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An Elementary Comment on the Zeros of The Zeta Function (on the Riemann's Conjecture)

Luiz C.L. Botelho
Departamento de Matemática Aplicada,
Instituto de Matemática, Universidade Federal Fluminense,
Rua Mario Santos Braga
24220-140, Niterói, Rio de Janeiro, Brazil
e-mail: botelho.luiz@ig.com.br

Abstract

We present an equivalent conjecture to the original Riemann's conjecture on the localization on the Complex Plane of the zeros of the Zeta Function. This new proposed conjecture is hoped to be more monagable for a classical complex variable analysis for this *long standing one century and half unsolved mathematical problem, besides of being a potential clue for the reason of why Riemann was naturally led to his conjecture - a historical gap in the history of the Riemann's conjecture*.

Key-words: Zeta function; Riemman conjecture.

1 Introduction – "Elementary may be deep" ([1])

The Riemann's series $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$ converges uniformly for all real numbers x greater then or equal to a any given (fixed) abscissa $\bar{x}$: $x > \bar{x} > 1$. It is well-known that the complex valued (meromorphic) continuation to complex values ($z = x+iy$) throughout the Complex Plane $z \in \mathbb{C}$ is obtained from standard analytic (finite-part) complex variables methods applied to the integral representation ([1]))

$$\begin{aligned}\zeta(z) &= \frac{i\Gamma(1-z)}{2\pi}\left(\int_C \left(\frac{(-w)^{z-1}}{e^w - 1}\right) dw\right) = \frac{1}{\Gamma(z)}\left(\int_0^\infty w^{z-1} \times \left[\frac{1}{2^w - 1} - \frac{1}{w}\right] dw\right) \\ &= \frac{i\Gamma(1-z)}{2\pi}\left\{\int_C \left[(-w)^{z-2} - \frac{1}{2}(-w)^{z-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+z-2} B_n / w^{z+2n-2}}{(2n)!}\right]\right\}, \end{aligned} \quad (1)$$

here B_n are the Bernoulli's numbers and C is any contour in the Complex Plane, coming from positive infinity and encircling the origin once in the positive direction.

An important relationship resulting from eq.(1) is the so called functional equation satisfied by the Zeta function, holding true for any $z \in \mathbb{C}$

$$\frac{\zeta(z)}{\zeta(1-z)} = 2^z \cdot \pi^{z-1} \cdot \operatorname{sen}\left(\frac{\pi z}{2}\right) \cdot \Gamma(1-z) \quad (2)$$

In applications to Number Theory, where this Special function plays a special role, it is a famous conjecture proposed by B. Riemann (1856) that the only non trivial zeros of the Zeta function lie in the so-called critical line Real $(z) = x = \frac{1}{2}$.

In the next section we intend to propose an equivalent conjecture, hoped to be more suitable for handling the Riemann's problem by the standard methods of Classical Complex Analysis ([2]), besides of proving a historical clue for the reason that led B. Riemann to propose his conjecture ([2]).

2 On the Equivalent Conjecture 1

Let us state our conjecture

Conjecture: In each horizontal line of the complex Plane of the form Imaginary $(z) = y = b =$ constant, the Zeta function $\zeta(z)$ *posseses at most a unique zero.*

We show now that the above written conjecture leads elementarly to a proof of the Riemann's conjecture.

Theorem. (The Riemann's Conjecture) *All the non-trivial zeros of the Riemann Zeta function lie on the critical line Real* $(z) = x = \frac{1}{2}$.

Proof: Let us consider a given non-trivial zero $\bar{z} = \bar{x} + i\bar{y}$ on the open strip $0 < x < 1$, $-\infty < y < +\infty$. It is a direct consequence of the Schwartz's reflection principle since $\zeta(x)$ is a real function in $0 < x < 1$ that $(\bar{z})^* = \bar{x} - i\bar{y}$ is another non-trivial zero of the Riemann function on the above pointed out open strip. The basic point of our proof is to show that $1 - (\bar{z})^* = (1 - \bar{x}) + i\bar{y}$ is another zero of $\zeta(z)$ in the same horizontal line $Im(z) = \bar{y}$. This result turns out that $(1 - (\bar{z})^*) = \bar{z}$ on the basis of the validity of our Conjecture. As a consequence we obtain straightforwardly that $1 - \bar{x} = \bar{x}$. In others words Real $(\bar{z}) = \frac{1}{2}$.

At this point, we call the reader attention, on the result that if $\bar{z}$ is a zero of the Riemann Zeta function, then $1-\bar{z}$ must be another non-trivial zero is a direct consequence of the functional eq.(2), since sen $\left(\frac{\pi z}{2}\right)$ and $\Gamma(1 - z)$ never vanishe both on the open strip $0 < \text{Real}(z) < 1$.

At this point of our note, we want to state clearly that the significance of replacing the Riemann's original conjecture by your complex oriented conjecture rests on the possibility of progress in producing sound results for its proof, which is not claimed in our elementary note. However, we intend to point out directions (arguments) in its favor.

By firstly, it is worth call the reader attention on the validity of the elementary expansion below on the open interval $0 < x < 1$

$$\zeta(x + ib) = \zeta(x) \left\{ \prod_{\substack{\{x_n+iy_n \\ \in \text{zeros of} \zeta}} \left[1 - \frac{ib}{(\frac{1}{2} - x_n) + iy_n} \right] \right\} \tag{3}$$

which should reduces our conjecture about the zero uniquenness analysis of the Riemann Zeta function solely to the real axis $(b = 0)$.

At this point we wishe to remark the following weaker conjecture 2 equivalent to our strong conjecture 1

Conjecture 2: If there is a zero of the Riemann's Zeta function $t \in (0, 1)$ such that the set of real number $\left\{ \frac{d^n}{d^n x} \zeta(t) \right\}_{n=0,1,2,\ldots}$ is contained entierely on the positive or negative real exis (these numbers all have the same sgnal), then the Conjecture 1 holds true.

On basis of the above stated Conjecture 2, we can easily show a argument of our conjecture based in the power series exponsion around the fixed zero t ($\bar{x} > t$)

$$\zeta(\bar{x}) = 0 = \sum_{n=1}^{\infty} \frac{\zeta^{(n)}(t)}{n!}(\bar{x} - t)^n > 0 \tag{4}$$

here $\bar{x}$ is the supposed different zero of $\zeta(x)$ on the open interval $(0, 1)$.

Finally let us point out the following formula of ours, related to the zeros of the Riemann Zeta Function on the complex plane $\mathbb{C}$.

Lemma 1. *Let $\{z_n\}_{n=0,1,2,...}$ denote the zeros of the Riemann's Zeta function $\zeta(z)$. We have thus the following result, for any integer $p > 0$.*

$$\begin{aligned} &-\left\{ \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{1}{(z_n - z_k)^{p+1}} \right\} \\ &= \lim_{z\to z_k} \left\{ \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(\frac{1}{z-1} + \frac{\zeta'(z)}{\zeta(z)} \right) (z_k + e^{i\theta}) \cdot (z_k - z)^{p+1} e^{-ip\theta} \right\} \end{aligned} \tag{5}$$

Proof: This result can be seen from the fact that $h(z) = (z-1)\zeta(z)$ is an integral function and from any integral function of order k, with zeros $\bar{z}_1, \bar{z}_2, \ldots, (h(0) \neq 0)$

$$\lim_{z\to z_k} \left[(z - \bar{z}_k)^{p+1} \left(\frac{d}{dz} \right)^p \left\{ \frac{h'(z)}{h(z)} \right\} \right] = -p! \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{1}{(\bar{z}_n - \bar{z}_k)^{p+1}} \tag{6}$$

By re-writing the derivative term in the left-hand side of eq.(6) by means of the Cauchy Theorem

$$\left\{ \frac{h'(z)}{h(z)} \right\}^{(p)} = \frac{p!}{2ni} \left\{ \int_{\partial D(z,1)} dw \left[\frac{h'(w)}{h(w)} \right] [(w - z)^{-(p+1)}] \right\}, \tag{7}$$

where $D(z, 1) = \{w \mid |w-z| \leq 1\}$, and arranging terms of the form $(z-\bar{z}_k)^{p+1}/(w-z)^{(p+1)}$ inside eq.(6)–eq.(7), we obtain eq.(5).

Lemma 2. *Let $h(z) = \zeta(z)(z - 1)$ be an (analytic) function on the strip $0 < Real(z) < 1$. Let us consider its expression on the unit disk conformally equivalent to the above considered strip, including the boundaries correspondence.*

$$h\left(\frac{1}{\pi i} lg \left(\frac{i(i - w)}{i + w} \right) \right) = h(g(w)) = \bar{h}(w) \tag{8}$$

We have that, the Generalized Jensen's formula

$$\int_0^1 \frac{\eta(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(lg|\bar{h}(e^{i\theta})| \right) \left(\frac{g'}{g} \right) (e^{i\theta}) d\theta \tag{9}$$

where $\eta(x)$ is the number of zeros of the Zeta Function on the region $g^{-1}(|w| < x)$, contained on the strip $0 < \text{Real}(z) < 1$.

Proof: Since $h(z)$ is an analytic function in the strip $0 < \text{Real}(z) < 1$, we are going to prove the general result for a region Ω conformally equivalent to a disc of radius 1, including its boundary correspondence, namely

$$\begin{aligned} g\colon \Omega &\to D(0,1) \\ g(\partial\Omega) &= \partial D(0,1) \end{aligned} \tag{10}$$

with $w = g(z)$ a conformal mapping applying $\partial\Omega$ in $\partial D(0,1)$ diffeomorphically.

Let $\eta(x)$ denote the number of zeros of $f(g^{-1}(w)) = h(w)$ on $D(0,x)$. By the usual Jensen's formula for $r < 1$, with the hypothesis of $h(w)$ has no zero in $\partial D(0,r)$.

$$\int_0^r \frac{\eta(x)}{x}dx = \frac{1}{2\pi i}\int_{\partial D(0,r)} \frac{h'(w)}{h(w)}dw \tag{11}$$

Since $\eta(x)$ coincides with the number of zeros of the function $f(z)$ in the region $g^{-1}(D(0,1))$, we obtain the Jensen's formula in Ω in the general case

$$\int_0^r \frac{\eta(x)}{x}dx = \left(\frac{1}{2\pi i}\int_{\partial\Omega} lg\left\{|l(z)|\frac{g'(z)}{g(z)}\right\}\cdot dz\right) - lg|f(0)| \tag{12}$$

Equation (8) comes from the fact the strip $0 < \text{Real}(z) < 1$ is conformally mapped - with the boundary correspondence - on the unit circle $\{w|j \mid |w| \leq 1\}$ by the function $z = \frac{1}{\pi i}lg\left(\frac{i(i-w)}{i+h}\right)$.

References

[1] Kato, T. "Analysis Workshop", Univ. of California at Berkeley (1976), USA.

[2] Any standard classical book on the subject as for instance

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