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INVARIANCE OF FIELD THEORY UNDER TIME INVERSION

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The invariance of Field Theory under Lorentz transformations and in particular under time inversion, was extensively analyzed by the author in his Princeton Thesis*. The present paper reproduces the main part of the second chapter of the referred thesis, except for some details.

A general analysis of the proper Lorentz transformations in Field Theory is made in section A, mainly for the purpose of illustrating how to formulate the problem of invariance in Field Theory.

In Section B the antiunitary transformation for time inversion of the type suggested by Wigner[1, 2] is introduced (transformation I) and in section C, after an analysis of the invariance of Quantum Electrodynamics under transformation I, a new form of time inver-

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* J. Tiomno, "Theories on Neutrino and the Double Beta Decay", Princeton Thesis, September, 1950. The numbers of the formulae used in the present paper refer to those of Part II of the Thesis which this paper is intended to divulgate. Some changes of order in the presentation were made, and some details were suppressed. No references to posterior work on the same subject were added.

sion is introduced (transformation II) which differs from the previou one by a charge conjugation. In section D two other possibilities (transformations III and IV) are shown to be possible for zero mass particles. The introduction of phase factors in the improper Lorentz transformations is considered in section E and some consequences are analyzed. Finally, a reexamination of the covariant field quantities on these lines is made in section F.

A. Preliminary concepts and results.

Following the lines of Wightman and Wigner[3] it is convenient, for simplicity of language, to introduce the concepts of bodily identity and subjective identity. A given physical system (or several equivalent systems) can be viewed by several observers using different coordinate frames. The system and its state, although in different relations to the several observers, is then said to be bodily identical for all of them. On the other hand two different systems, each observed by a different observer, are said to be subjectively identical if they each bear the same relation, to the corresponding observers. In what follows we shall interpret the Lorentz transformation as a passive transformation, i.e., a change in the coordinate system given by:

$$x^{\mu} \longrightarrow \tilde{x}^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} , \qquad (1a)$$

with

$$a^{\mu}{}_{\nu}\, a_{\mu\lambda} = g_{\nu\lambda} . \qquad (1b)$$

The tilde will be used to indicate the quantities (coordinates, wave functions, operators) used by the new observer.

As a consequence of transformation (1) the wave function Ψ, observable quantities and auxiliary field quantities (ψ,A) will be in general also transformed. The new quantities (with tilde) refer, of course, to the bodily identical system. We shall use here the usual condition for relativistic invariance: the invariance of the form of the equations of motion.

Among the many possible formulations of the Lorentz Transformation in Quantum Theory there are two especially simple, the Heisenberg and the Schrodinger[3]. These derive their name as a result of their similarity with the corresponding representations of the Field Theory.

1). Schrödinger type of Lorentz transformation.

Here the two observers use the same wave function for two subjectively identical systems (and thus, different wave functions for bodily identical systems). On the other hand both observers will work with the same set of operators.

Thus we have the transformation:

$$\psi[\sigma] \longrightarrow \tilde{\psi}[\sigma] = U\psi[\sigma] \tag{2a}$$

where U is in general a unitary operator (we exclude here the anti-unitary case of time inversion, which we consider in section B.

U will be chosen in such a way that we have

$$U\psi(\ )U^{-1} = \Lambda^{-1}\psi(x) \tag{2b}$$

for spinor fields ψ, where the operator Λ (acting on the spinor space) is such that:[4]

$$\Lambda\gamma^{\mu}\Lambda^{-1} = a^{\mu}_{\nu}\,\gamma^{\nu} \tag{2c}$$

and

$$UA_{\mu}(x)U^{-1} = a^{\nu}_{\mu}A_{\nu}(\tilde{x}) \tag{2d}$$

for vector fields; the generalization to other types of fields is obvious. We use throughout Schwinger's notation [5].

2). Heisenberg type of Lorentz transformation.

In this case the same wave function is used by the two observers for bodily identical systems, say, for the same physical system as viewed by both of them, although they will use different observables and auxiliary field quantities. The transformations are now, instead of (2):

$$\psi[\sigma] \longrightarrow \tilde{\psi}[\sigma] = \psi[\sigma] \tag{3a}$$

$$\psi(x) \longrightarrow \tilde{\psi}(\tilde{x}) = \Lambda\psi(x) \tag{3b}$$

$$A_\mu(x) \longrightarrow \tilde{A}_\mu(\tilde{x}) = a^\nu{}_\mu A_\nu(x) \tag{3c}$$

The justification of the last two transformations is as follows. The equivalence of the Heisenberg and Schrödinger forms of Lorentz transformation imposes the equality of the expectation values of corresponding quantities:

$$\left(\psi[\sigma], \tilde{F}(\tilde{x})\psi[\sigma]\right)_H = \left(\tilde{\psi}[\sigma], F(\tilde{x})\tilde{\psi}[\sigma]\right)_S \tag{4}$$

where the indices H and S appear only to indicate to us that we use, respectively, the Heisenberg and Schrodinger transformed quantities. From (4) there results

$$\tilde{F}(\tilde{x}) = U^{-1} F(\tilde{x}) U, \tag{5}$$

which was used in obtaining (3b,c) from (2).

It should be observed that in the Heisenberg form of the Lorentz transformation the spinor field quantities $\psi(x)$ transform in the same way as the wave functions $\varphi(x)$ in the one particle Dirac equation [4].

3). Relation between the transformation (2b) for the field operator $\psi(x)$ and the transformation of the one particle Dirac wave function $\varphi(x)$.

In order to justify the transformation (2b) which we assumed as valid in the Schrödinger type of Lorentz transformation we should verify that it leads to the usual transformation for the one particle Dirac wave function $\varphi(x)$ in configuration space:

$$\varphi(x) \longrightarrow \tilde{\varphi}(\tilde{x}) = \Lambda \varphi(x) \tag{6}$$

This can be done in the following way. We first express the quantum wave function Ψ for a system with one particle (say, electron) as:

$$\Psi = \int_\sigma d\sigma_\mu \bar{\psi} \gamma^\mu \varphi(x) \Omega_0 \tag{7}$$

where $\psi_+(x)$ is the creation operator for electrons and Ω_0 is the vacuum wave function, which satisfies the relation

$$\psi_+(x)\, \Omega_0 = 0 \tag{8}$$

$\psi_+(x)$ being the anihilation operator for electrons.

Now, if we write the new wave function in the same form as (7):

$$\tilde{\Psi} = \int_\sigma d\tilde{\sigma}_\mu \overline{\psi_+}(\tilde{x}) \gamma^\mu \tilde{\varphi}(\tilde{x}) \tilde{\Omega}_0 \tag{9}$$

we see, by application of (2) to (7) and comparison with (9), that the positive energy wave function $\varphi(x)$ transforms according to (6). The extension of this method to integer spin fields is obvious.

B. Time inversion.

In addition to the requirements of invariance of our field theories under the proper, restricted, Lorentz group, we require also invariance under the improper Lorentz Group.

The requirement of invariance under space reflexions leads to the well known, and very useful, concept of parity. The condition of invariance under time inversion is equivalent to the assumption of the Principle of Microscopic Reversibility[6].

For the transformation, by time inversion, of the one particle Dirac wave function in configuration representation we adopt Wigner's anti-unitary transformation[1,2]

$$\varphi(x) \longrightarrow \tilde{\varphi}(\tilde{x}) = \Lambda \varphi^{*}(x) = \gamma_5 C \varphi^{*}(x) \tag{10}$$

which transforms positive energy wave functions into positive energy ones. The matrix C in (10) is defined by:

$$C \gamma^{\mu} C = - \gamma^{\mu}_{T} \tag{10a}$$

$$C^{+} = C^{-1} = C_T = C \tag{10b}$$

where the indice T is used to indicate the transposed matrix. Now, by the method described in the subsection (A-3) we find, for the Schrodinger type of transformation in Field Theory, that assuming

$$\Psi[\sigma] \longrightarrow \tilde{\Psi}[\sigma] = U \Psi^{*}[\sigma] \tag{11a}$$

the field operator $\psi(x)$ should transform as:

$$U \psi^{*}(x) U^{-1} = \Lambda^{-1} \psi(\tilde{x}) \tag{11b}$$

with

$$\Lambda = \gamma_5 C \tag{11c}$$

In (11b) $\psi^{*}(x)$ is the operator which is represented by the complex conjugate (not hermitian conjugate!) of the matrix which corresponds to $\psi(x)$ in the same representation.

Now, in order to find the Heisenberg form of the time-inversion transformation, equivalent to (7) we impose again condition (4) for the invariance of expectation values and find, instead of (5):

$$\tilde{F}(\tilde{x}) = \left[U^{-1} F(x) U \right]_t \tag{12}$$

where the index t means the transposed of the operator in the bracket. (In other words, F_t is the operator which is represented by the transposed matrix of that which represents F).

Thus we find, for the Heisenberg form of Lorentz transformation:

$$\Psi[\sigma] \longrightarrow \tilde{\Psi}[\sigma] = \Psi[\sigma] \tag{13a}$$

$$\psi(x) \longrightarrow \tilde{\psi}(\tilde{x}) = \Lambda \psi^{+}(x) = \beta \gamma_5 C \bar{\psi}(x) \tag{13b}$$

The transformation (13b), for the field operator $\psi(x)$, is formally the same as (10) for the wave function $\varphi(x)$, as in the case of unitary transformations. However, in the case of transformations (13) there is an additional operation, coming from the application of (12), in the case where F is a product of field operators:

$$F = F_1 F_2 \ldots F_n \tag{14}$$

Then, in view of (12) we find:

$$\tilde{F} = (U^{-1} F_1 U \cdot U^{-1} F_2 U \ldots U^{-1} F_n U) = (\tilde{F}_1{}^t \tilde{F}_2{}^t \ldots \tilde{F}_n{}^t)_t$$

or:

$$\tilde{F} = \tilde{F}_n \ldots \tilde{F}_2 \tilde{F}_1 \tag{15}$$

Thus we have to add to transformations (13) the operation[7]

$$F_1(x) F_2(x) \ldots F_n(x) \longrightarrow \tilde{F}_n(\tilde{x}) \ldots \tilde{F}_2(\tilde{x}) \tilde{F}_1(\tilde{x}) \tag{13c}$$

For instance*:

$$\bar{\Psi}(x)\,\psi(x) \longrightarrow \tilde{\psi}(x)\,\tilde{\bar{\psi}}(\tilde{x})$$

From now on we shall refer to the time inversion transformation (13) as transformation I. Other forms of transformation will be considered in the following sections.

C. Invariance of Quantum Electrodynamics under time inversion.

It is simpler to make this analysis in the Heisenberg representation, in which the equations of motion of the electron field $\psi(x)$ and electromagnetic field $A_\mu(x)$ are the following[5]:

$$[\gamma^\mu(\frac{\partial}{\partial x^\mu} + ie A_\mu(x)) + m]\,\psi(x) = 0 \tag{16}$$

$$\Box A_\mu(x) = j_\mu(x) \tag{17}$$

with

$$j_\mu(x) = \frac{ie}{2}\left[\bar{\psi}(x)\gamma_\mu\psi(x) - \psi(x)\gamma_\mu^T\bar{\psi}(x)\right] \tag{17a}$$

$$\frac{\partial A^\mu(x)}{\partial x^\mu}\,\Phi = 0 \tag{18}$$

These equations should be invariant under the Lorentz transformation (we restrict ourselves from now on to the Heisenberg form of Lorentz transformation). This is the case if we add to the transformation (13b) for $\psi(x)$:

$$\psi(x) \longrightarrow \tilde{\psi}(\tilde{x}) = \beta\gamma_5 C\,\bar{\psi}(x) \tag{19a}$$

the following transformation for $A_\mu(x)$:

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* One should notice that transformation (13b) together with (13c) is equivalent to

$$\psi \longrightarrow \gamma_5 C\,\psi^*$$

with the additional rule that all matrices representing physical observables should be transposed.

$$A_0(x) \longrightarrow \widetilde{A}_0(x) = A_0^{\dagger}(x) = A_0(x) \;;\; \vec{A}(x) \longrightarrow -\vec{A}(x) \tag{19b}$$

To these should be added equation (13c):

$$F_1(x)\, F_2(x) \,\ldots\, F_n(x) \longrightarrow \widetilde{F}_n(\tilde{x}) \,\ldots\, \widetilde{F}_2(\tilde{x})\, \widetilde{F}_1(\tilde{x}) \tag{19c}$$

It should be observed that transformation (19b) differs from the usual one for a covariant vector by a sign. The same is also true for the transformation of $j_\mu(x)$, thus making $j^\mu(x)\, A_\mu(x)$ a scalar density. Therefore the quantity $\frac{\partial A^\mu}{\partial x^\mu}$ is not invariant in a time inversion, as it changes sign. However, this is unimportant as this last quantity has always an expectation value zero, in view of the auxiliary condition (18).

Equation (19a) should better be written:

$$\psi_+(x) \longrightarrow \beta\gamma_5 C\, \overline{\psi}_+(x) \tag{20a}$$

$$\psi_-(x) = C\,\psi_+'(x) \longrightarrow \beta\gamma_5 C\,\overline{\psi}_-(x) = \beta\gamma_5\,\psi_+'(x) \tag{20b}$$

$$\psi_+'(x) = C\,\overline{\psi}_-(x) \longrightarrow \beta\gamma_5 C\,\overline{\psi}_+'(x) \tag{20c}$$

where

$$\psi'(x) = C\,\overline{\psi}(x)$$

is the charge conjugate field.

In transformations (20) $\psi_+(x)$ and $\psi_+'(x)$ are, respectively, operators of absorption of electrons and positrons. We see from these transformations that:

1) Positive energy operators go into positive energy ones.
2) Electrons (or positrons) go into electrons (or positrons).

3) Absorption operators go into emission operators and vice-versa.

Now, we see that besides the transformation (20) for time inversion, which is the one that we have called transformation I in sec. A, another type of time inversion exists, such as:

$$\psi_+(x) \rightarrow \beta\gamma_5 C \overline{\psi}'(x) \tag{21a}$$

$$\psi_+(x) \rightarrow \beta\gamma_5 C \overline{\psi}_+(x) \tag{21b}$$

or, instead of (19a)

$$\psi(x) \rightarrow \tilde{\psi}(\tilde{x}) = \beta\gamma_5 C \overline{\psi}'(x) \tag{22a}$$

Now the equations of motion (16, 18) will be invariant if we add to (22a), instead of (19b):

$$A_0(x) \longrightarrow -A_0(x) \;;\; \vec{A}(x) \longrightarrow \vec{A}(x) \tag{22b}$$

and keep (as is necessary [8]) equation (19):

$$F_1(x)\, F_2(x) \longrightarrow \tilde{F}_2(\tilde{x})\, \tilde{F}_1(\tilde{x}) \tag{22c}$$

We should observe that now the transformation (22b) is the usual one for covariant vectors. We shall refer to the transformation (22), for time inversion as transformation II. In this case we see from (21) or (22a) that positive energy operators go into positive energy operators and absorption operators into emission ones, as in the case of transformation I. However, now, electron operators go into positron operators and vice-versa.

If we notice that (22a) can be written as:

$$\psi(x) \longrightarrow \beta\gamma_5\psi(x) \tag{23}$$

We see that this type of time inversion is formally the same as the one given by Pauli[4], in the c-number theory, except for the additional operation (22c). This operation is, however, fundamental in the q-number theory as the anticommutation of the operators $\psi(x)$ and $\overline{\psi}(x)$ would produce otherwise a change of sign in the second member of (17) which destroys the invariance to time inversion.

The classical analogy of these two possible types of transformation, (19) and (22), for time inversion is well known:

The classical equations for the electro-magnetic field and for the charged particle are:

$$\frac{\partial F^{\mu}_{\nu}(x)}{\partial x^{\mu}} = e\int_{-\infty}^{+\infty}\frac{dy_{\nu}}{ds}\,\delta(x-y)\,ds \tag{24}$$

$$m\frac{d^2x^{\mu}}{ds^2} = eF^{\mu}_{\nu}(x)\frac{dx^{\nu}}{ds} \tag{25}$$

where

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} \tag{26}$$

$$ds = \sqrt{-dx^{\mu}dx_{\mu}} \tag{27}$$

Now we require (as in the quantum case) that after the transformation for time inversion

$$x_0 \rightarrow \tilde{x}_0 = -x_0 \; ; \quad \vec{x} \longrightarrow \tilde{\vec{x}} = \vec{x} \tag{28a}$$

the energy, which was given by $m\frac{dx^0}{ds}$ in the old system and is now $m\frac{d\tilde{x}^0}{d\tilde{s}}$, still remain positive. We must then assume that the proper time s transforms according to:

$$ds \longrightarrow d\tilde{s} = -d_s \qquad (23)$$

(or change s into -s).

Then the equations (24) and (25) will remain invariant in either of the following two cases:

a) $\tilde{A}_0(\tilde{x}) = A_0(x)$; $\tilde{\vec{A}}(\tilde{x}) = -\vec{A}(x)$.

In this case the charge, e, does not change sign.

b) $\tilde{A}_0(\tilde{x}) = -A_0(x)$; $\tilde{\vec{A}}(\tilde{x}) = \vec{A}(x)$,

and also change e into -e, i.e., interchange positrons and electrons.

D. Other possibilities for time inversion of zero mass spinor fields.

If a given massive particle of spin 1/2 is charged, or has a magnetic moment, then the possible transformations under time inversion are the ones given by (19) and (22) which we shall refer to, from now on, as I and II, respectively (for the moment we consider only interaction with the electromagnetic field). This arbitrariness of choice between I and II, which exists when we have only one type of charged spinor fields disappears when there are several of them (charged or with an anomalous magnetic moment) for the following reason. The choice of I or II foe one of these fields (say, the electron field) determines the transformation of the electro-magnetic field (as either (22b) or (19b)) and thus all the other (charged or with anomalous magnetic moment) fields should transform in the same way as the first one (at least up to a phase factor ± 1 or $\pm i$ as will be analyzed in section E.

Now if another spinor field has a zero mass we find, besides transformations I and II, two more possibilities for the transformation under time inversion for this field. These are listed in Table I and will be referred to respectively as <u>transformations III</u> and <u>IV</u>.

TABLE I

POSSIBLE TYPES OF TRANSFORMATIONS UNDER TIME INVERSION

Name of the transformation	Transformation of the spinor field	Restriction on the mass of the spin 1/2 field	Resulting form of the transformation of the electromagnetic potential $A_\mu(x)$ and of a pseudo-scalar charged field B(x), if $\psi(x)$ is a charged field.
I	$\psi \to \beta\gamma_5 C \bar{\psi}$	No restriction	$A_\mu(x)$ $-\hat{A}_\mu(x)$ $B(x)$ $-B(x)^\dagger$
II	$\psi \to \beta\gamma_5 \bar{\psi}'$	No restriction	$A_\mu(x)$ $+\hat{A}_\mu(x)$ $B(x)$ $-B'(x)^\dagger$
III	$\psi \to \beta C \bar{\psi}$	Zero mass	Same as for transformation II
IV	$\psi \to \beta C \bar{\psi}'$	Zero mass	Same as for transformation I

In all these cases the additional reordering operation (22c) should be performed.

The quantity $\hat{A}_\mu(x)$ used in the fourth column of Table I is defined by:

$$\hat{A}_0(x) = -A_0(x) \; ; \quad \hat{\vec{A}}(x) = \vec{A}(x). \qquad (29)$$

The results given in the fourth column of Table I were found as follows:

1) The transformation of $A_\mu(x)$ should be the same as that of:

$$j_\mu(x) = \frac{ie}{2}\left[\bar{\psi}(x)\gamma_\mu\psi(x) - \psi(x)\gamma_\mu^T\bar{\psi}(x)\right] \qquad (17)$$

if $\psi(x)$ is a charged field, in order that the wave equations (16) (17) (eventually with $m = 0$) of the field ψ in interaction with the electromagnetic fields would be invariant[9].

2) The transformation of a charged pseudoscalar field (B(x) which also interacts with the electromagnetic field was obtained by the following condition. The current density vector

$$j_\mu(x) = \frac{ie}{2}\left(B(x)^\dagger \frac{\partial B(x)}{\partial x^\mu} - B(x)\frac{\partial B(x)^\dagger}{\partial x^\mu}\right) \qquad (30)$$

should transform in the same way as $A_\mu(x)$. Now the transformation of $A_\mu(x)$ was conditioned by that of $\psi(x)$. Thus if both $\psi(x)$ and $B(x)$ interact with the electromagnetic field $A_\mu(x)$ then the transformation of $B(x)$ under time invertion is conditioned by that of $\psi(x)$ in the way given in Table I.

It should be observed that $B'(x)$ used in Table I is the charge conjugate field to $B(x)$:

$$B'(x) = B(x)^\dagger \qquad (31)$$

Also, in close analogy with the corresponding transformation for $\psi(x)$ positive energy absorption operators of the field $B(x)$ go into positive energy emission operators. This is both true if either $B_+(x) \longrightarrow -B_+(x)^\dagger$ (transformations I, IV) in which case positive particles go into positive particles, or

$B_+(x) \longrightarrow -B_+'(x)^\dagger$ (transformations II, III),

when positive particles go into negative particles.

E. Phase factors in the improper Lorentz transformations of spinor fields.

The possibility of introducing a phase factor ±1 or ± i in the same transformations, under the improper Lorentz group, of a spinor field ψ :

$$\psi \longrightarrow \Lambda \psi \qquad (47a)$$

was first shown by Racah [10]. He argued that Pauli's condition [11]:

$$\det \Lambda = 1 \qquad (47b)$$

would still be satisfied if we substitute Λ by $f\Lambda$ in (47a), if the phase factor f is equal to ± 1 or ± i. Such a phase factor could not be introduced for the continuous restricted transformations as the limiting value of Λ for no change of coordinates should be the unit matrix. However, Racah introduced a symmetry principle [10], whose physical meaning is not clear to us, by which the antiparticle field ($\psi' = C\bar{\psi}$) should transform in the same way as the particle field (ψ).

Recently this question of phase factors in the improper transformations was reanalyzed by Yang and the present author [12], Racah's principle discarded, and inferences were drawn for the interaction of several fields.

This analysis will be repeated, in more detail, in the present section.

1) Phase factors and conservation of particles.

Besides Racah's justification of the introduction of the phase factors in the improper transformations of spinor fields there is another argument which is more appropriate as a justification of this possibility for the case of antiunitary time inversions. It is known that the proper transformations for spinor fields are double valued as a consequence of the fact that for a space rotation of 360 degrees, which is physically equivalent to no rotation at all, we have the transformation:

$$\psi \longrightarrow -\psi \tag{48a}$$

in contrast to the case of no rotation when we have

$$\psi \longrightarrow \psi \tag{49b}$$

(We use, as before, Heisenberg type of transformation).

In other words, two consecutive rotations of 180°, which are physically equivalent to no rotation at all, bring about a change of sign in the field operators $\psi(x)$. Thus there is no reason why we should exclude the similar possibility of a change of sign of $\psi(x)$ after two improper transformations [13], which are physically equivalent to no transformation at all.

Thus we see that, if we take for the improper transformations [14]

$$\psi \longrightarrow f\beta\psi \qquad \text{(space reflexion)} \tag{49}$$

$$\psi \longrightarrow i f_I \beta\gamma_5 C \bar{\psi} \qquad \text{(time inversion I)} \tag{50a}$$

$$\psi \longrightarrow i f_{II} \beta\gamma_5 C \bar{\psi}' \qquad \text{(time inversion II)} \tag{50b}$$

(where f, f_I and f_{II} are not necessarily the same for different fields). We find that, after two identical transformations of one of these forms the field operator $\psi(x)$ will change as:

$$\psi(x) \longrightarrow f^2\psi(x) \quad \text{or} \quad \psi(x) \longrightarrow f_{II}^2\,\psi(x) \tag{51a}$$

in the cases of (49) and (50b) respectively, and:

$$\psi(x) \longrightarrow -f_I^* f_I\,\psi(x) \tag{51b}$$

in the case of (50a).

Thus we see that for (49) and (50b) $\psi(x)$ will be unchanged if:

$$f = \pm 1 \quad \text{or} \quad f_{II} = \pm 1, \tag{52a}$$

respectively, and will change sign if:

$$f = \pm i \quad \text{or} \quad f_{II} = \pm i, \tag{52b}$$

respectively. In the case of (50a) $\psi(x)$ will change sign, regardless of the value of f_I, which should, however, be unimodular for the conservation of normalization:

$$f_I^* \quad f_I = 1 \tag{53}$$

We did not consider the cases of transformations III and IV because in view of the argument given in the preceding section, they are of no interest for our future analysis.

From now on we shall arbitrarily assume, transformation II, i.e., (50b) for the charged particles (and neutrons), in view of the fact that in this case the electromagnetic potential $A_\mu(x)$ transforms under time inversion, as an ordinary covariant vector (see Table I). This is not, however, a restriction of generality as, it is easy to see, the assumption of transformation I, i.e., (50a) for these particles would not change the qualitative results of the following analysis.

Transformation I will be retained, as a possibility, for the neutrino case. As a consequence of the condition for invariance of the interaction hamiltonian $\mathcal{H}(x)$ we will be then restricted to the values ± 1, $\pm i$, for f_I.

Finally, in order not to introduce scalar-like quantities which would be invariant under space reflexion (time inversion) and change sign under time inversion (space reflexion) we take from now on (also arbitrarily in principle):

$$f_I = \pm f \quad ; \quad f_{II} = \pm f \tag{54}$$

It should be observed here that the usual assumption that a phase factor in the field operators $\psi^{(\ell)}(x)$ is irrelevant applies only to the cases when there is <u>conservation of particles</u>, i.e., when in a given process a certain particle disappears either another particle could be anihilated instead, or an antiparticle (not a particle) could be created. This because the mathematical expression of such a conservation principle is just that $\mathcal{H}(x)$ should be invariant under a transformation by a phase factor η, <u>the same for all particles</u> and η^* for the antiparticles. Now an interaction of the type:

$$\mathcal{H}(x) = g\,\bar{\psi}_P(x)\,\psi_N(x)\,\psi_e(x)\left[\psi_\nu(x) + C\bar{\psi}_\nu(x)\right] + h.c. \tag{55}$$

which is invariant under, say, (49) and (50b) with $f = \pm i$ ($f_{II} = f$) for all particles, but not with $f = \pm 1$, is surely not invariant under a phase transformation of the type above referred to (say, for instance with $\eta = i$). The physical meaning of this fact is that interaction (55) leads to no conservation of particles as either a neutrino or antineutrino can be emitted, in this case, in a given process. However, interaction (55) still satisfies the conditions of conservation of charge and conservation of nucleons, mathematically expressed by its invariance under an arbitrary phase transformation for the charged particles or for the nucleons, respectively. This example should be enough to show the importance of the phase factors f in the improper transformations.

2) Phase factors in the improper transformations for nucleons and the symmetry properties of the π meson.

As it was shown by Yang and Tiomno [12] the fact that we do not know the relative signs of the improper transformation for the proton and neutron field would leave undetermined the reflexion properties of the π meson field even if the form of the interaction of these particles would be experimentally found. As an example, if we assume an interaction of the type:

$$A_{\pi} \bar{\Psi}_P \gamma^5 \Psi_N + \text{h.c.} \qquad (56)$$

thus the π meson would be pseudoscalar, if Ψ_P and Ψ_N transform in the same way under the improper Lorentz group, or scalar, if they transform with opposite signs. This because $\bar{\Psi}_P \gamma^5 \Psi_N$ which is a pseudoscalar quantity in the first case, is a scalar in the second case.

3) Signs of the improper Lorentz transformations and the representations of the Dirac equations for a system of two or more interacting fields:

It is interesting to relate the results of the previous section to the fact that when there are two interacting spinor fields there are two different representations for the Dirac equations of these fields (we consider the interaction representation): that in which the mass terms in the equations for the two fields:

$$\left(\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} \pm M_{(r)} \right) \psi^{(r)}(x) = 0 \quad (r = 1,2) \qquad (57)$$

have the same sign and that in which they have opposite signs.

Now we see that if we start from the case when $\psi^{(1)}$ and $\psi^{(2)}$ do satisfy Dirac equations with the same sign of the mass term, but transform under the improper group (say by I) with opposite signs and make a transformation:

$$\psi^{(1)} \longrightarrow \psi^{(1)} \quad , \psi^{(2)} \longrightarrow \gamma_5 \psi^{(2)} \qquad (57)$$

the new fields will transform now in the same way but will satisfy Dirac equations with opposite signs of the mass term. Also the quantity $\bar{\Psi}^{(1)} \gamma_5 \psi^{(2)}$, which was a scalar in the first representation will go into $\bar{\psi}^{(1)}\psi^{(2)}$, a scalar in the new representation.

Another interesting example is the comparison of the Fermi theories characterized by the interactions:

$$\mathcal{H}_1(x) = g\bar{\psi}_p(x)\psi_N(x)\bar{\psi}_e(x)\psi_\nu(x) + h.c \tag{58}$$

$$\mathcal{H}_2(x) = g\bar{\psi}_p(x)\psi_N(x)\bar{\psi}_e(x)\gamma^5\psi_\nu(x) + h.c \tag{59}$$

when the same sign of the mass term is taken for all particles. $\mathcal{H}_1(x)$ is a scalar density if, say, all particles transform by II with the same phase factor. $\mathcal{H}_2(x)$ is an invariant if in the improper transformation for the neutrino the opposite sign is used in relation to those for the other fields. Now if we make the transformation:

$$\psi_\nu \longrightarrow \gamma_5\psi_\nu \tag{60}$$

then $\mathcal{H}_2(x)$ will go into $\mathcal{H}_1(x)$ but the mass term of the Dirac equation for the neutrino will change sign. Thus we see that the theories characterized by $\mathcal{H}_1(x)$ and $\mathcal{H}_2(x)$ will be equivalent if the mass of the neutrino is zero.

F. <u>Covariant field quantities</u>.

Let us consider a general situation in which there are several spin 1/2 particles which interact with the electromagnetic field (via its charge or anomalous magnetic moment), a charged meson field (pseudoscalar) and a spinor field of zero mass.

The first group of spinor fields (proton, neutron, electron and μ-meson fields), transform necessarily by I or II. Also it should be remembered that if one of them transforms, under time inversion, by I (or II) then all the others transform in the same way. The Boson (pseudoscalar) field, whose transformation is conditioned by that of the charged fields in the way indicated in Table I, will be identified to the π-meson in some examples. The zero mass field of spin 1/2 (neutrino) may transform by any one of I, II, III or IV, under time inversion.

The interaction representation of Field Theory will be used here. The condition of relativistic invariance is now, besides that of invariance of the free field equations, the invariance of the interaction hamiltonian $\mathcal{H}_0(x)$ under the considered transformation.

We consider now the possible cases of combination of these transformations, the resulting covariant quantities and the restrictions which may result on the form of the interactions from the assumption of a special combination of transformations. We quote here only the following results from the thesis:

1) If the neutrino field ψ_ν transforms by IV and the fields $\psi(x)$ by II, then no form of Fermi interaction is possible which is invariant under the complete Lorentz group.

2) If we use transformation III for the neutrino and I for the other fields then we are restricted to Fermi interactions of the type (in the scalar case):

$$\mathcal{H}(x) = g\,\bar{\psi}_P\,\psi_N\,\bar{\psi}_e\,(\psi_\nu + \gamma_5\, c\,\bar{\psi}_\nu) + h.c. \qquad (34)$$

3) If all fields (including neutrino) transform in the same way, we find for the usual covariant quantities, according to the type I or II of the transformation, the one-field quantities given in Table II.

TABLE II

	Transformation I	Transformation II
Scalar	$\bar{\psi}\psi$	$\bar{\psi}\psi + \bar{\psi}'\psi'$
Vector	$\bar{\psi}\gamma_\mu\psi$	$\bar{\psi}\gamma_\mu\psi - \bar{\psi}\gamma_\mu\psi'$
Tensor	$\bar{\psi}\gamma_{\mu\nu}\psi$	$\bar{\psi}\gamma^{\mu\lambda}\psi - \bar{\psi}'\gamma'^{\mu\lambda}\psi'$
Pseudovector	$\bar{\psi}\gamma_5\gamma_\lambda\psi$	$\bar{\psi}\gamma_5\gamma^\mu\psi + \bar{\psi}'\gamma^5\gamma'^\mu\psi'$
Pseudoscalar	$\bar{\psi}\gamma_5\psi$	$\bar{\psi}\gamma^5\psi + \psi'\gamma'^5\psi'$

We see that Case's conclusion [7] that condition of invariance of field theory under time inversion imposes the form given in the last column for the covariant quantities is only correct for the type II of transformation under time inversion.

For the two field quantities the results are of the type here indicated for the scalar quantity:

(a) If both fields transform by I, the scalar is

$$\bar{\psi}\psi_\nu + \bar{\psi}_\nu\psi$$

(b) If we adopt transformation II we find

$$\bar{\psi}\psi_\nu \longrightarrow -\psi_\nu\bar{\psi} = \bar{\psi}\psi_\nu$$

the last step being true only if the two different fields <u>anticommute</u>.

4.) If ψ_ν transforms by I and ψ by II, the scalar is now

$$\bar{\psi}(\psi_\nu + c\,\bar{\psi}_\nu)$$

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3) The concept used in section A and results of items 1 and 2 are mainly due to A.S.Wightman and E.P. Wigner (unpublished).

4) W.Pauli, Handbuck der Physik, Vo. 241, p.259, Rev. Mod. Phys. 13, 203, 1941.

5) J. Schwinger, Phys. Rev. 74, 1939, 1948; 75, 651, 1949; 76, 790, 1949.

6) E. P. Wigner, private communication.

7) The rule for reverting the order of the field operators in connection with time inversion was first obtained by K.M. Case by imposing the invariance of the commutation relations (private communication). Application of this was made in his paper in Phys. Rev. 76, 1, 1949.

8) This is because in the Schrodinger form of the Lorentz transformation we use again an anti-unitary transformation of the type (11a), as is necessary in order that positive energy electron (positron) wave functions in the one particle configuration space will go into positive energy positron (electron) wave function.

9) In the case of transformation I we could substitute $A_\mu(x) \rightarrow -\hat{A}_\mu(x)$ by equivalent transformation:
$A_\mu(x) \rightarrow A_\mu(x)$; $e \rightarrow -e$ (change in sign of the charge).

10) G. Racah, Nuovo Cim., 14, 322, 1937.

11) W. Pauli, Ann. Inst. H. Poincaré, 6, 109, 1936.

12) C. N. Yang and J. Tiomno, Phys. Rev. 79, 495, 1950.

13) It is of immediate verification that if $\Psi_{\alpha}(x)$ changes sign after two such transformations, the wave functions $\varphi_{\alpha}(x)$, $\varphi_{\alpha_1\alpha_2}(x_1 x_2)$, etc., in configuration representation change sign for the sub-spaces of odd number of particles and is unaltered for thos of even number of such particles.

14) It should be kept in mind that for time inversion the additional operation $F_1(x)\, F_2(x) \longrightarrow \tilde{F}_2(\tilde{x})\, \tilde{F}_1(\tilde{x})$ should be performed together with (50).