and the poles of the S-matrix? (b) How does the behaviour of the system depend on the excitation?

The first example (section 2) is the problem of a harmonic oscillator coupled with a vibrating string, a special case of which was solved by Lamb [4]. This is one of the simplest illustrations of the theory, and it is particularly suitable for explaining the method. The second example (section 3) is Thomson's problem of the perfectly conducting sphere, which is of special interest in connection with antenna theory. The third example (section 4) is the analogue of Thomson's problem in non-relativistic quantum mechanics, i.e., the initial-value problem for a hard sphere.

In each case, we shall find the general solution of the initial-value problem. The solution will be expressed in terms of propagators, which are very convenient for visualizing the results. The The only parameters which appear in the solution are the poles of the

19. See G. Hohler: Z. Physik, 152, 546 (1958), where further references are given.

S-matrix. Their role is similar to that of the "complex eigenfrequencies" of discrete systems. A propagator may be associated with each pole of the S-matrix. These "propagators of transient modes" are closely related to the "complex-frequency wave functions" which are employed in the method of complex eigenvalues, but the excitation at a definite instant introduces a cut-off factor, which eliminates the exponential catastrophe.

The idea of exponential decay, which is usually associated with the method of complex eigenvalues, may be applied, in general, only to the propagators, and not to the actual wave function. In the case of short-lived modes (e.g. in Thomson's problem or in the hard-sphere problem),the decay law depends very strongly on the excitation. In the Schrodinger case, one must also take into account the effect of the spreading of wave packets; as will be shown in section 4, this introduces further limitations on the domain of validity of the exponential decay law.

The origin of the "transient modes" will also be discussed. It will be seen that there are significant differences, in this respect, between the first example and the others.

It will be shown in the second part that the "expansion in transient modes" introduced in this paper may have as a limiting case an expansion in stationary states. In general, however, the transient modes are not even approximately orthogonal, and one cannot asscribe an independent physical meaning to each term in the expansion.

2. - Vibrating string and oscillator.

The problem which will be treated in this section may be formulated as follows: a harmonic oscillator is attached to the extremity of a semi-definite string; given the initial displacement and velocity of the oscillator and the string, it is required to determine the subsequent motion. This differs only slightly from Lamb's example [4]. However, while Lamb restricted himself to the case in which the string is at rest, and a sudden blow is given to the oscillator, we shall consider an arbitrary initial excitation.

Let the rest position of the string coincide with the positive x-axis, and let $y(x,t)$ denote the transverse displacement of the string. We shall assume that the oscillator is constrained to move only in the y direction, so that $y(0,t)$ represents the displacement of the oscillator. Let m denote the mass of the oscillator, and ω_o its natural frequency. Let T be the tension of the string, and let us define $\gamma = T/m$. If we take the wave velocity in the string to be unity, its equation of motion is

(1) $$\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0 \quad (x > 0).$$

The equation of motion of the oscillator

(2) $$\left(\frac{d^2}{dt^2} + \omega_o^2\right) y(0,t) = \gamma \frac{\partial y}{\partial x}(0,t)$$

may be considered as a boundary condition for the motion of the string. Let the initial conditions be

(3) $$y(x,0) = u(x) \; ; \quad \frac{\partial y}{\partial t}(x,0) = v(x) \qquad (x \geqslant 0).$$

The stationary solutions of (1) and (2) may be written as (up to a constant factor)

(4) $$y(x,t,\omega) = \exp\left[-i\omega(x+t)\right] - S(\omega)\exp\left[i\omega(x-t)\right] \quad ,$$

where

$$(5) \qquad S(\omega) = \frac{\omega^2 - \omega_0^2 - i\gamma\omega}{\omega^2 - \omega_0^2 + i\gamma\omega} = \frac{(\omega + \omega_1)(\omega + \omega_2)}{(\omega - \omega_1)(\omega - \omega_2)}$$

is the S-matrix (in this case, an ordinary function of ω), which satisfies the well-known unitarity and symmetry conditions. The parameters

$$(6) \qquad \omega_{1,2} = \pm(\omega_0^2 - \tfrac{1}{4}\gamma^2)^{\frac{1}{2}} - \tfrac{1}{2} i \gamma$$

are the poles of $S(\omega)$, which, in agreement with causality, are located in the lower half of the complex ω-plane. According to the method of complex eigenvalues, they represent the "complex eigenfrequencies" of the system.

The general solution of (1) and (2) may be expressed as a superposition of stationary solutions (Fourier integral). The expansion coefficients have to be determined by the requirement that the solution must satisfy the initial conditions (3). It may be seen that although the stationary solutions do form a complete set [20], they are not orthogonal in this case. The physical reason for this is the additional degree of freedom due to the presence of the oscillator. In spite of the non-orthogonality, however, it is still possible to find formulae for the evaluation of the expansion coefficients.

Perhaps the most obvious way of finding the connection with the method of complex eigenvalues would be to deform the path of integration in the Fourier integral into the complex ω-plane. The connection would appear by taking the residues of the integrand at the poles of $\underline{S}(\omega)$. We shall, however, follow a different procedure, which, besides being much simpler, leads more directly to the physical interpretation. This procedure is an extension of d'Alembert's clas-

20. N.G.Van Kampen: Physica, 21, 127 (1955).

sical solution of the Cauchy problem for the vibrating string.

The general solution of (1) is

(7) $$y(x,t) = f(x-t) + g(x+t),$$

where f and g are arbitrary functions of their arguments. To satisfy the initial conditions (3), it suffices to take

(8) $$f(x) = \tfrac{1}{2} u(x) - \tfrac{1}{2} \int_0^x v(x')\, dx' \qquad (x \geqslant 0),$$

(9) $$g(x) = \tfrac{1}{2} u(x) + \tfrac{1}{2} \int_0^x v(x')\, dx' \qquad (x \geqslant 0).$$

We may add an arbitrary constant to f , and subtract the same constant from g , without modifying the results. However, the choice of the lower limit 0 in the above integrals simplifies the subsequent calculations.

Since we are interested in the solution of the initial - value problem for $t > 0$, the function $g(x+t)$ in (7) is completely defined by (9). However, $f(x-t)$ is defined by (S) only if $x \geqslant t$. In this case, (7) becomes

(10) $$y(x,t) = \tfrac{1}{2}\left[u(x+t) + u(x-t)\right] + \tfrac{1}{2} \int_{x-t}^{x+t} v(x')dx' \quad (x \geqslant t).$$

which is the well-known d'Alembert solution, corresponding to free propagation in the string.

In order to determine the solution for $x < t$, we need the continuation of f to negative values of its argument. Physically, this means that we must find the "reflected wave", i.e. that part of the outgoing wave which has interacted with the oscillator. For this purpose, we must employ the boundary condition. Replacing (7) in (2), and introducing the notation $\overline{f}(t) = f(-t)$, we find

(11) $\bar{f}''(t) + \gamma \bar{f}'(t) + \omega_o^2 \bar{f}(t) = -g''(t) + \gamma g'(t) - \omega_o^2 g(t) \quad (t>0)$,

where the primes stand for derivatives with respect to the argument. According to (9), the second member of (11) is known, so that this is an ordinary differential equation for the unknown function $\bar{f}(t)$.

Since the displacement of the oscillator is given by $y(0,t) = y_o(t) = \bar{f}(t) + g(t)$, (11) may be rewritten as

(12) $y_o''(t) + \gamma y_o'(t) + \omega_o^2 y_o(t) = 2\gamma g'(t) = \gamma[u'(t) + v(t)]$.

This is the equation of motion of a damped harmonic oscillator, with natural frequency ω_o and damping constant γ, subject to the external driving force $2m\gamma g'(t)$ (*). Thus, <u>the effect of the coupling to the string on the motion of the oscillator is equivalent to a damping term (radiation damping) and a given external force, due to the incoming waves</u>.

Introducing the notations $u(0) = u_o$ and $v(0) = v_o$ for the initial displacement and initial velocity of the oscillator, respectively, we find from (8) and (9)

(13) $\bar{f}(0) = g(0) = \frac{1}{2}u_o$; $\bar{f}'(0) = \frac{1}{2}[v_o - u'(0)]$; $g'(0) = \frac{1}{2}[v_o + u'(0)]$.

The solution of (11) subject to the conditions (13) is a typical problem of the theory of transients in discrete systems, to

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(*) The parameter γ plays the role of a coupling constant between the oscillator and the string. According to (6), for $\gamma \ll 2\omega_o$ (weak coupling), the poles of $S(\omega)$ are located very close to the real axis, below the points $\pm\omega_o$. When γ increases from 0 to $2\omega_o$, they approach the negative imaginary axis (moving along the half-circle of radius ω_o and center at the origin), joining each other at the point $-i\omega_o$ for $\gamma = 2\omega_o$ (critical damping). For $\gamma > 2\omega_o$, the poles move in opposite directions along the negative imaginary axis; one of them approaches the origin, while the other tends to $-i\infty$.

which the standard Laplace transformation method [1] may be applied. If $F(p)$, $G(p)$, denote the Laplace transforms of $\bar{f}(t)$, $g(t)$, respectively, the Laplace transform of (11) becomes

$$(14)\quad F(p) = -\frac{(p+p_1)(p+p_2)}{(p-p_1)(p-p_2)}\, G(p) + \frac{pu_o + v_o}{(p-p_1)(p-p_2)} =$$

$$= \left[-1 + \sum_{j=1}^{2} \frac{a_j}{(p-p_j)}\right] G(p) - \tfrac{1}{2}\,(p_1+p_2)^{-1} \sum_{j=1}^{2} \frac{a_j}{(p-p_j)} \left(u_o + \frac{v_o}{p_j}\right),$$

where

$$(15)\qquad p_j = -i\,\omega_j \quad (j = 1, 2),$$

and

$$(16)\quad a_j = -2p_j(p_j+p_k)(p_j-p_k)^{-1} = i \cdot \text{residue}\,[S(\omega)]_{\omega=\omega_j}\ (j=1,2;\ k\neq j)$$

The inverse Laplace transform of (14) is

$$(17)\quad \bar{f}(t) = -g(t) + \sum_{j=1}^{2} a_j \exp(p_j t) * g(t) - \tfrac{1}{2}(p_1+p_2)^{-1}\,\cdot$$

$$\cdot \sum_{j=1}^{2} a_j\left(u_o + \frac{v_o}{p_j}\right)\exp(p_j t),$$

where

$$(18)\quad f_1(t)*f_2(t) = \int_0^t f_1(t')f_2(t-t')dt' = \int_0^t f_1(t-t')f_2(t')dt'$$

is the convolution product.

It follows from (7), (9) and (17) that

$$(19)\quad y(x,t) = \tfrac{1}{2}\left[u(t+x)-u(t-x)\right] + \tfrac{1}{2}\int_{t-x}^{t+x} v(x')dx' + \tfrac{1}{2}\sum_{j=1}^{2} a_j \exp[p_j(t-x)]\,\cdot$$

$$\cdot\left\{\int_0^{t-x} \exp(-p_j x')\left[u(x') + \frac{v(x')}{p_j}\right]dx' - (p_1+p_2)^{-1}\left(u_o + \frac{v_o}{p_j}\right)\right\} \quad (0 \leq x < t).$$

Equations (10) and (19) give the general solution of the problem. The case treated by Lamb, in which the string is initially at rest and an impulse is given to the oscillator, corresponds to the initial conditions

(20) $u(x) = 0 \ (x \geqslant 0); \quad v(x) = 0 \ (x > 0); \quad v_o \neq 0$.

Substituting this in the solution, we find, in exact agreement with Lamb's result,

(21) $$y(x,t) = -\tfrac{1}{2}(p_1 + p_2)^{-1} v_o \sum_{j=1}^{2} \frac{a_j}{p_j} H(t-x)\exp[p_j(t-x)] \ ,$$

where $H(t)$ denotes Heaviside's step function, $H(t) = 0 \ (t < 0; \ H(t) = 1 \ (t > 0)$. For $x < t$, each term of (21) is of the form usually associated with a "complex-frequency wave function". However, there is no exponential catastrophe, because the step function introduces a sharp cut-off at the wave front. This corresponds to the excitation at a definite instant.

We may consider the situation described by (21) as the analogue of an emission process, in which the kinetic energy initially concentrated in the oscillator is gradually propagated to the string.

According to (12), the terms which contain u_o and v_o in (19) may be regarded as the ordinary transients associated with the initial displacement and velocity of the oscillator, which are transmitted to the string. To interpret the remaining terms, which arise from the forced motion of the oscillator, it suffices to consider the case in which $u_o = v_o = 0$. Under these circumstances, (10) and (19) may be rewritten as follows:

(22) $$y(x,t = \int_o^{\infty} [G_-(x,x',t)\ f(x') + G_+(x,x',t)g(x')]dx' \ ,$$

where f and g are given by (8) and (9),

(23) $$G_-(x,x',t) = \delta(x-x'-t) ,$$

(24) $$G_+(x,x',t) = \delta(x-x'xt) - \delta(x+x'-t) + \sum_{j=1}^{2} a_j H(t-x-x')\exp[p_j(t-x-x')] ,$$

and $\delta(x)$ is Dirac's delta function.

By means of (8) and (9), the initial wave function is decomposed into an "outgoing part" f and an "incoming part" g. The subsequent behaviour of f and g is determined by the kernels G_- and G_+ of the integral transformation (22). For this reason, we shall call G_- the propagator of the outgoing part, and G_+ the propagator of the incoming part *.

If we take the special (purely symbolic) initial conditions

(25) $$u(x) = \delta(x-x_o) \; ; \; v(x) = 0 ,$$

(22) becomes

(26) $$y(x,t) = \tfrac{1}{2} G_-(x,x_o,t) + \tfrac{1}{2} G_+(x,x_o,t) .$$

Thus, in this case, the initial pulse splits into two identical pulses, which move in opposite directions. According to (23), the outgoing pulse propagates freely (as it would do in an unlimited string), as ought to be expected. According to (24), the same applies to the incoming pulse before it strikes the oscillator ($t < x_o$). For $t > x' = x_o$ the first term in the second member of (24) vanishes. The remaining terms represent the reflected wave, which consists of two parts: (a) an inverted "mirror image" of the incoming pulse; (b) an "exponential

* It should be understood that the names "outgoing" and "incoming" refer only to to the initial situation; subsequently, the "incoming part" gives rise to an outgoing (reflected) wave.

tail", similar to (21), which is due to the excitation of the oscillator transients by the incoming pulse [(+)].

These results may also be visualized by introducing an "image space", i.e., a fictitious continuation of the string for $x < 0$. The initial situation in real space and in image space corresponding to (26) is depicted in figure 1, which shows the break-up of the initial pulse at x_o, and the mirror image of the incoming pulse at $-x_o$, followed by the exponential tail. If we let the different parts of this initial configuration propagate freely (in the direction of the arrows in figure 1), the resulting wave function in real space is identical to (26).

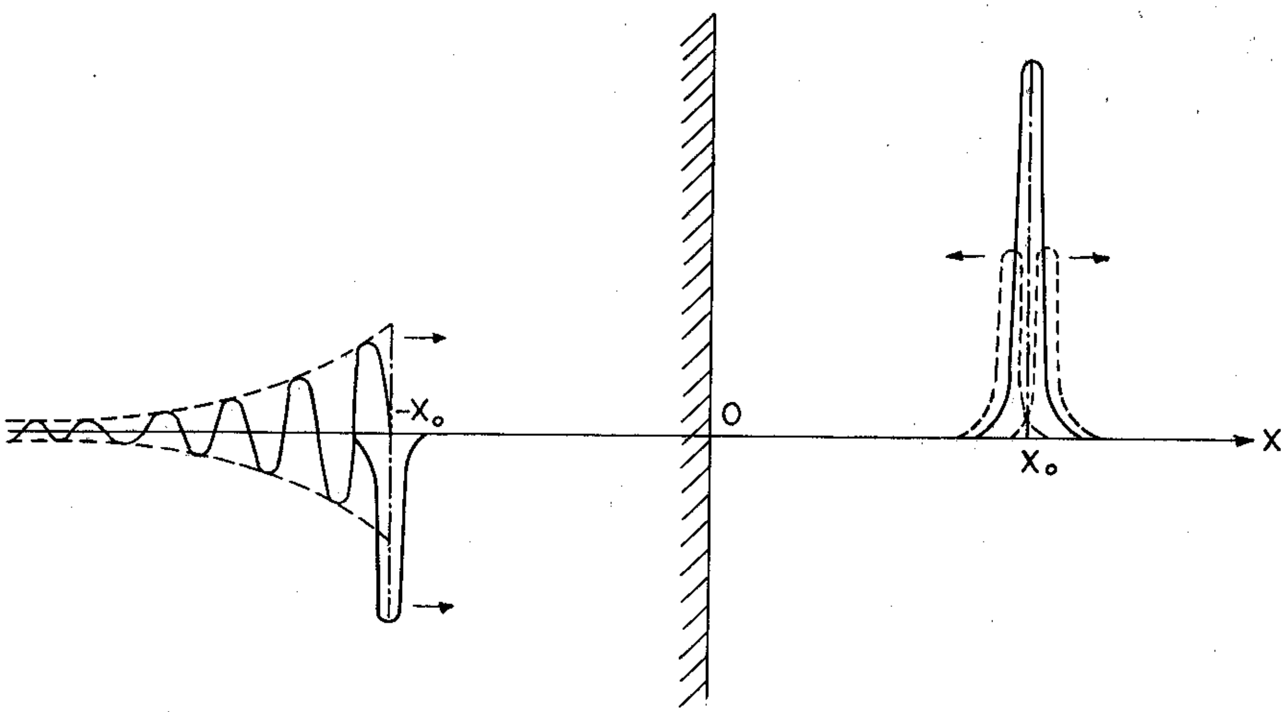

Fig. 1 - An initial pulse at x_o in real space splits into two equal pulses which propagate in opposite directions; the incoming pulse gives rise in image space to a mirror image at $-x_o$, followed by an exponential tail.

(+) It may readily be verified that, in the limiting case $m \to \infty$ ($m \to 0$), which corresponds to a fixed extremity (free extremity), we obtain from (24) the ordinary reflection with change of sign (without change of sign). In the case $m \to 0$ we must employ the relation: $\lim_{\gamma \to \infty} [\gamma H(z) \exp(-\gamma z)] = \delta(z)$.

According to (24), to each pole ω_j of $\underline{S}(\omega)$ corresponds a term of the form

$$G_j(x,x',t) = a_j H(t-x-x')\exp[p_j(t-x-x')] \quad (27)$$

in the propagator of the reflected wave. We shall call $G_j(x,x',t)$ <u>the propagator of the transient mode</u> associated with the pole ω_j. Notice that, according to (16), the factor a_j is completely determined by the poles of the $\underline{S}$-matrix. The remarks which were made in connection with (21) may be applied just as well to (27). In this way, we may give a rigorous meaning to the method of complex eigenvalues, for an arbitrary initial excitation: <u>the "complex-frequency wave functions", with a suitable cut-off factor (which eliminates the exponential catastrophe), correspond precisely to the propagators of the transient modes</u>.

Owing to the "exponential tail" in (24), <u>the form of the reflected wave at a given moment depends on the whole previous history, i.e., on the entire portion of the incoming wave which has stricken the oscillator up to that moment</u>. It is only in very special cases that terms of the form of (27) will appear in the reflected wave. This happens in Lamb's illustrations of an "emission process" (cf. 21)). According to (25) and (26), it also happens in the case of excitation by a very sharp pulse. More generally, if the initial disturbance vanishes for $x > x_0$, the reflected wave will be of this form for $t-x > x_0$, i.e., after the whole incoming wave packet has stricken the oscillator. However, these are special cases, and we may conclude that, as a rule (for an arbitrary excitation), no trace of exponential behaviour will appear in the reflected wave.

3. - Spherical antenna.

The next problem which we shall consider is Thomson's problem of the "free oscillations" of a perfectly conducting spherical antenna. We shall study the decay of an arbitrary initial field. Our problem is therefore to find a solution of Maxwell's equations in the exterior of a perfectly conducting sphere, which satisfies the boundary conditions on the surface of the sphere and given initial conditions.

The electromagnetic field outside the sphere may be represented in terms of two scalar functions, the well-known Debye potentials [21,22]. The general solution of Maxwell's equations in free space is a superposition of "electric" (E) and "magnetic" (M) solutions, corresponding to the Debye potentials $\Pi^E(\underline{r},t)$ and $\Pi^M(\underline{r},t)$, respectively. The general form of the E-solution is (we take $c = 1$)

$$(28) \qquad \underline{E}(\underline{r},t) = \text{rot rot}(\underline{r}\,\Pi^E); \; \underline{H}(\underline{r},t) = \frac{\partial}{\partial t}\,\text{rot}(\underline{r}\,\Pi^E).$$

The general form of the M-solution is obtained from (28) by substituting:

$$(29) \qquad \Pi^E \rightarrow \Pi^M \; ; \; \underline{E} \rightarrow \underline{H} \; ; \; \underline{H} \rightarrow -\underline{E} \, .$$

Both Debye potentials satisfy the scalar wave equation

$$(30) \qquad (\Delta - \partial^2/\partial t^2)\,\Pi^{E,M}(\underline{r},t) = 0.$$

We shall employ spherical coordinates, $\underline{r} = (r, \theta, \varphi)$, with origin at the center of the sphere. Let a be the radius of the sphere. The boundary condition at the surface of the sphere (vanishing of the tangential component of the electric field) may be expres-

21. P.J.W. Debye: Ann. Physik, 30, 57 (1909).
22. C.J. Bouwkamp & H.B.G. Casimir: Physica, 20, 539 (1954).

sed as follows:

(31) $\Pi^M(\underset{\sim}{r},t) = 0$ for $r = a$,

(32) $\frac{\partial}{\partial r}\left[r\,\Pi^E(\underset{\sim}{r},t)\right] = 0$ for $r = a$.

According to (30), the initial-value problem is determined by the following initial conditions.

(33) $\Pi^{E,M}(\underset{\sim}{r},0) = U^{E,M}(\underset{\sim}{r})$; $\frac{\partial \Pi^{E,M}}{\partial t}(\underset{\sim}{r},0) = V^{E,M}(\underset{\sim}{r})$ $(r \geqslant a)$.

Let us introduce the multipole expansion of the Debye potentials

(34) $\Pi^{E,M}(\underset{\sim}{r},t) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\psi^{E,M}_{\ell m}(r,t)Y_{\ell m}(\theta,\varphi)$.

It follows from (28) and (29) that

(35) $rE_{\underset{\sim}{r}}(r,t) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\ell(\ell+1)\psi^{E}_{\ell m}(r,t)Y_{\ell m}(\theta,\varphi)$,

(36) $rH_r(r,t) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\ell(\ell+1)\psi^{M}_{\ell m}(r,t)Y_{\ell m}(\theta,\varphi)$,

so that the initial-value problem may be formulated just as well in terms of the radial components of $\underset{\sim}{E}$ and $\underset{\sim}{H}$, which are closely related to the Debye potentials [22].

The initial values (33) have corresponding expansions

(37) $U^{E,M}(\underset{\sim}{r}) = \sum_{\ell=1}\sum_{m=-\ell}u^{E,M}_{\ell m}(r)Y_{\ell m}(\theta,\varphi)$; $V^{E,M}(\underset{\sim}{r}) =$

$$= \sum_{\ell=1}\sum_{m=-\ell}v^{E,M}_{\ell m}(r)Y_{\ell m}(\theta,\varphi),$$

provided that we restrict ourselves to functions having zero mean value when averaged over all solid angles(*). This restriction, as

(*) We thereby exclude from our consideration the trivial case of the electrostatic field due to a charged sphere.

well as the corresponding omission of the term $\ell = 0$ from (34) to (37) is related to the non-existence of radiating monopoles [22].

Substituting (34) and (37) in (30) to (33), we find separate initial-value problems for each value of ℓ and m. We may restrict ourselves in what follows to one particular value of ℓ and m. In order not to encumber the notation, we shall omit both of these subscripts. We shall also omit the superscripts E and M, except where it is necessary to draw the attention to differences between electric and magnetic multipoles; most of the results apply equally well to both cases.

The initial-value problem for an electric or magnetic multipole of order ℓ then becomes:

(38) $$\left[\frac{\partial^2}{\partial r^2} - \ell(\ell+1)r^{-2} - \frac{\partial^2}{\partial t^2}\right][r\,\psi(r,t)] = 0 \qquad (r > a);$$

(39) $$\psi(r,0) = u(r)\ ; \quad \frac{\partial\psi}{\partial t}(r,0) = v(r) \qquad (r \geqslant a);$$

(40) $$\psi^M(a,t) = 0\ ;$$

(41) $$\left\{\frac{\partial}{\partial r}\left[r\,\psi^E(r,t)\right]\right\}_{r=a} = 0\ .$$

The normalized stationary solutions of (38) which satisfy the boundary conditions are

(42) $$F(k,r)\exp(-ikt) = (2\pi)^{-\frac{1}{2}}[h_\ell^{(2)}(kr) + S(k)h_\ell^{(1)}(kr)]\ \exp(-ikt),$$

where $h_\ell^{(1)}(z)$, $h_\ell^{(2)}(z)$ are the spherical Hankel functions [23].
The S-functions for magnetic and electric multipoles are given by

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23. P. M. Morse & H. Feshbach: Methods of Theoretical Physics (New York, 1953) p. 1573.

(43) $$S^M(k) = -[h_\ell^{(2)}(ka)]/[h_\ell^{(1)}(ka)] ,$$

(44) $$S^E(k) = -[kah_\ell^{(2)}(ka)]'/[kah_\ell^{(1)}(ka)]' ,$$

where the primes stand for derivatives with respect to ka.

The well-known relations

(45) $$S^*(k)S(k^*) = S(k)S(-k) = 1$$

follow from (43) and (44) and from the properties of the spherical Hankel functions.

We have

(46) $$h_\ell^{(1)}(z) = z^{-1}\exp(iz) \sum_{n=0}^{\ell} \frac{i^{n-\ell-1}(\ell+n)!}{n!(\ell-n)!}(2z)^{-n} =$$

$$= z^{-\ell-1}p_\ell(z)\exp(iz) ,$$

where $p_\ell(z)$ is a polynomial of degree ℓ in z. Similarly

(47) $$[zh_\ell^{(1)}(z)]' = z^{-\ell-1}q_{\ell+1}(z)\exp(iz) ,$$

where $q_{\ell+1}(z)$ is a polynomial of degree $\ell + 1$ in z.

It follows from (43) to (47) that $S^M(k)$, $S^E(k)$ are meromorphic functions of the complex variable k. Their poles are roots of the equations

(48) $$h_\ell^{(1)}(ka) = 0 \qquad (M),$$

(49) $$[kah_\ell^{(1)}(ka)]' = 0 \qquad (E) .$$

According to (46) and (47), (48) has exactly ℓ roots, and (49) has exactly $\ell + 1$ roots.

It will be shown in appendix A that all the roots of (48) and (49) are simple, and that they are located in the lower half of the k-plane. Therefore, $S^M(k)$ has exactly ℓ poles and $S^E(k)$ has exactly $\ell + 1$ poles; all the poles are simple and lie in the lower half-plane. The non-existence of poles in the upper half-plane also follows from the causality conditions [24]. According to (45), if k_n is a pole, so is $-k_n^*$, while k_n^* and $-k_n$ are zeros. Therefore, the poles lie symmetrically with respect to the imaginary axis (*).

Let k_j^M ($j = 1, 2, \ldots, \ell$) and k_j^E ($j = 1, 2, \ldots, \ell + 1$) be the poles of $S^M(k)$ and $S^E(k)$, respectively. Then, it follows from (43) to (47) that

(50) $$S^M(k) = (-1)^{\ell} \exp(-2ika) \prod_{j=1}^{\ell} (k + k_j^M)(k - k_j^M)^{-1} ,$$

(51) $$S^E(k) = (-1)^{\ell+1} \exp(-2ika) \prod_{j=1}^{\ell+1} (k + k_j^E)(k - k_j^E)^{-1} .$$

These are the canonical product expansions of the S function, for a scatterer of range a and a finite number of poles [24].

The poles of $S^M(k)$ and of $S^E(k)$ for the first few values of ℓ are given in table I [25].

TABLE I

The poles of $S^M(k)$ and of $S^E(k)$

ℓ =	1	2	3
ak_j^M =	-1	$\pm \frac{1}{2}\sqrt{3} - \frac{3}{2} i$	$-2.26i$ $\pm 1.75 - 1.87i$
ak_j^E =	$\pm \frac{1}{2}\sqrt{3} - \frac{1}{2}i$	$-1.60i$	$\pm 0.87 - 2.17i$
		$\pm 1.81 - 0.70i$	$\pm 2.77 - 0.83i$

(*) The results which have been proved so far on the poles of $S^M(k)$ are particular cases of theneral theorems on the zeros of $H_\nu^{(1)}(z)$ for $\nu \geq 0$, which are due to H. Falkenberg & E. Hilb (Göttinger Nachrichten, 190 (1916)) and H. Falkenberg (Math. Z., 35, 457 (1932). It also follows from their investigations that $S^M(k)$ has no poles on the negative imaginary axis for even ℓ, whereas, for odd ℓ, it has one and only one imaginary pole.

24. N.G. Van Kampen: Phys.Rev., 90, 1072 (1953).
25. J.A. Stratton: Electromagnetic Theory (New York, 1941), p. 559. A graphical re

The functions $krF(k,r)(0 \leqslant k < \infty)$, where $F(k,r)$ has been defined in (42), form a complete orthonormal set in $a \leqslant r \leqslant \infty$. This allows us to solve the initial-value problem by means of an expansion in stationary states. The general solution of (38) which satisfies the boundary conditions is

$$(52) \qquad \psi(r,t) = \int_0^\infty [a(k)\cos(kt) + b(k)\sin(kt)]F(k,r)k^2 dk .$$

The initial conditions (39) will be satisfied if we take

$$(53) \qquad a(k) = \int_a^\infty u(r')F^*(k,r')r'^2 dr', kb(k) = \int_a^\infty v(r')F^*(k,r')r'^2 dr'.$$

Replacing these results in (52), we get

$$(54) \qquad \psi(r,t) = \tfrac{1}{2}\int_{-\infty}^{+\infty} dk \int_a^\infty dr'(kr')^2[u(r') + ik^{-1}v(r')] \cdot$$
$$\cdot F^*(k,r')F(k,r)\exp(-ikt).$$

This is equivalent to the result obtained by Hill and Gelbaum [6].

The connection with the method of complex eigenvalues may be found by inverting the order of integration in (54) and evaluating the integral with respect to k by contour integration; this gives rise to residues at the poles of $S(k)$. It is much simpler, however, to apply the method of section 2, which leads to an extension of Love's treatment [5].

The general solution of the multipole wave equation (38) is [26]

$$(55) \qquad \psi(r,t) = r^\ell D_r^\ell [r^{-1}\varphi(r,t)] ,$$

presentation of the poles of $S^M(k)$ may be found in E. Jahnke & F. Emde: Tables of Functions (New York, 1945), p.243, fig. 129:

26. H. Lamb: Hydrodynamics, 6th ed. (Cambridge, 1953), p. 522

where D_r^ℓ is the differential operator

(56) $$D_r^\ell = \left(\frac{\partial}{r\partial r}\right)^\ell$$

and $\varphi(r,t)$ is the general solution of the one-dimensional wave equation

(57) $$\varphi(r,t) = f(r-t) + g(r+t) .$$

Equation (55) is related to the well-known process of generating a multipole by repeated differentiation of a monopole.

The function $\varphi(r,t)$ is not uniquely determined by (55), for we may add to it an arbitrary solution $\chi(r,t)$ of the homogeneous equation

(58) $$D_r^\ell \left[r^{-1}\chi(r,t)\right] = 0 .$$

The general solution of this equation is

(59) $$\chi(r,t) = \sum_{n=0}^{\ell-1} A_n(t) r^{2n+1} ,$$

where $A_o(t),\ldots, A_{\ell-1}(t)$ are arbitrary functions of t. If we restrict ourselves to solutions of the form (57), there is still some arbitrariness in the choice of the functions f and g. In fact, according to (59), the pair $f(r-t)$, $g(r+t)$ is equivalent to the pair

(60) $$f(r-t) + \sum_{n=0}^{2\ell}(-1)^{n+1}C_n(r-t)^n , \quad g(r+t) + \sum_{n=0}^{2\ell} C_n(r+t)^n ,$$

where $C_o,\ldots, C_{2\ell}$ are $2\ell+1$ arbitrary constants. We may take advantage of this arbitrariness to choose the functions f and g in such a way that the solution takes the simplest possible form. We shall determine these functions by the following $2\ell+1$ conditions:

(61) $$f^{(j)}(a) = g^{(j)}(a) = 0 \quad (j = 0, 1,\ldots, \ell-1);$$

$$f^{(\ell)}(a) - g^{(\ell)}(a) = 0,$$

where $f^{(j)}(a)$ denotes the jth derivative of f, evaluated at the point a, and similarly for $g^{(j)}(a)$.

Replacing (55) to (57) in the initial conditions (39), we get

(62) $$D_r^{\ell}\left\{r^{-1}[f(r)+g(r)]\right\} = r^{-\ell}u(r) \quad (r \geqslant a),$$

(63) $$D_r^{\ell+1}[g(r)-f(r)] = r^{-\ell}v(r) \quad (r \geqslant a).$$

The choice of the homogeneous conditons (61) greatly simplifies the solution of these equations. If we define the operator $D_r^{-\ell}$ applied to a function $w(r)$ by

(64) $$D_r^{-\ell}[w(r)] = \int_a^r r_1 dr_1 \int_a^{r_1} r_2 dr_2 \ldots \int_a^{r_{\ell-1}} w(r_\ell) r_\ell \, dr_\ell \quad (\ell \text{ integrations}),$$

it follows from (61) to (63) that

(65) $$f(r) = \tfrac{1}{2} r D_r^{-\ell}[r^{-\ell}u(r)] - \tfrac{1}{2} D_r^{-\ell-1}[r^{-\ell}v(r)] \quad (r \geqslant a),$$

(66) $$g(r) = \tfrac{1}{2} r D_r^{-\ell}[r^{-\ell}u(r)] + \tfrac{1}{2} D_r^{-\ell-1}[r^{-\ell}v(r)] \quad (r \geqslant a).$$

Equation (66) determines the function $g(r+t)$ for all $t \geqslant 0$, whereas $f(r-t)$ is determined by (65) only for $r-t \geqslant a$. To find the "reflected wave" $f(r-t)$ $(r-t < a)$, we must apply the boundary condition.

It follows from (55) to (57) that

(67) $$\psi(r,t) = \sum_{n=0}^{\ell} (-1)^n c_{\ell n} r^{-n-1}\left[f^{(\ell-n)}(r-t) + g^{(\ell-n)}(r+t)\right],$$

where

(68) $$c_{\ell n} = [2^n n!(\ell-n)!]^{-1}(\ell+n)! \;.$$

In the case of magnetic multipoles, we must apply (40), which gives

(69) $$\sum_{n=0}^{\ell} c_{\ell n} a^{-n-1} f^{(\ell-n)}(t) = (-1)^{\ell+1} \sum_{n=0}^{\ell} (-1)^n c_{\ell n} a^{-n-1} g^{(\ell-n)}(t),$$

where

(70) $\bar{f}(t) = f(a-t)$; $\bar{g}(t) = g(a+t)$.

According to (61), $\bar{f}(t)$ and $\bar{g}(t)$, as well as their derivatives up to the order $\ell-1$, vanish for $t = 0$. To solve (69) under these conditions we shall apply the Laplace transformation. Let $F(p)$, $G(p)$, be the Laplace transforms of $\bar{f}(t)$, $\bar{g}(t)$. Then, according to (43) and (46), the Laplace transform of (69) may be written as follows:

(71) $$F(p) = -\exp(-2ap)S^M(iap)G(p) .$$

Taking into account (50), we get

(72) $$\exp(-2ap)S^M(iap) = (-1)^\ell \prod_{j=1}^{\ell} (p+p_j^M)(p-p_j^M)^{-1} = (-1)^\ell - \sum_{j=1}^{\ell} a_j^M(p-p_j^M)^{-1} ,$$

where

(73) $$p_j^M = -ik_j^M ,$$

(74) $$a_j^M = (-1)^{\ell+1} 2p_j^M \prod_{k\neq j}^{\ell} (p_j^M + p_k^M)(p_j^M - p_k^M)^{-1} = i \cdot \text{residue}[\exp(2ika)S^M(k)]k=k_j .$$

Replacing (72) in (71), and applying the inverse Laplace transformation, we finally obtain

(75) $$\bar{f}^M(t) = (-1)^{\ell+1}\bar{g}^M(t) + \sum_{j=1}^{\ell} a_j^M \exp(p_j^M t) * \bar{g}^M(t) \quad (t>0),$$

where we have employed the notation for the convolution product, defined in (18). Equations (66), (70) and (75) determine the reflected wave.

An entirely similar calculation gives, for electric multipoles,

(76) $$\bar{f}^E(t) = (-1)^\ell \bar{g}^E(t) + \sum_{j=1}^{\ell+1} a_j^E \exp(p_j^E t) * \bar{g}^E(t) \quad (t > 0),$$

where p_j^E and a_j^E may be obtained from (73) and (74) by replacing M by E and ℓ by $\ell+1$ (cf. (51)).

Equations (55) to (57), (65), (66), (75) and (76) give the general solution of our initial-value problem. As was done in the previous section, we may rewrite the solution in terms of propagators:

(77) $$\varphi(r,t) = \int_a^\infty [G_-(r-a),r'-a,t)f(r') + G_+(r-a,r'-a,t)g(r')]dr' ,$$

where the "outgoing part" f and the "incoming part" g are defined by (65) and (66), and the corresponding propagators G_- and G_+ are given by

(78) $$G_-(r,r't) = \delta(r-r't),$$

(79) $$G_+^M(r,r',t) = \delta(r-r'+t) + (-1)^{\ell+1} \delta(r+r'-t) + \sum_{j=1}^{\ell} G_j^M(r,r',t),$$

(80) $$G_+^E(r,r',t) = \delta(r-r'+t) + (-1)^\ell \delta(r+r'-t) + \sum_{j=1}^{\ell+1} G_j^E(r,r',t),$$

with

(81) $$G_j(r,r',t) = a_j H(t-r-r')\exp[p_j(t-r-r')] .$$

These results may again be visualized by means of an "image space" $(r < a)$, in which the role of the mirror is played by the surface of the sphere. An incoming pulse gives rise to a "mirror image" (the sign of which depends on the type and order of the multipole), followed by an "exponential tail" of transient modes. The expression (81) for the propagator of a transient mode is similar to that found in the previous section; according to (73) and (74), it is entirely determined by the poles of the $\underline{S}$-matrix. Thus, it is possible to give a rigorous meaning to Thomson's "natural modes of oscillation" (cf. section 1), by introducing a cut-off factor (which eliminates the exponential catastrophe) and identifying them with the propagators of the transient modes.

The apparent lack of symmetry in the propagators is due to the fact that (78) to (81) apply only for $t > 0$. For $t < 0$, the unknown in (69) is $g(a+t)$, whereas $f(a-t)$ is given by (65). The complete expression for the propagators, valid both for $t > 0$ and for $t < 0$, is easily found. The result is that (79) and (80) remain unchanged, while

(82) $$G_-(r,r',t) = G_+(r,r',-t) ,$$

so that the symmetry of the propagators is restored. The solution for $t < 0$ is related to the solution for $t > 0$ by time inversion, so that the "emission modes" are replaced by "absorption modes".

The results obtained in the present problem show a close formal analogy with those obtained in section 2. It must be emphasized, however, that, from the physical point of view, the transient phenomena found in these two problems have essentially different origins. In the case of section 2, we have to deal with an interaction between a field (string) and a discrete mechanical system (oscillator). As a consequence of the energy exchange between string and oscillator, part of the energy localized in the field may become temporarily stored in the mechanical system. This process is easy to visualize and does not require any further explanation.

In the case of the perfectly conducting sphere, however, the field forms a closed system, which can be described by a complete orthogonal set of stationary wave functions. Thus, the transient modes which have been found in this case cannot be attributed to an interaction with another system (such as the oscillator in the former example)

As will be shown below, we have to deal with an interaction between different regions of the field.

It might appear, at first sight, that the surface currents and charges on the sphere should be considered as an independent part of the system, which may absorb field energy and store it for some time. However, since the sphere is a perfect conductor, the normal component of Poynting's vector on its surface must vanish, and no energy can be accumulated on the surface. The energy accumulation which gives rise to the transient modes must therefore be localized in the field itself. Thus, we are led to look for the physical process by means of which energy can temporarily be stored outside of the sphere, apparently in free space. We shall refer to this phenomenon as "antenna effect".

In order to understand the antenna effect, it should first be observed that energy can be accumulated in those regions of a wave field where "repulsive forces" act upon it, i.e., where the "refractive index" decreases or becomes imaginary. This occurs, for instance, in the interior of a potential barrier * or in the interior of a superconductor.

In the present case, the repulsive forces (which act in the neighborhood of the sphere) are centrifugal forces, which correspond to the "centrifugal potential" $\ell(\ell+1)r^{-2}$ in the multipole wave equation

$$(83) \qquad \left[\frac{\partial^2}{\partial r^2} - \ell(\ell+1)r^{-2} + k^2 \right] [r\,\psi(k,r)] = 0 \,.$$

It can easily be verified that the antenna modes occur only in frequen

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* A characteristic example of energy accumulation inside a potential barrier has recently been studied by M. MALOGOLOWKIN (to be published).

cy regions which satisfy the condition

(84) $$k^2 \lesssim \ell(\ell + 1/a^2 ,$$

which means that the centrifugal forces must be taken into account. In particular, as will be seen in the next section, no antenna modes appear in the case of s-waves [27].

Centrifugal forces give rise to "non-asymptotic" terms in the solutions of the field equations. In the case of an electric dipole wave, for instance, the solution of Maxwell's equations corresponding to an incoming spherical wave packet has the form

(85) $$E_r = 2r^{-2}\left[g'(r+t) - r^{-1}g(r+t)\right]\cos\theta ,$$

(86) $$E_\theta = -r^{-1}\left[g''(r+t) - r^{-1}g'(r+t) + r^{-2}g(r+t)\right]\sin\theta ,$$

(87) $$H_\varphi = r^{-1}\left[g''(r+t) - r^{-1}g'(r+t)\right]\sin\theta .$$

As long as the wave packet is at large distances from the sphere, only asymptotic terms (in r^{-1}) have to be considered, and both the energy density and the Poynting vector are proportional to $\sin^2\theta$, so that they vanish at the poles ($\theta = 0, \pi$). As the wave packet approaches the sphere, however, non-asymptotic terms become increasingly more important, and the energy current is deviated towards the polar regions, where an energy storage takes place.

In general terms, the deviation of Poynting's vector under the action of inertial forces can be considered as a consequence of Einstein's conservation laws for the electromagnetic field in curvilinear coordinates

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27. Cf. also H.M. Nussenzveig: Nuclear Physics, 11, 499 (1959), sect. 4.2(b).

$$\frac{\partial \sqrt{-g}\ S_\mu^\nu}{\partial x^\nu} = \frac{1}{2}\ \frac{\partial g_{\lambda\nu}}{\partial x^\mu}\ \sqrt{-g}\ \ S^{\lambda\nu}\ ,$$

where $g_{\lambda\nu}$ and S_μ^ν are the components of the metric tensor and of the electromagnetic energy-momentum tensor, respectively. It follows from this that the way in which inertial forces act on the electromagnetic field is very similar to the way in which they act on a mechanical system. We conclude, therefore, that the antenna effect is a direct consequence of the inertia of the electromagnetic field, and that it results from the action of the inertial field.

The antenna effect is a very general phenomenon, which comes into play whenever a propagating field meets obstacles with curved surfaces. It also plays a role in diffraction phenomena, giving rise to optical border effects in the neighbourhood of sharp edges. A more detailed discussion of the general character of the antenna effect and its connection with inertial forces will be given in another paper.

It must be strongly emphasized, in connection with the physical interpretation of the transient modes, that no special significance can be attached, in general, to the "amplitude of excitation" of each separate mode. In fact, the transient modes are not orthogonal, so that the total energy is not a sum of terms associated with the separate modes, and it is not possible to excite one particular mode independently of the others. A similar situation exists in the case of transients in discrete systems.

The remarks which were made at the end of section 2 concerning the dependence of the decay on the excitation may be extended to the present case.

The perfect conductor, which we have considered in this section, is an ideal limiting case, which may be approximately realized by a super-conductor or by a very good normal conductor, In the latter case, however, the presence of the ohmic losses renders the problem considerably more complicated, and additional effects, which are not apparent in the limit of infinite conductivity, may have to be taken into account.

4. - Hard sphere.

The counterpart in non-relativistic quantum mechanics of the problem treated in section 3 is the following problem: given an arbitrary (normalizable) initial wave packet in the exterior of a "hard sphere" of a radius a, it is required to determine its subsequent behaviour. The solution of this problem for s-waves has been given by MOSHINSKY[28]; however, as will be seen below, the S-function has no poles in this case, so that we shall be interested in higher angular momenta.

The initial-value problem may be formulated as follows: to find a solution of Schrödinger's equation (we take $\hbar = m = 1$)

$$-i \frac{\partial}{\partial t} \Psi(\underline{r},t) = \tfrac{1}{2} \Delta \Psi(\underline{r},t) \tag{88}$$

which satisfies the boundary condition

$$\Psi(r,t) = 0 \quad \text{for} \quad r = a \tag{89}$$

and the initial condition

$$\Psi(\underline{r},0) = U(\underline{r}) \qquad (r \geqslant a). \tag{90}$$

If we expand the wave function and its initial value in par

28. M. Moshinsky: Rev. Mex. Fis., 1, 28 (1952).

tial waves,

(91) $$\Psi(\underline{r},t) = \sum_{\ell=0} \sum_{m=-\ell} \psi_{\ell m}(r,t)Y_{\ell m}(\theta, \varphi),$$

(92) $$U(\underline{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}(r)Y_{\ell m}(\theta, \varphi),$$

we find the following initial-value problem for the ℓth partial wave:

(93) $$\left[\frac{\partial^2}{\partial r^2} - \ell(\ell+1)r^{-2} + 2i \frac{\partial}{\partial t}\right][r\,\psi(r,t)] = 0,$$

(94) $$\psi(a,t) = 0,$$

(95) $$\psi(r,0) = u(r) \quad (r \geqslant a),$$

where the subscripts ℓ and m have been dropped for convenience.

The stationary scattering states are given by

(96) $$F(k,r)\exp(-iEt) = (2\pi)^{-\frac{1}{2}}[h_\ell^{(2)}(kr) + S(k)h_\ell^{(1)}(kr)]\exp(-iEt),$$

where

(97) $$E = \tfrac{1}{2} k^2$$

and

(98) $$S(k) = -[h_\ell^{(2)}(ka)]/[h_\ell^{(1)}(ka)] ,$$

which is identical to the S-function for magnetic multipole waves of order ℓ (cf. (43)). The only difference is that $\ell = 0$ is allowed here. The corresponding S-function is

(99) $$S(k) = \exp(-2ika) \quad (\text{for } \ell = 0),$$

which has no poles.

The solution of the initial-value problem by means of the stationary-state expansion is similar to that given by the previous section. The main differences are the dispersion formula (97) and the absence of a condition on the initial time derivative. The results may be obtained by means of an appropriate modification of (54):

$$(100)\qquad \psi(r,t) = \tfrac{1}{2}\int_{-\infty}^{+\infty} dk \int_a^{\infty} dr'(kr')^2 u(r') F^*(k,r')\cdot \cdot \exp(-\tfrac{1}{2}ik^2 t) .$$

The connection with the method of complex eigenvalues may be found by an extension of the treatment given in section 3.

The general solution of the radial equation (93) is

$$(101)\qquad \psi(r,t) = r^{\ell} D_r^{\ell}\,[r^{-1}\varphi(r,t)],$$

where D_r^{ℓ} is the differential operator (56), and $\varphi(r,t)$ is the general solution of the one-dimensional free-particle Schrodinger equation which is given by [29]

$$(102)\qquad \varphi(r,t) = \int_{-\infty}^{+\infty} U(r-r',t)f(r')dr' = \int_{-\infty}^{+\infty} U(r',t)f(r+r')dr' \;;$$

where $f(r')$ is an arbitrary function, and

$$(103)\qquad U(r,t) = \exp(-i\pi/4)(2\pi t)^{-\frac{1}{2}}\exp(ir^2/2t)$$

is the free-particle Schrodinger propagator ("heat pole" solution). We have

$$(104)\qquad \lim_{t\to 0} U(r,t) = \delta(r),$$

so that the initial condition (95) becomes

$$(105)\qquad D_r^{\ell}[r^{-1}f(r)] = r^{-\ell}u(r) \quad (r \geq a).$$

The function $f(r)$ is not uniquely determined by (101) and (102); we may add to it any solution of the equation

$$(106)\qquad D_r^{\ell}\left\{r^{-1}[f(r+r') + f(r-r')]\right\} = 0.$$

The general solution of this equation is

$$(107)\qquad f(r) = \sum_{n=0}^{\ell-1} C_n r^{2n+1} ,$$

29. W. Pauli: Handbuch der Physik, XXIV/1, 2.Aufl.(Berlin, 1933), p.103.

where $C_0, C_1, \ldots, C_{\ell-1}$ are arbitrary constants. We may profit from this to choose f in such a way that the solution takes the simplest possible form. This is achieved by imposing the supplementary conditions *

(108) $$f^{(j)}(a) = 0 \quad (j = 0, 1, \ldots, \ell-1).$$

The solution of (105) subject to these conditions is

(109) $$f(r) = rD_r^{-\ell}[r^{-\ell}u(r)] \qquad (r \geqslant a),$$

where $D_r^{-\ell}$ is defined by (64). The problem is now reduced to the determination of the "reflected wave" $f(r)$ $(r < a)$ in (102). Substituting (101) and (102) in the boundary condition (94), we obtain the differential equation

(110) $$\sum_{n=0}^{\ell}(-1)^n c_{\ell n} a^{-n-1}\left[f^{(\ell-n)}(a-r)+f^{(\ell-n)}(a+r)\right] = 0 \quad (r > 0)$$

where $f(a-r)$ is the unknown, and $c_{\ell n}$ is defined by (68).

Equation (110) is identical to (69), with t replaced by r and $f = g$ (but not $\bar{f} = \bar{g}$!). The supplementary conditions (108) are related in the same way to (61). Therefore, according to (75), the solution of (110) is

(111) $$f(a-r) = (-1)^{\ell+1}f(a+r) + \sum_{j=1}^{\ell} a_j \exp(p_j r) * f(a+r) \quad (r > 0),$$

where p_j and a_j are given by (73) and (74), respectively.

Equations (101), (102), (109) and (111) give the general solution of the initial-value problem. The solution may be rewritten in the following form

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* These conditions do not exclude the possibility that terms of the form of (107), which would give rise to singular integrals in (102), may appear in the solution. However, according to (106), such terms may be dropped.

(112) $$\varphi(r,t) = \int_a^\infty G(r-a,r'-a,t)f(r')dr' ,$$

where $f(r')$ is given by (109), and $G(r,r',t)$, which may be called the propagator of the φ-wave, is given by

(113) $$G(r,r',t) = U(r-r',t) + (-1)^{\ell+1}U(r+r',t) + \sum_{j=1}^{\ell} G_j(r,r',t) ,$$

where

(114) $$G_j(r,r',t) = a_j M(r+r',k_j,t),$$

and

(115) $$M(x,k,t) = \int_{-\infty}^{+\infty} H(x'-x)\exp[ik(x-x')]U(x',t)dx' .$$

If we introduce an "image space", so that (113) is defined for $-\infty < r < +\infty$, it follows from (104) and (112) to (114) that $G(r,r',t)$ is that solution of the one-dimensional free-particle Schrödinger equation which, for $t = 0$, reduces to

(116) $$G(r,r',0) = \delta(r-r') + (-1)^{\ell+1}\delta(r+r') + \sum_{j=1}^{\ell} a_j H(-r-r')\exp[ik_j(r+r')] .$$

According to (79) and (81), this is identical to $G_+^M(r,r',0)$. Thus, we may say again that a pulse in "real space" at $t = 0$ gives rise in "image space" to a mirror image, followed by an exponential tail of transient modes. If we let this initial configuration propagate freely (according to Schrödinger's equation), the result is described by $G(r,r',t)$.

There is a far-reaching formal analogy between the present problem and that of the previous section (for magnetic multipoles). The main difference lies in the nature of free propagation, which is described in one case by the wave equation, and in the other by Schrödin ger's equation. Notice that, since there is no limiting velocity for Schrödinger particles, the "reflected pulse" in this case appears at once in real space, whereas in the electromagnetic case it appears only after the incident pulse has stricken the surface of the sphere.

The propagator associated with a pole of the $\underline{S}$-matrix,

(117) $k = k' - iK \quad (K > 0)$,

is given by (114) and (115). The function $M(x,k,t)$ was introduced by MOSHINSKY [15]. It may be expressed in terms of the error function of a complex argument, by means of the formula

(118) $M(x,k,t) = \frac{1}{2}v(x,k,t)\mathrm{erfc}(c^{-ix/4}w)$,

with

(119) $w = (2t)^{-\frac{1}{2}}(x-kt)$,

(120) $\mathrm{erfc}(z) = 2\pi^{-\frac{1}{2}} \int_{z}^{\infty} \exp(-\zeta^2)d\zeta$,

and

(121) $v(x,k,t) = \exp[i(kx-Et)]$,

where E is the "complex energy", given by

(122) $E = E' - \frac{1}{2}i\Gamma = \frac{1}{2}k^2 = \frac{1}{2}(k'^2 - K^2) - ik'K$.

According to (115), $M(x,k,t)$ is that solution of the free-particle Schrödinger equation which, for $t = 0$, reduces to

(123) $M(x,k,0) = H(-x)\exp(ikx)$,

which is a wave packet with a sharp front. For $t > 0$, the front becomes diffuse, and is replaced by a transitional region, in which $|w| \lesssim 1$. According to (119), the width of this region is given by $|x-kt| \lesssim (2t)^{\frac{1}{2}}$ (or, in ordinary units, $(2\hbar t/m)^{\frac{1}{2}}$). This "blurring" of the initially sharp edge is a purely quantal "diffraction" effect.* We shall be interested in the behaviour of $M(x,k,t)$ outside of the transitional region, i.e. either beyond or behind the wave front, but not very close to it. Thus, we want to find the behaviour of (118) for $|w| \gg 1$. For this purpose, we shall employ the asymptotic expansion of the error function, which is given in appendix B.

Let $\underline{A}$ and $\underline{B}$ denote the regions of the complex plane above and below the second bisector, respectively, so that $-\pi/4 < \arg w < 3\pi/4$ if $w \in \underline{A}$, and $3\pi/4 < \arg w < 7\pi/4$ if $w \in \underline{B}$. Then, it follows from (118) and from the results given in appendix B that

(124) $$M(x,k,t) = M_A(x,k,t) = it(x-kt)^{-1}U(x,t)[1-\tfrac{1}{2}iw^{-2}+\ldots+ (-\tfrac{1}{2}i)^n(2n-1)!!w^{-2n}+R_n(w)] \quad \text{if} \quad w \in \underline{A},$$

(125) $$M(x,k,t) = M_B(x,k,t) = v(x,k,t) + M_A(x,k,t) \quad \text{if} \quad w \in \underline{B},$$

where $U(x,t)$ and $v(x,k,t)$ are defined by (103) and (121), respectively, and

(126) $$|R_n(w)| \leqslant \pi^{\frac{1}{2}}\, 2^{-n-1}(2n+1)!!\, |w|^{-2n-1} \quad (n = 0, 1, \ldots).$$

For $|w| \gg 1$, $M_A(x,k,t)$ differs from the free-particle propagator exxentially by a factor of squared modulus

(127) $$t^2[(x-k't)^2+(Et)^2]^{-1},$$

which has a peak of width Et around the point $x = k't$. M_B differs from M_A by the additional term $v(x,k,t)$. This term corresponds to the

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* For real k, (123) may be thought of as representing a beam of particles of "velocity" k confined to the half-space $x<0$ by a perfectly absorbing shutter, which is suddenly removed at $t = 0$. According to classical mechanics, the behaviour of the current at a point $x > 0$ as a function of time would be given by a step function, with a sharp rise at $t = x/k$ (time of flight). For Schrödinger particles, the current begins to rise immediately after $t=0$, and approaches the classical value for $t \gg x/k$. In the neighbourhood of $t = x/k$, there appear oscillations in the current, which resemble the Fresnel diffraction pattern of a straight edge in optics. Moshinsky has called this effect "diffraction in time" (M. Moshinsky: Phys. Rev., **88**, 625 (1952)).

"complex-energy wave function" which is usually associated with a decaying state" in the method of complex eigenvalues. However, it appears in the propagator only for a special class of poles, and only for a limited range of values of x and t.

To show this, let us consider the behaviour of $M(x,k,t)$ in "real space" ($x > 0$), as a function of x, for fixed t ($t > 0$). It may be seen in figure 2 that, if $k \in \underline{B}$, then $w \in \underline{A}$ for all $x > 0$. On the other hand, if $k \in \underline{A}$, then $w \in \underline{B}$ if

(128) $\quad 0 < x < (k' - K)t$,

and $w \in \underline{A}$ if $(k'-K)t < x$. Thus, it is only in the case of poles located above the second bisector, and only within the range of values of x and t defined by (128), that the term usually associated with a "de-

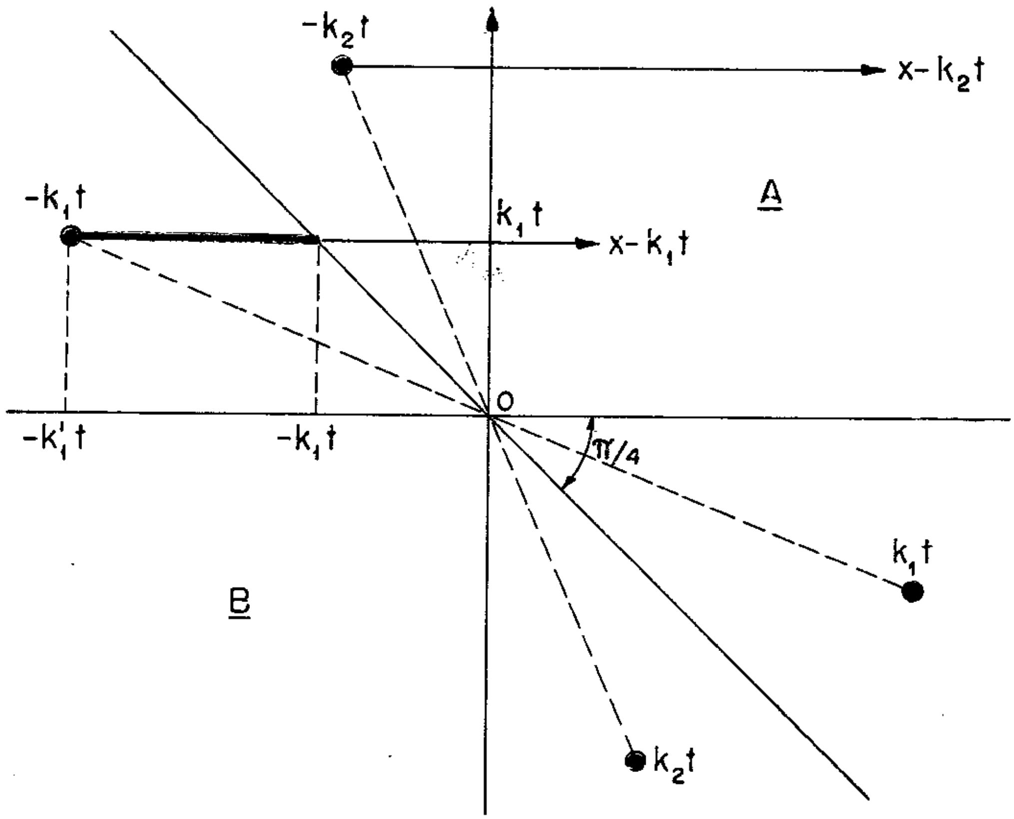

Fig. 2. - For a pole k_2 belonging to $\underline{B}$, $x-k_2t$ lies in $\underline{A}$. For a pole k_1 belonging to $\underline{A}$, $x-k_1t$ lies in $\underline{B}$ within the range indicated by the thick line, and it lies in $\underline{A}$ outside of this range.

caying state" appears in the propagator.

It is readily verified that the real part of the exponent in (121) is always negative within the range (128), so that there is no exponential catastrophe. This also follows from the conservation of probability, since (123) is a normalizable wave packet. The essential point is the presence of the cut-off factor in (123).

The distinction between poles located above or below the second bisector is implicitly contained in HEITLER and HU's criterion[30], according to which only those poles for which the real part E' of the "complex energy" (122) is positive give rise to "decaying states". It has been pointed out elesewhere[31] that Heitler and Hu's justification of this criterion is not satisfactory.

In fact, according to (128), there is a continuous transition between poles belonging to A and poles belonging to B, the range (128) becoming smaller and smaller as the second bisector is approached. Moreover, if we compare the order of magnitude of the two terms of (125) as a function of time, within the range (128), we find that $M_A(x,k,t)$ always predominates over $v(x,k,t)$ after a sufficient lapse of time. Since

(129) $$|v(x,k,t)| = \exp[K(x-k't)] ,$$

we may say that the term $v(x,k,t)$ is associated with the propagation of the initial wave packet, without change of shape (with velocity k'). However, as is well known, a free-particle Schrödinger wave packet always undergoes a broadening in the course of time. This "spreading effect" is contained in the term $M_A(x,k,t)$ of (125). It follows from

30. W. Heitler & N. Hu: Nature, 159, 776 (1947).
31. H. M. Nussenzveig: Nuclear Physics, 11, 499 (1959).

(123) that the width of the initial wave packet is of the order of K^{-1}. According to the uncertainty relation, this corresponds to a momentum spread of the order of K, so that the wave packet will have spread by an amount of the order of Kt after a time t (cf. (127)). The effect of spreading becomes important when this quantity is comparable with the initial width, i.e., for $t \gtrsim t_s = K^{-2}$. On the other hand, according to (129), the lifetime is given by $\tau = \Gamma^{-1} = \frac{1}{2}(k'K)^{-1}$. It follows that, as we approach the second bisector, t_s and τ become of the same order so that the spreading effect predominates over the exponential decay within a single half-life. There is no time, so to speak, for the exponential law to manifest itself. This explains the special role which is played by the second bisector.

It also follows from the above discussion that, no matter what may be the position of the pole, the exponential law cannot remain valid for arbitrarily large times: it must ultimately be superseded by the decay law for a free-particle wave packet. In this case, as is well known [32], the probability distribution at a fixed point behaves like t^{-3} for $t \to \infty$. This may be verified in the present example; similar results have been obtained by other authors [19, 33].

In the case of a pole which is close to the real axis ("long - lived" transient mode), a careful discussion is required to determine the range of validity of the exponential decay law and the dependence of the decay on the excitation. However, in the hard-sphere problem, there are no poles satisfying this condition. Although there are poles

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32. W. Brenig & R. Haag: Fortschr. Phys., 7, 183 (1959).
33. J. Petzold: Z. Physik, 155, 422 (1959).

above the second bisector for $\ell \geqslant 4$, they are still far from the real axis (see JAHNKE-EMDE [25]). This may be understood by considering the origin of the transient modes in this case. They are clearly related to the presence of the "centrifugal barrier": for sufficiently large angular momentum, it is possible for a wave packet to remain "trapped" near the surface of the sphere for a short time (cf. the discussion on the role of the centrifugal forces at the end of section 3). However, the centrifugal barrier alone is too transparent to allow the formation of long-lived modes.

So far, we have considered only the behaviour of the propagator for $t > 0$. However, (112) to (115) may be applied just as well for $t < 0$. According to (103), we may take

(130) $U(r,-t) = U^*(r,t)$.

It follows from (115) and (130 that

(131) $M(x,k,-t) = M^*(x,-k^*,t)$,

so that poles which are symmetrically placed with respect to the imaginary axis exchange their roles under time inversion. In particular, for $t < 0$, the term (121) may appear in the propagator only in the case of poles located above the first bisector *. It corresponds to the reverse of an "emission mode", so that it may be called an "absorption mode"[30]. Thus, the absorption modes appear in connection with "final-value problems", i.e., when we want to describe how a given situation was built up.

Taking into account the symmetry of the pole distribution about the imaginary axis, it follows from (113) to (115), (130) and (131) that

(132) $G(r,r',t) = G^*(r,r',t)$.

* We are indebted for this remark to Professor L. Van Hove.

If we denote the solution of the initial-value problem, which is a functional of the initial value $u(r)$, by $\psi(r,t,[u])$, it follows from (101), (112) and (132) that

$$\Psi(r,-t,[u]) = \psi^{*}(r,t,[u^{*}]) . \tag{133}$$

The second member of (133) describes the "time-reversed motion" corresponding to the solution for $t > 0$ [34]. This result could of course have been anticipated.

By going over from the stationary-state expansion (100) to the "expansion in transient modes" (113), we have effectively replaced a function given on the real axis, the S-function, by a set of complex parameters, the poles of the S-function. This transformation may be very useful: it gives us greater insight into the behaviour of the solution, and it clearly displays the role of the excitation conditions. However, it must be stressed that, on account of the non-orthogonality of the transient modes, it is not possible, in general, to ascribe an independent physical meaning to each term of the expansion. Thus, the price that has to be paid for what is accomplished by this transformation is the loss of some definiteness in the physical interpretation.

The transient modes occupy an intermediate position between stationary states and free-particle wave packets, sharing some of the properties of both. In the case of poles which are far from the real axis, the free-particle features predominate. On the other hand, in the case of poles which are close to the real axis, it may be possible to give an approximate description of the system, during a long time interval, by means of concepts taken over from the theory of stationary states. A more complete discussion of this case will be given in the

34. E.P. Wigner: Göttinger Nachrichten, 31, 546 (1932) .

second part.

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Appendix

A. - The roots of equations (48) and (49)

In this appendix, we shall prove the following results:

(a) The roots of the equations

(M) $$h_\ell^{(1)}(ka) = 0,$$

(E) $$[kah_\ell^{(1)}(ka)]' = 0,$$

are located in the lower half of the k-plane; (b) all the roots are simple.

To prove (a), we start from the following identity, which results from the differential equation of the spherical Hankel functions:

(A1) $$(k^2-k^{*2})r^2|h_\ell^{(1)}(kr)|^2 = \frac{d}{dr}\left\{ r^2 h_\ell^{(1)}(kr)\frac{d}{dr}[h_\ell^{(1)}(kr)]^* - r^2[h_\ell^{(1)}(kr)]^*\frac{d}{dr}[h^{(1)}(kr)]\right\} ,$$

and we integrate both members over the interval from a to r. If k is a root of (M) or (E), the contribution to the second member from the lower limit a vanishes, and we are left with

(A2) $$(k^2-k^{*2})\int_a^r |h_\ell^{(1)}(kr')|^2 r'^2 dr' = r^2\left\{ h_\ell^{(1)}(kr)\frac{d}{dr}[h_\ell^{(1)}(kr)]^* - [h_\ell^{(1)}(kr)]^*\frac{d}{dr}[h_\ell^{(1)}(kr)]\right\} .$$

For sufficiently large r, we have $|kr| \gg \ell$, so that the spherical Hankel functions in the second member of (A2) may be replaced by their asymptotic expansions, leading to

(A3) $$(k^2-k^{*2})\int_a^r |h_\ell^{(1)}(kr')|^2 r'^2 dr' \approx -i|k|^{-2}(k+k^*)\exp[i(k-k^*)r] .$$

Let us assume first that $k = k'-iK$, with $k' \neq 0$. Then, it follows from (A3) that

(A4) $$K \approx \tfrac{1}{2}|k|^{-2}\exp(2Kr)\left[\int_a^r |h_\ell^{(1)}(kr')|^2 r'^2 dr'\right]^{-1} > 0 ,$$

so that (a) is proved, except in the case of purely imaginary roots. To complete the proof, we must show that there cannot be any roots on the positive imaginary axis. This follows from the fact that, for $v > 0$, $ip_\ell(iv)$ in (46) and $-iq_{\ell+1}(iv)$ in (47) are polynomials in v with real positive coefficients.

To prove statement (b), it suffices to show that $[h_\ell^{(1)}(ka)]' \neq 0$ or $[kah_\ell^{(1)}(ka)]'' \neq 0$, if k is a root of (M) or (E), respectively. This follows from the non-vanishing of the Wronskian determinants

(A5) $$h_\ell^{(2)}(z)h_\ell'^{(1)}(z) - h_\ell^{(1)}(z)h_\ell'^{(2)}(z) = 2iz^{-2} ,$$

(A6) $$[zh_\ell^{(2)}(z)]'[zh_\ell^{(1)}(z)]'' - [zh_\ell^{(1)}(z)]'[zh_\ell^{(2)}(z)]'' = 2i[1 - \ell(\ell+1)z^{-2}] .$$

B. - Asymptotic expansion of the error function *.

The function erfc(z), where z is a complex variable, is defined by

$$\text{(B1)} \quad \operatorname{erfc}(z) = 2\pi^{-\frac{1}{2}} \int_z^{\infty} \exp(-\zeta^2) d\zeta \ .$$

The path of integration in (B1) may be deformed in an arbitrary way, provided that it remains within the quadrant $-\pi/4 \leq \arg \zeta \leq \pi/4$ for $|\zeta| \rightarrow \infty$.

We have

$$\text{(B2)} \quad \operatorname{erfc}(z) + \operatorname{erfc}(-z) = \operatorname{erfc}(-\infty) = 2,$$

so that it suffices to consider the half-plane $\operatorname{Re} z \geqslant 0$.

To find the asymptotic expansion of erfc(z) in this half-plane, we choose as path of integration a straight line parallel to the real axis. A straightforward application of the method of integration by parts yields

$$\text{(B3)} \quad \operatorname{erfc}(z) = \pi^{-\frac{1}{2}} z^{-1} \exp(-z^2)\left[1 - \tfrac{1}{2}z^{-2} + \frac{1.3}{2^2}z^{-4} - \ldots + (-\tfrac{1}{2})^n (2n-1)!!\, z^{-2n} + R_n(z)\right],$$

where

$$\text{(B4)} \quad (2n-1)!! = 1.3.5\ldots(2n-1),$$

$$\text{(B5)} \quad R_n(z) = (-\tfrac{1}{2})^n (2n+1)!!\, z \exp(z^2) \int_z^{\infty + i\Im m z} \exp(-\zeta^2)\, \zeta^{-2n-2} d\zeta \ .$$

It follows from (B5) that

$$\text{(B6)} \quad |R_n(z)| \leqslant \pi^{\frac{1}{2}}\, 2^{-n-1} (2n+1)!!\, |z|^{-2n-1} \qquad (\operatorname{Re} z \geqslant 0).$$

The asymptotic expansion of erfc(z) in the left half-plane follows from (B2) and (B3).

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* The results of this appendix have been given in a less complete form by Moshinsky [15].