CBPF-NF-017/00

Comment on the Path-Integral Quantization for One-Dimensional Particle*

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I correct the previous Notas de Física CBPF-NF-001/00 above entitled.

In this comment, I point out that the study previously presented in CBPF-NF-001/00 is not correct. The correct approach is presented in my previous paper already published Luiz C.L. Botelho and Edson P. Silva, "Quantum Geometry for the Brownian Particle", Modern Physics Letters B, vol. 12, n. 28 (1998) 1191-1134 (see Appendix A). Also, I should point out that Dr. E.P. Silva is not a co-author of the CBPF-NF-001/00.

*by Luiz C.L. Botelho, Jorge C. Mello and Edson P. Silva

Appendix A

Quantum Geometry for the Brownian Particle

Luiz C.L. Botelho and Edson P. Silva

One of the most interesting problems in non-relativistic quantum mechanics of one particle system is how apply the Feynman path integral theory directly to the classical dissipative action system in order to define quantum (non-unitary) transition amplitudes.

In this letter, we intend to propose a (formal) solution for this problem by considering the quantization of a classical particle subject to damping and an external stochastic potential,[1] by means of a Feynman path integral as formulated in Ref. 2.

Let us start our analysis by proposing the following classical Langevin equation in a form appropriate to apply the Feynman path integral formalism of Ref. 2, i.e., a generalized Hamilton-Jacobi equation with damping term and a stochastic potential.3

$$\frac{\partial S(\boldsymbol{r},t)}{\partial t}+\frac{1}{2m}|\nabla S(\boldsymbol{r},t)^2=-\lambda S(\boldsymbol{r},t)+V(\boldsymbol{r},t)+\phi(\boldsymbol{r},t)\ , \tag{A.1}$$

here $S(r,t)$ is the proposed classical equation for the Brownian particle, $V(\boldsymbol{r},t)$ is the deterministic potential $-\lambda S(\boldsymbol{r},t)$ with $\lambda>0$ and denoting the term which is related to the damping effects on the motion of the particle. Finally, $\phi(\boldsymbol{r},t)$ is the (intrinsic) stochastic Gaussian noise potential responsible for the classical stochastic behavior of the Brownian particle.

In the Feynman formalism, one should define as a quantum transition amplitude for the quantum version of the classical dissipative system, Eq. (1), the following sum over quantum trajectories:

$$G((\boldsymbol{r},t);(\boldsymbol{r}',t'))=\sum_{\{\ell\}}\left\langle exp\left(\frac{i}{\hbar}S[(\boldsymbol{r},t);(\boldsymbol{r}',t')]\right)\right\rangle_{\phi}\ . \tag{A.2}$$

Here ℓ is some trajectory of the classical system, Eq. (1), $S(\boldsymbol{r},t)$;$(\boldsymbol{r}',t')$ is the classical generalized action of the system and $\langle\cdots\rangle_\phi$ denotes the stochastic average over all realizations of the random potential $\phi(\boldsymbol{r},t)$.[3]

The above formula, Eq. (2), is symbolic, but while in the case of non-damping $\lambda=0$ and no stochasticity, i.e., $\phi(\boldsymbol{r},t)\equiv 0$, we know how to decipher and compute it, in the general dissipative case, Eq. (1), such knowledge is not available presently.[2] It is our purpose to overcome, at least partly, this drawback.

As a first step, we solve the generalized Hamilton-Jacobi equation, Eq. (1), by means of the ansatz

$$S(\boldsymbol{r},t) = e^{-\lambda t} S^{(0)}(\boldsymbol{r},t) \tag{A.3}$$

with $S^{(0)}(\boldsymbol{r},t)$ satisfying the usual Hamilton-Jacobi equation with time dependent parameters, including the mass term, i.e.,

$$\frac{\partial}{\partial t} S^{(0)}(\boldsymbol{r},t) + \frac{1}{2me^{\lambda t}} |\nabla S(\boldsymbol{r},t)|^2 = e^{\lambda t}(\phi(\boldsymbol{r},t) + V(\mathbf{r},t)) \ . \tag{A.4}$$

An exact solution of Eq. (4), in terms of the usual action, is easily given in terms of the Caldirola-Kanai action[2]

$$S^{(0)}(\boldsymbol{r},t) = \int_{t'}^{t} d\sigma e^{\lambda\sigma} \left\{ \frac{m}{2} \left(\frac{d\boldsymbol{r}}{d\sigma} \right)^2 - (\phi(\boldsymbol{r}(\sigma)) + V(\boldsymbol{r}(\sigma),\sigma)) \right\} \ . \tag{A.5}$$

which, by its turn, leads to the following expression for our complete phase factor, Eq. (3):

$$S(\boldsymbol{r},t) = \int_{t'}^{t} d\sigma e^{\lambda(\sigma - t)} \left\{ \frac{1}{2} m \left(\frac{d\boldsymbol{r}}{d\sigma} \right)^2 - [\sigma(\boldsymbol{r}(\sigma),\sigma) + V(\boldsymbol{r}(\sigma),\sigma))] \right\} \ . \tag{A.6}$$

It is worth point out that such modified Caldirola-Kanai Lagrangian was firstly considered in similar (but different) quantization situation in Ref. 4.

Following now the procedure exposed in Ref. 2, we consider the discretized version of Eq. (6), i.e.,

$$\begin{aligned} &\hat{S}((\mathrm{x}_{k+1}, t_{k+1}); (\mathrm{x}_k, t_k)) \\ &= e^{\lambda(t_k - t_{k+1})} \left[\frac{1}{2} m \frac{(\mathrm{x}_{k+1} - \mathrm{x}_k)^2}{\varepsilon^2} - \varepsilon V(\mathrm{x}_k, t_k) - \varepsilon \phi(\mathrm{x}_k, t_k) \right] \ . \end{aligned} \tag{A.7}$$

At this point our study, we remark that the short-time transition amplitude, in the Feynman path integral and propagator formalism, is given explicitly by the asymptotic result, i.e.

$$G((\mathrm{x}_{k+1}, t_{k+1}); (\mathrm{x}_k, t_k)) \cong A(t_{k+1}, t_k) exp \left[\frac{i}{\hbar} \hat{S}((\mathrm{x}_{k+1}, t_{k+1}); (\mathrm{x}_k, t_k)) \right] \ , \tag{A.8}$$

where $t_{k+1} - t_k \to 0$.

The pre-factor in Eq. (7) is easily obtained from the $t \to 0^+$ condition, i.e.,

$$\lim_{(t_{k+1}-t_k)\to 0} G((\mathrm{x}_{k+1}, t_{k+1}); (\mathrm{x}_k, t_k)) = \delta^{(D)}(\mathrm{x}_{k+1} - \mathrm{x}_k) \ , \tag{A.9}$$

and leading, thus, to the exact result:

$$A(t_{k+1}, t_k) = e^{D\lambda(t_k - t_{k+1})/2} \left[\frac{m}{2\pi\hbar(t_{k+1} - t_k)}\right]^{D/2} \ . \tag{A.10}$$

As a consequence of the above displayed formulae, we obtain the finite time propagator, i.e.

$$\begin{aligned} & G((\boldsymbol{r}, t); (\boldsymbol{r}', t')) \\ & \lim_{N\to\infty} \int \left(\prod_{k=0}^{N-1} d\boldsymbol{r}_k\right) exp\left\{\frac{D\lambda}{2}\left[\sum_{k=0}^{N}\left(t' + \frac{t-t'}{N}k\right) - \left(t' + \frac{t-t'}{N}(k+1)\right)\right]\right\} \\ & \times \prod_{k=0}^{N-1}\left[\frac{m}{2\pi\hbar(t_{k+1} - t_k)}\right]^{D/2} \\ & exp\left\{\frac{i}{\hbar}\prod_{k=0}^{N} \varepsilon e^{\lambda(t_k - t_{k+1})/2}\left[\frac{(\boldsymbol{r}_{k+1} - \boldsymbol{r}_k)^2}{\varepsilon^2} + V(\boldsymbol{r}_k, t_k) + \phi(\boldsymbol{r}_k, t)\right]\right\} \ . \end{aligned} \tag{A.11}$$

Now it is easy to evaluate the sum in Eq. (1), where D is the space-time dimension:

$$e^{D\lambda[\sum_{k=0}^{N-1}(t'+\varepsilon k) - (t'+\varepsilon(k+1))]/2} = e^{-D\lambda(t-t')/2} \ , \tag{A.12}$$

and thus arrive at the following computable Feynman path integral (without making the evaluation of the classical stochastic average over the random potentials yet), i.e.,

$$\begin{aligned} G_\phi((\boldsymbol{r}, t); (\boldsymbol{r}, t')) = e^{-D\lambda(t-t')/2} \int_{\boldsymbol{r}(t')=\boldsymbol{r};\boldsymbol{r}(t)=\boldsymbol{r}} \\ \times \mathcal{D}^F[\boldsymbol{r}(\sigma)] e^{(i/h)\int_{t'}^{t} d\sigma e^{\lambda(\sigma - t)}[(1/2)m(d\boldsymbol{r}/d\sigma)^2 - V(\boldsymbol{r}(\sigma),\sigma) - \lambda(\boldsymbol{r}(\sigma),\sigma)]} \ . \end{aligned} \tag{A.13}$$

Note that, in contrast to previous studies (Refs. 2 and 4), the dissipative anomaly in Eq. (13) decays to zero at the equilibrium limit $t \to \infty$, without any regularization assumption.

At this point we take the average of Eq. (13) in the ensemble of the classical stochastic potentials $\{\phi(\boldsymbol{r}, t)\}$ with the result

$$\left\langle e^{(i/h)} \int_{t'}^{t} d\sigma e^{\lambda(\sigma - t)\phi(\boldsymbol{r}(\sigma),\sigma)} \right\rangle_\phi = e^{-(\gamma^2/\hbar^2)\int_{t'}^{t} d\sigma d\sigma' e^{\lambda(\sigma+\sigma')} e^{-2\lambda t} f(\sigma-\sigma') V(\boldsymbol{r}(\sigma) - (\boldsymbol{r}(\sigma'))} \ . \tag{A.14}$$

The complete propagator takes, thus, the final form

$$
\begin{aligned}
&G((\boldsymbol{r},t);(\boldsymbol{r},t')) = e^{-D\lambda(t-t')/2} \\
&\times \int_{\boldsymbol{r}(t')=\boldsymbol{r}';\boldsymbol{r}(t)=\boldsymbol{r}} \mathcal{D}^F[\boldsymbol{r}(\sigma)] e^{(i/h)\int_{t'}^{t} d\sigma e^{\lambda(\sigma-t)[(1/2)m(d\boldsymbol{r}/d\sigma)^2} - V(\boldsymbol{r}(\sigma),\sigma)]} \\
&\times e^{-(\gamma^2/\hbar^2)\int_{t'}^{t} d\sigma d\sigma' \ e^{\lambda(\sigma+\sigma')} e^{-2\lambda t} f(\sigma-\sigma') V(\boldsymbol{r}(\sigma)-\boldsymbol{r}(\sigma'))} \ . \qquad \text{(A.15)}
\end{aligned}
$$

Note that this is very similar, in its mathematical structure, to the Feynman path integral for the polaron problem.[5]

Acknowledgement

Luiz C.L. Botelho is thankful to Professor Helayël-Neto from CBPF/UCP for warm hospitality.

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