CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Rio de Janeiro

Notas de Física

An explanation for the tiny value of the cosmological constant and the low vacuum energy density

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GOVERNO FEDERAL

BRASIL

PÁTRIA EDUCADORA

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Abstract The paper aims to provide an explanation for the tiny value of the cosmo-logical constant and the low vacuum energy density to represent the dark energy. To accomplish this, we will search for a fundamental principle of symmetry in space-time by means of the elimination of the classical idea of rest, by including an invariant minimum limit of speed in the subatomic world. Such a minimum speed, unattainable by particles, represents a preferred reference frame associated with a background field that breaks down the Lorentz symmetry. The metric of the flat space-time shall include the presence of a uniform vacuum energy density, which leads to a negative pressure at cosmological length scales. Thus, the equation of state for the cosmological constant [p(pressure)$= -\epsilon$ (energy density)] naturally emerges from such a space-time with an energy barrier of a minimum speed. The tiny values of the cosmological constant and the vacuum energy density will be successfully obtained, being in agreement with the observational results of Perlmutter, Schmidt and Riess.

1 Introduction

Driven by a search for new fundamental symmetries in Nature [1,2] the paper attempts to implement a uniform background field into the flat space-time. Such a background field connected to a uniform vacuum energy density represents a preferred reference

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frame, which leads us to postulate a universal minimum limit of speed for particles with very large wavelengths (very low energies).

The idea that some symmetries of a fundamental theory of quantum gravity may have non trivial consequences for cosmology and particle physics at very low energies is interesting and indeed quite reasonable. So, it seems that the idea of a universal minimum speed as one of the first attempts of Lorentz symmetry violation could have the origin from a fundamental theory of quantum gravity at very low energies (very large wavelengths).

The hypothesis of the lowest non-null limit of speed (V) for low energies ($v << c$) in space-time results in the following physical reasoning:

- In non-relativistic quantum mechanics, the plane wave wave-function ($Ae^{\pm ipx/\hbar}$) which represents a free particle is an idealisation that is impossible to conceive under physical reality. In the event of such an idealized plane wave, it would be possible to find with certainty the reference frame that cancels its momentum ($p = 0$), so that the uncertainty on its position would be $\Delta x = \infty$. However, the presence of an unattainable minimum (non-zero) limit of speed emerges in order to prevent the ideal case of a plane wave wave-function ($p = constant$ or $\Delta p = 0$ with $\Delta x = \infty$). This means that there is no perfect inertial motion ($v = constant$) such as a plane wave, except the privileged reference frame of a universal background field connected to an unattainable minimum limit of speed V, where p would vanish. However, since such a minimum speed V (universal background frame S_V) is unattainable for the particles with low energies (large wavelengths), their momentum can actually never vanish when one tries to be closer to such a preferred frame (V), as it will be shown that there is an insuperable energy barrier when one tries to decelerate a particle very close to the vacuum regime of the background frame S_V, which represents a fundamental zero-point energy for $v \rightarrow V$ (see Sect. 5).

On the other hand, according to special relativity (SR), the momentum cannot be infinite since the maximum speed c is also unattainable for a massive particle, except the photon ($v = c$) as it is a massless particle.

This reasoning allows us to think that the electromagnetic radiation (photon:"$c - c$" $= c$) as well as the massive particle ("$v - v$" $> V$ for $v < c$) are in equal-footing in the sense that it is not possible to find a reference frame at rest ($v_{relative} = 0$) for both through any speed transformation in a space-time with a maximum and a minimum limit of speed. Thus such a doubly special relativity with an invariant minimum speed will be denominated as symmetrical special relativity (SSR). We will look for new speed transformations of SSR in the next section.

In a future paper, we will investigate the origin of the minimum speed V, which could have a direct connection with the Planck length, i.e., the minimum length $l_P = \sqrt{G\hbar/c^3}(\sim 10^{-35}m)$ in a quantum gravity theory.

The dynamics of particles in the presence of a universal background reference frame connected to V is within a context of the ideas of Sciama [3], Schrödinger [4] and Mach [5], where there should be an "absolute" inertial reference frame in relation to which we have the inertia of all moving bodies. However, we must emphasize that the approach used here is not classical as machian ideas, since the lowest (unattainable) limit of speed V plays the role of a privileged (inertial) reference frame of background field instead of the "inertial" frame of fixed stars.

It is very curious to notice that the idea of a universal background field was sought in vain by Einstein [6,7] motivated firstly by Lorentz. It was Einstein who coined the term *ultra-referential* as the fundamental aspect of reality to represent a universal background field [8]. Based on such a concept, let us call *ultra-referential S_V* to be the universal background field of a fundamental inertial reference frame connected to V.

The present doubly special relativity (SSR) is a kind of deformed special relativity (DSR) with two invariant scales, namely the speed of light c and a minimum speed V. DSR theory was first investigated by Camelia et al. [9–12]. It contains two invariant scales: the speed of light c and a minimum length scale (Planck length l_P of quantum gravity). An alternative approach to DSR theory, inspired by that of Camelia, was proposed later by Smolin and Magueijo [13–15].

Another extension of SR is known as triply special relativity, which is characterized by three invariant scales, namely the speed of light c, a mass k and a length R [16]. Still another generalization of SR is the quantizing of speeds [17], where Barrett-Crane spin foam model for quantum gravity with positive cosmological constant was considered, encouraging the authors to look for a discrete spectrum of velocities and the physical implications of this effect, namely an effective deformed Poincaré symmetry.

2 Transformations of space-time and velocity in the presence of the ultra-referential S_V

The classical notion we have about the inertial (galilean) reference frames, where the system at rest exists, is eliminated in SSR where $v > V(S_V)$ (Fig. 1). However, if we consider classical systems composed of macroscopic bodies, the minimum speed V is neglected ($V = 0$) and so we can reach a vanishing velocity ($v = 0$), i.e., in the classical approximation ($V \rightarrow 0$), the ultra-referential (background frame) S_V is eliminated and simply replaced by the galilean reference frame S connected to a classical system at rest.

Since we cannot consider a reference system made up of a set of infinite points at rest in quantum space-time with an invariant minimum speed, then we should define a new status of referentials, namely a non-galilean reference system, which is given essentially as a set of all the particles having the same state of movement (speed v) with respect to the ultra-referential S_V (preferred reference frame of the background field), so that $v > V$, V being unapproachable and connected to S_V. So, a set of particles with the same speed v with respect to the ultra-referential S_V provides a given non-galilean framework. Hence, SSR should contain three postulates, namely:

Fig. 1 S' moves with a speed $v(> V)$ with respect to the background field of the covariant ultra-referential S_V. If $V \rightarrow 0$, S_V is eliminated (empty space) and, thus, the galilean frame S takes place, recovering the Lorentz transformations
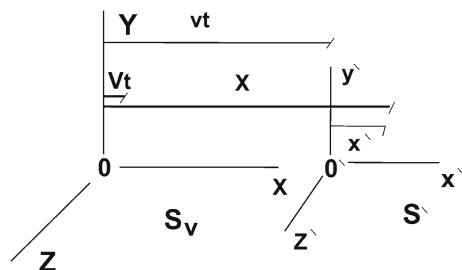
(1) The non-equivalence (asymmetry) of the non-galilean reference frames due to the presence of the background frame S_V that breaks down Lorentz symmetry, i.e., we cannot exchange v for $-v$ by means of inverse transformations, since we cannot achieve a rest state ($v = 0$) for a certain non-galilean reference frame in order to reverse the direction of its velocity only for one spatial dimension;

(2) the invariance of the speed of light (c);

(3) the covariance of the ultra-referential S_V (background framework) connected to an invariant and unattainable minimum limit of speed V, i.e., all the non-galilean reference frames with speeds $V < v \leq c$ experience the same background frame S_V, in the sense that the background energy (vacuum energy) at S_V does not produce a flow $-v$ at any of these referentials. Thus, S_V does not work like the newtonian absolute space filled by luminiferous (galilean) ether in the old (classical) sense, in spite of S_V's being linked to a background energy that works like a *non-galilean "ether"*, leading to the well-known vacuum energy density (cosmological constant), as we will show later.

The third postulate is directly connected to the second one. Such a connection will be clarified by investigating the new velocity transformations to be obtained soon.

Of course if we consider $V = 0$, we recover the well-known two postulates of SR, i.e., we get the equivalence of inertial reference frames, where one can exchange v for $-v$ with appropriate transformations and, consequently, this leads to the absence of such a background field (S_V); however, the constancy of the speed of light is still preserved.

Let us assume the reference frame S' with a speed v in relation to the ultra-referential S_V according to Fig. 1.

So, to simplify, consider the motion only at one spatial dimension, namely $(1+1)D$ space-time with the background field S_V. So we write the following transformations:

$$dx' = \Psi(dX - \beta_* c\, dt) = \Psi(dX - v\, dt + V\, dt), \tag{1}$$

where $\beta_* = \beta\epsilon = \beta(1-\alpha)$, being $\beta = v/c$ and $\alpha = V/v$, so that $\beta_* \to 0$ for $v \to V$ or $\alpha \to 1$.

$$dt' = \Psi\left(dt - \frac{\beta_* dX}{c}\right) = \Psi\left(dt - \frac{v\, dX}{c^2} + \frac{V\, dX}{c^2}\right), \tag{2}$$

being $\vec{v} = v_x \mathbf{x}$. We have $\Psi = \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\beta^2}}$. If we make $V \to 0$ ($\alpha \to 0$), we recover the Lorentz transformations, where the ultra-referential S_V is eliminated and simply replaced by the galilean frame S at rest for a classical observer.

In order to get the transformations in Eqs. (1) and (2) above, let us consider the following more general transformations: $x' = \theta\gamma(X - \epsilon_1 vt)$ and $t' = \theta\gamma(t - \frac{\epsilon_2 vX}{c^2})$, where θ, ϵ_1 and ϵ_2 are factors (functions) to be determined. We hope all these factors depend on α, such that, for $\alpha \to 0$ ($V \to 0$), we recover Lorentz transformations as a particular case ($\theta = 1$, $\epsilon_1 = 1$ and $\epsilon_2 = 1$). By using those transformations to perform $[c^2 t'^2 - x'^2]$, we find the identity: $[c^2 t'^2 - x'^2] = \theta^2\gamma^2[c^2 t^2 - 2\epsilon_1 vtX + 2\epsilon_2 vtX - \epsilon_1^2 v^2 t^2 + \frac{\epsilon_2^2 v^2 X^2}{c^2} - X^2]$. Since the metric tensor is diagonal, the crossed

terms must vanish and so we assure that $\epsilon_1 = \epsilon_2 = \epsilon$. Due to this fact, the crossed terms $(2\epsilon vt X)$ are cancelled between themselves and finally we obtain $[c^2 t'^2 - x'^2] = \theta^2 \gamma^2 (1 - \frac{\epsilon^2 v^2}{c^2})[c^2 t^2 - X^2]$. For $\alpha \to 0$ ($\epsilon = 1$ and $\theta = 1$), we reinstate $[c^2 t'^2 - x'^2] = [c^2 t^2 - x^2]$ of SR. Now we write the following transformations: $x' = \theta\gamma(X - \epsilon vt) \equiv \theta\gamma(X - vt + \delta)$ and $t' = \theta\gamma(t - \frac{\epsilon vX}{c^2}) \equiv \theta\gamma(t - \frac{vX}{c^2} + \Delta)$, where we assume $\delta = \delta(V)$ and $\Delta = \Delta(V)$, so that $\delta = \Delta = 0$ for $V \to 0$, which implies $\epsilon = 1$. So, from such transformations we extract: $-vt + \delta(V) \equiv -\epsilon vt$ and $-\frac{vX}{c^2} + \Delta(V) \equiv -\frac{\epsilon vX}{c^2}$, from where we obtain $\epsilon = (1 - \frac{\delta(V)}{vt}) = (1 - \frac{c^2\Delta(V)}{vX})$. As ϵ is a dimensionaless factor, we immediately conclude that $\delta(V) = Vt$ and $\Delta(V) = \frac{VX}{c^2}$, so that we find $\epsilon = (1 - \frac{V}{v}) = (1 - \alpha)$. On the other hand, we can determine θ as follows: θ is a function of α ($\theta(\alpha)$), such that $\theta = 1$ for $\alpha = 0$, which also leads to $\epsilon = 1$ in order to recover the Lorentz transformations. So, as ϵ depends on α, we conclude that θ can also be expressed in terms of ϵ, namely $\theta = \theta(\epsilon) = \theta[(1 - \alpha)]$, where $\epsilon = (1 - \alpha)$. Therefore we can write $\theta = \theta[(1 - \alpha)] = [f(\alpha)(1 - \alpha)]^k$, where the exponent $k > 0$. The function $f(\alpha)$ and k will be estimated by satisfying the following conditions:

(i) as $\theta = 1$ for $\alpha = 0$ ($V = 0$), this implies $f(0) = 1$.

(ii) the function $\theta\gamma = \frac{[f(\alpha)(1-\alpha)]^k}{(1-\beta^2)^{\frac{1}{2}}} = \frac{[f(\alpha)(1-\alpha)]^k}{[(1+\beta)(1-\beta)]^{\frac{1}{2}}}$ should have a symmetrical behavior, that is to say it goes to zero closer to V ($\alpha \to 1$) in the same way it goes to infinite closer to c ($\beta \to 1$). In other words, this means that the numerator of the function $\theta\gamma$, which depends on α should have the same shape of its denumerator, which depends on β. Due to such conditions, we naturally conclude that $k = 1/2$ and $f(\alpha) = (1 + \alpha)$, so that $\theta\gamma = \frac{[(1+\alpha)(1-\alpha)]^{\frac{1}{2}}}{[(1+\beta)(1-\beta)]^{\frac{1}{2}}} = \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-\beta^2)^{\frac{1}{2}}} = \frac{\sqrt{1-V^2/v^2}}{\sqrt{1-v^2/c^2}} = \Psi$, where $\theta = \sqrt{1-\alpha^2} = \sqrt{1-V^2/v^2}$.

The transformations shown in Eqs. (1) and (2) are the direct transformations from S_V [$X^\mu = (X, ct)$] to S' [$x'^\nu = (x', ct')$], where we have $x'^\nu = \Lambda^\nu_\mu X^\mu$ ($x' = \Lambda X$), so that we obtain the following matrix of transformation:

$$\Lambda = \begin{pmatrix} \Psi & -\beta(1-\alpha)\Psi \\ -\beta(1-\alpha)\Psi & \Psi \end{pmatrix}, \tag{3}$$

such that $\Lambda \to L$ (Lorentz matrix of rotation) for $\alpha \to 0$ ($\Psi \to \gamma$).

We obtain $\det\Lambda = \frac{(1-\alpha^2)}{(1-\beta^2)}[1 - \beta^2(1-\alpha)^2]$, where $0 < \det\Lambda < 1$. Since V (S_V) is unattainable ($v > V$), this assures that $\alpha = V/v < 1$ and therefore the matrix Λ admits inverse ($\det\Lambda \neq 0$ (>0)). However, Λ is a non-orthogonal matrix ($\det\Lambda \neq \pm 1$) and so it does not represent a rotation matrix ($\det\Lambda \neq 1$) in such a space-time due to the presence of the privileged frame of background field S_V that breaks strongly the invariance of the norm of the 4-vector [(limit $v \to V$ in Eqs. (30) or (31)]. Actually such an effect ($\det\Lambda \approx 0$ for $\alpha \approx 1$ or $v \approx V$) emerges from a new relativistic physics of SSR for treating much lower energies at ultra-infrared regime closer to S_V (very large wavelengths).

We notice that $\det\Lambda$ is a function of the speed v with respect to S_V. In the approximation for $v \gg V$ ($\alpha \approx 0$), we obtain $\det\Lambda \approx 1$ and so we practically reinstate

the rotation behavior of Lorentz matrix L as a particular regime for higher energies. If we make $V \to 0$ ($\alpha \to 0$), we recover $\det \Lambda \approx \det L = 1$ (rotation condition). This subject will be explored with more details in Sect. 3, where we will verify whether Eq. (3) forms a group. We will explore deep physical implications of such a result that will also lead to the tiny positive value of the cosmological constant (Sect. 6).

The inverse transformations (from S' to S_V) are

$$dX = \Psi'(dx' + \beta_* cdt') = \Psi'(dx' + vdt' - Vdt'), \tag{4}$$

$$dt = \Psi'\left(dt' + \frac{\beta_* dx'}{c}\right) = \Psi'\left(dt' + \frac{vdx'}{c^2} - \frac{Vdx'}{c^2}\right). \tag{5}$$

In matrix form, we have the inverse transformation $X^\mu = \Lambda^\mu_\nu x'^\nu$ ($X = \Lambda^{-1} x'$), so that the inverse matrix is

$$\Lambda^{-1} = \begin{pmatrix} \Psi' & \beta(1-\alpha)\Psi' \\ \beta(1-\alpha)\Psi' & \Psi' \end{pmatrix}, \tag{6}$$

where we can show that $\Psi' = \Psi^{-1}/[1-\beta^2(1-\alpha)^2]$, so that we must satisfy $\Lambda^{-1}\Lambda = I$.

Indeed we have $\Psi' \neq \Psi$ and therefore $\Lambda^{-1} \neq \Lambda(-v)$. This aspect of Λ has an important physical implication. In order to understand such an implication, let us first consider the rotation aspect of Lorentz matrix in SR. Under SR, we have $\alpha = 0$ ($V = 0$), so that $\Psi' \to \gamma' = \gamma = (1-\beta^2)^{-1/2}$. This symmetry ($\gamma' = \gamma$, $L^{-1} = L(-v)$) happens because the galilean reference frames allow us to exchange the speed v (of S') for $-v$ (of S) when we are at rest at S'. However, under SSR, since there is no rest at S', we cannot exchange v (of S') for $-v$ (of S_V) due to that asymmetry ($\Psi' \neq \Psi$, $\Lambda^{-1} \neq \Lambda(-v)$). Due to this fact, S_V must be covariant, namely V remains invariant for any change of reference frame in such a space-time. Thus we can notice that the paradox of twins, which appears due to that symmetry by exchange of v for $-v$ in SR should be naturally eliminated in SSR, where only the reference frame S' can move with respect to S_V. So, S_V remains covariant (invariant for any change of reference frame). Such a covariance will be verified soon.

We have $\det \Lambda = \Psi^2[1-\beta^2(1-\alpha)^2] \Rightarrow [(\det \Lambda)\Psi^{-2}] = [1-\beta^2(1-\alpha)^2]$. So we can alternatively write $\Psi' = \Psi^{-1}/[1-\beta^2(1-\alpha)^2] = \Psi^{-1}/[(\det \Lambda)\Psi^{-2}] = \Psi/\det \Lambda$. By inserting this result in Eq. (6) to replace Ψ', we obtain the relationship between the inverse matrix Λ^{-1} and $\Lambda(-v)$, namely $\Lambda^{-1} = \Lambda(-v)/\det \Lambda$.

By dividing Eq. (1) by Eq. (2), we obtain the following speed transformation:

$$v_{rel} = \frac{v' - v + V}{1 - \frac{v'v}{c^2} + \frac{v'V}{c^2}}, \tag{7}$$

where we have considered $v_{rel} = v_{relative} \equiv dx'/dt'$ and $v' \equiv dX/dt$. v' and v are given with respect to S_V, and v_{rel} is the relative velocity between v' and v. Let us consider $v' > v$ (see Fig. 2).
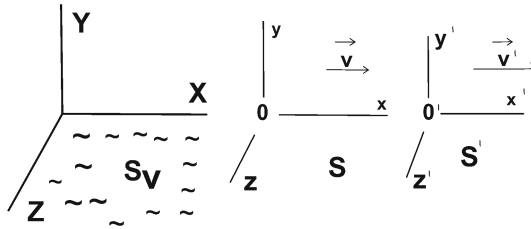
Fig. 2 S_V is the covariant ultra-referential of background field related to the vacuum energy. S represents the reference frame for a massive particle with speed v in relation to S_V, where $V < v < c$. S' represents the reference frame for a massive particle with speed v' in relation to S_V. In this case, we consider $V(S_V) < v \leq v' \leq c$

If $V \to 0$, the transformation in Eq. (7) recovers the Lorentz velocity transformation where v' and v are given in relation to a certain galilean frame S_0 at rest. Since Eq. (7) implements the ultra-referential S_V, the speeds v' and v are now given with respect to the background frame S_V, which is covariant (absolute). Such a covariance is verified if we assume that $v' = v = V$ in Eq. (7). Thus, for this case, we obtain $v_{rel} = \text{"}V - V\text{"} = V$.

Let us also consider the following cases in Eq. (7):

(a) $v' = c$ and $v \leq c \Rightarrow v_{rel} = c$. This just verifies the well-known invariance of c.

(b) if $v' > v(= V) \Rightarrow v_{rel} = \text{"}v' - V\text{"} = v'$. For example, if $v' = 2V$ and $v = V \Rightarrow v_{rel} = \text{"}2V - V\text{"} = 2V$. This means that V really has no influence on the speed of the particles. So V works as if it were an "*absolute zero of movement*", being invariant and having the same value in all directions of space of the isotropic background field.

(c) if $v' = v \Rightarrow v_{rel} = \text{"}v - v\text{"}(\neq 0) = \dfrac{V}{1 - \frac{v^2}{c^2}(1 - \frac{V}{v})}$. From (**c**) let us consider two specific cases, namely:

 – (c$_1$) assuming $v = V \Rightarrow v_{rel} = \text{"}V - V\text{"} = V$ as verified before.
 – (c$_2$) if $v = c \Rightarrow v_{rel} = c$, where we have the interval $V \leq v_{rel} \leq c$ for $V \leq v \leq c$.

This last case (c) shows us in fact that it is impossible to find the rest for the particle on its own reference frame S', where $v_{rel}(v) (\equiv \Delta v(v))$ is a function that increases with the increasing of v. However, if we make $V \to 0$, then we would have $v_{rel} \equiv \Delta v = 0$ and therefore it would be possible to find the rest for S', which would become simply a galilean reference frame of SR.

By dividing Eq. (4) by Eq. (5), we obtain

$$v_{rel} = \frac{v' + v - V}{1 + \frac{v'v}{c^2} - \frac{v'V}{c^2}} = \frac{v' + v^*}{1 + \frac{v'v^*}{c^2}}, \tag{8}$$

where we define the notation $v^* = v\epsilon = v(1 - \alpha) = v(1 - V/v) = v - V$.

In Eq. (8), if $v' = v = V \Rightarrow$ "$V + V$"$= V$. Indeed V is invariant, working like an *absolute zero state* in SSR. If $v' = c$ and $v \leq c$, this implies $v_{rel} = c$. For $v' > V$ and considering $v = V$, this leads to $v_{rel} = v'$. As a specific example, if $v' = 2V$ and assuming $v = V$, we would have $v_{rel} = $ "$2V + V$"$= 2V$. And if $v' = v \Rightarrow v_{rel} = $"$v + v$"$= \frac{2v - V}{1 + \frac{v^2}{c^2}(1 - \frac{V}{v})}$. In newtonian regime ($V << v << c$), we recover $v_{rel} = $"$v + v$"$= 2v$. In relativistic (einsteinian) regime ($v \to c$), we reinstate Lorentz transformation for this case ($v' = v$), i.e., $v_{rel} = $"$v + v$"$= 2v/(1 + v^2/c^2)$.

By joining both transformations in Eq. (7) and in Eq. (8) into just one, we write the following compact form:

$$v_{rel} = \frac{v' \mp \epsilon v}{1 \mp \frac{v'\epsilon v}{c^2}} = \frac{v' \mp v(1 - \alpha)}{1 \mp \frac{v'v(1-\alpha)}{c^2}} = \frac{v' \mp v \pm V}{1 \mp \frac{v'v}{c^2} \pm \frac{v'V}{c^2}}, \tag{9}$$

being $\alpha = V/v$ and $\epsilon = (1 - \alpha)$. For $\alpha = 0$ ($V = 0$) or $\epsilon = 1$, we recover Lorentz speed transformations.

Transformations for $(3 + 1)D$ in SSR will be treated elsewhere. In the next section, we will check whether the new transformations given by Eqs. (1) and (2) form a group, giving a physical explanation for such a result.

3 Do the space-time transformations with an invariant minimum speed form a group? What are their deep implications?

It is well-known that the Lorentz transformations form a group ($L = L(v)$), since they obey the following conditions, namely: a) $L_2 L_1 = L(v_2)L(v_1) = L(v_3) = L_3 \in L(v)$ (Closure condition); b) $L_1(L_2 L_3) = (L_1 L_2)L_3$ (Associativity); c) $L_0 L = L L_0 = L$, such that $L_0 = L(0) = I$ (Identity element); d) $L^{-1}L = L L^{-1} = L_0$, being $L^{-1} = L(-v)$ (Inverse element).

Our goal is to make an analysis of the new transformations in Eqs. (1) and (2) with regard to the conditions above in order to verify whether they form a group and discuss deeply the results. So, to do that, we first rewrite the matrix Λ (Eq. 3), namely:

$$\Lambda = \begin{pmatrix} \Psi & -\Psi\beta^* \\ -\Psi\beta^* & \Psi \end{pmatrix}, \tag{10}$$

where $\Psi = \frac{\sqrt{1 - V^2/v^2}}{\sqrt{1 - v^2/c^2}}$. We have defined the notation $\beta^* = \beta\epsilon = \beta(1 - \alpha) = (v/c)[1 - V/v]$. If $V \to 0$ or $\alpha \to 0$, we recover the Lorentz matrix, i.e., $\Lambda(v) \to L(v)$, since $\Psi \to \gamma$ and $\beta^* \to \beta$.

Now, we have $\Lambda_1 = \Lambda(v_1)$ as being

$$\Lambda_1 = \begin{pmatrix} \Psi_1 & -\Psi_1\beta_1^* \\ -\Psi_1\beta_1^* & \Psi_1 \end{pmatrix} = \begin{pmatrix} \Psi_1 & -\Psi_1\frac{v_1^*}{c} \\ -\Psi_1\frac{v_1^*}{c} & \Psi_1 \end{pmatrix} \tag{11}$$

and $\Lambda_2 \equiv \Lambda(v_2)$ as being

$$\Lambda_2 = \begin{pmatrix} \Psi_2 & -\Psi_2\beta_2^* \\ -\Psi_2\beta_2^* & \Psi_2 \end{pmatrix} = \begin{pmatrix} \Psi_2 & -\Psi_2\frac{v_2^*}{c} \\ -\Psi_2\frac{v_2^*}{c} & \Psi_2 \end{pmatrix}, \tag{12}$$

so that $\Lambda_2\Lambda_1$ is

$$\Lambda_2\Lambda_1 = [\Psi_2\Psi_1(1 + \beta_2^*\beta_1^*)] \begin{pmatrix} 1 & -\frac{(\beta_1^*+\beta_2^*)}{1+\beta_2^*\beta_1^*} \\ -\frac{(\beta_1^*+\beta_2^*)}{1+\beta_2^*\beta_1^*} & 1 \end{pmatrix}, \tag{13}$$

where $\beta_1^* = \beta_1\epsilon_1 = \beta_1(1-\alpha_1) = (v_1/c)[1 - V/v_1]$ and $\beta_2^* = \beta_2\epsilon_2 = \beta_2(1-\alpha_2) = (v_2/c)[1 - V/v_2]$.

We obtain that the multiplicative term of the matrix in Eq. (13) is written as $\Psi_2\Psi_1(1 + \beta_2^*\beta_1^*) = \sqrt{(1 - V^2/v_2^2)(1 - V^2/v_1^2)}\frac{1+(v_1^*v_2^*/c^2)}{\sqrt{1-(v_1^2/c^2+v_2^2/c^2-v_1^2v_2^2/c^4)}}$. Now by inserting this term into Eq. (13), we rewrite Eq. (13) in the following way:

$$\Lambda_2\Lambda_1 = \frac{\sqrt{\left(1 - \frac{V^2}{v_2^2}\right)\left(1 - \frac{V^2}{v_1^2}\right)}\left(1 + \frac{v_1^*v_2^*}{c^2}\right)}{\sqrt{1 - \left(\frac{v_1^2}{c^2} + \frac{v_2^2}{c^2} - \frac{v_1^2v_2^2}{c^4}\right)}} \begin{pmatrix} 1 & -\frac{1}{c}\left(\frac{v_1^*+v_2^*}{1+\frac{v_1^*v_2^*}{c^2}}\right) \\ -\frac{1}{c}\left(\frac{v_1^*+v_2^*}{1+\frac{v_1^*v_2^*}{c^2}}\right) & 1 \end{pmatrix} \tag{14}$$

Now we should note that, if the Eq. (14) satisfies the closure condition, Eq. (14) must be equivalent to

$$\Lambda_2\Lambda_1 = \Lambda_3 = \Psi_3 \begin{pmatrix} 1 & -\frac{v_3^*}{c} \\ -\frac{v_3^*}{c} & 1 \end{pmatrix}, \tag{15}$$

where, by comparing Eq. (14) with Eq. (15), we must verify whether the closure condition is satisfied, i.e., $\Psi_3 \equiv \sqrt{(1 - V^2/v_2^2)(1 - V^2/v_1^2)}\frac{1+(v_1^*v_2^*/c^2)}{\sqrt{1-(v_1^2/c^2+v_2^2/c^2-v_1^2v_2^2/c^4)}}$ and $v_3^* \equiv (v_2^* + v_1^*)/[1 + (v_2^*v_1^*)/c^2]$. However, we first realize that such speed transformation, which should be obeyed in order to satisfy the closure condition, differs from the correct speed transformation (Eq. 8) that has origin from the space-time transformations with a minimum speed given in Eqs. (4) and (5). So, according to Fig. 2, if we simply redefine $v' = v_2$ and $v = v_1$, we rewrite the correct transformation (Eq. 8) as being $v_{rel} = v_3 = (v_2 + v_1^*)/[1 + (v_2v_1^*)/c^2]$ with $v_1^* = v_1 - V$. Now, we see that the correct transformation for v_3 (Eq. 8) is not the same transformation given in the matrix above (Eq. 14), i.e., we have $v_3 \neq (v_2^* + v_1^*)/[1 + (v_2^*v_1^*)/c^2]$.

One of the conditions for having the closure relation is that the components outside the diagonal of the matrix in Eqs. (14) or (15) must include v_3 given by Eq. (8), which

does not occur. Therefore, we are already able to conclude that such condition is not obeyed in a spacetime with a minimum speed (a preferred reference frame) at the subatomic level, i.e., we find $\Lambda_2\Lambda_1 \neq \Lambda_3$, which does not generate a group. In order to clarify further this question, we just make the approximation $V = 0$ or also $v_1 \gg V$ and $v_2 \gg V$ in Eq. (14), and thus we recover the closure relation of the Lorentz group, as follows:

$$(\Lambda_2\Lambda_1)_{V=0} = L_2 L_1 = \frac{\left(1+\frac{v_1 v_2}{c^2}\right)}{\sqrt{1-\left(\frac{v_1^2}{c^2}+\frac{v_2^2}{c^2}-\frac{v_1^2 v_2^2}{c^4}\right)}} \begin{pmatrix} 1 & -\frac{1}{c}\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right) \\ -\frac{1}{c}\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right) & 1 \end{pmatrix}$$

$$= L_3, \tag{16}$$

where

$$L_3 = \gamma_3 \begin{pmatrix} 1 & -\frac{v_3}{c} \\ -\frac{v_3}{c} & 1 \end{pmatrix}, \tag{17}$$

which is the closure condition of the Lorentz group, since now it is obvious that the Lorentz transformation of speeds appears outside the diagonal of the matrix in Eq. (16), i.e., we find $v_3 = (v_1 + v_2)/[1+(v_1 v_2)/c^2]$. And, in order to complete the verification of the closure condition above, it is easy to verify that the multiplicative term of the matrix (Eq. 16) is $\gamma_2\gamma_1(1 + \beta_2\beta_1) = \left(1+\frac{v_1 v_2}{c^2}\right)/\sqrt{1-\left(\frac{v_1^2}{c^2}+\frac{v_2^2}{c^2}-\frac{v_1^2 v_2^2}{c^4}\right)} = \gamma_3$. To do this, we have to consider $v_3 = (v_1 + v_2)/[1 + (v_1 v_2)/c^2]$, so that we use this transformation to be inserted into $\gamma_3 = 1/\sqrt{1 - v_3^2/c^2}$ and we finally show that $\gamma_3 = 1/\sqrt{1 - v_3^2/c^2} = \gamma_2\gamma_1(1 + \beta_2\beta_1)$. However, now starting from this same procedure for obtaining $\Psi_3 = \frac{\sqrt{1-V^2/v_3^2}}{\sqrt{1-v_3^2/c^2}}$, where we have to use the correct transformation for v_3 (Eq. 8), we verify that $\Psi_3 \neq \Psi_2\Psi_1(1 + \beta_2^*\beta_1^*)$ and thus we conclude definitively that the closure condition does not apply to the new transformations, i.e., indeed we have $\Lambda_2\Lambda_1 \neq \Lambda_3$.

Although we already know that the new transformations do not form a group, it is still important to provide a physical justification for such conclusion. To do this with more clarity, we also should investigate whether the identity element and the inverse element exist in such a spacetime with an invariant minimum speed, since these two conditions are relevant to give us a clear comprehension of the conception of motion in this spacetime.

3.1 Identity element

For the case $(1 + 1)D$, the Lorentz group provides the identity element $L_0 = I_{(2X2)}$, since $L_0 L = L L_0 = L$. As the Lorentz matrix is $L_{(2X2)} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$, it is easy to

see that, if we make $v = 0$ or $\beta = 0$ (rest condition), the Lorentz matrix recover the identity matrix $I_{(2X2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, since $\gamma_0 = \gamma(v = 0) = 1$. This trivial condition of rest plus the fact that $\det(L) = 1$ (rotation matrix L) shows us the indistinguishability of rest and inertial motion.

The new transformations are represented by the matrix $\Lambda = \begin{pmatrix} \Psi & -\beta(1-\alpha)\Psi \\ -\beta(1-\alpha)\Psi & \Psi \end{pmatrix}$, where we have $\beta^* = \beta\epsilon = \beta(1-\alpha)$, with $\alpha = V/v$. Now, it is important to notice that there is no any speed v that generates the identity matrix from the new matrix. We would expect that the hypothesis $v = V$ could do that, but, if we make $v = V$ ($\alpha = 1$) inside the new matrix, we find the null matrix, i.e., $\Lambda(V) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since $\Psi(V) = 0$. So, we obtain

$$\Lambda(V)\Lambda = \Lambda_V \Lambda = \Lambda\Lambda_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \Lambda, \text{ where we have } \Lambda = \Lambda(v > V). \text{ Thus,}$$

there is no identity element in this spacetime, which means that there should be a distinction of motion and rest, since there is a preferred reference frame (an invariant minimum speed) in respect to which, the motion $v(> V)$ is given, in view of the absence of the rest condition for particles in this spacetime.

3.2 Inverse element

It is well-known that the inverse element exists in Lorentz transformations that form a group, i.e., we have $L^{-1}(v) = L(-v)$, which means that we can exchange the observer in the reference frame S at rest by another observer in the reference frame S' with speed v in respect to S, so that the other observer at S' simply observes S with a speed $-v$. Such symmetry comes from the galilean relativity of motion, which it is essentially due to the indistinguishability of rest and inertial motion. Here we must stress that such indistinguishability is broken down in the new transformations, since the invariant minimum speed related to a background reference frame introduces a preferential motion $v(> V)$ that cannot be exchanged by $-v$ due to the distinction of motion and rest, since rest does not exist in this spacetime, where we get $\Lambda^{-1}(v) \neq \Lambda(-v)$, such that we obtain $\Lambda(-v)\Lambda(v) = \theta^2 \begin{pmatrix} \gamma & \beta(1-\alpha)\gamma \\ \beta(1-\alpha)\gamma & \gamma \end{pmatrix} \times$

$\begin{pmatrix} \gamma & -\beta(1-\alpha)\gamma \\ -\beta(1-\alpha)\gamma & \gamma \end{pmatrix} = \Psi^2 \begin{pmatrix} \left(1 - \frac{v^{*2}}{c^2}\right) & 0 \\ 0 & \left(1 - \frac{v^{*2}}{c^2}\right) \end{pmatrix} \neq I_{(2X2)}$. For

$V = 0 (\alpha = 0)$, we recover the inverse element of the Lorentz group, that is a rotation group.

In short, we have verified that the new transformations do not form a group and we have provided a physical explanation for such Lorentz violation in view of the existence of an invariant minimum speed that breaks down the indistinguishability of rest and motion.

We have also concluded that the new matrix Λ (Eq. 3) does not represent a rotation matrix (Sect. 2). In view of this, we can realize that such transformations are not related with the well-known rotation group $SO(3)$ (Lie group), whose elements $R(\vec{\alpha})$ and $R(\vec{\beta})$ should obey a closure condition $R(\vec{\alpha})R(\vec{\beta}) = R(\vec{\gamma})$, such that $\vec{\gamma} = \gamma(\vec{\alpha}, \vec{\beta})$, with $R(\vec{\gamma})$ being a new rotation that belongs to the group, so that $\det(R) = +1$ (rotation condition), while we find $\det(\Lambda) = \theta^2 \gamma^2 \left[1 - \frac{v^2(1-\alpha)^2}{c^2} \right]$, where $0 < \det(\Lambda) < 1$, violating the rotation condition.

Although there could be a more complex mathematical structure in order to encompass the new transformations, which should be deeply investigated, at least, here we will make some interesting mathematical approximations on Λ in order to help us to understand further the nature of the new transformations.

In a certain approximation, let us show that Λ is a combination of rotation and deformation of the space-time interval ds, reminding the polar decomposition theorem in linear algebra for a "rigid" body that rotates and deforms. Intuitively, the polar decomposition separates a certain matrix A into a component that stretches the space along a set of orthogonal axes, represented by P, and a rotation (with possible reflection) represented by U, i.e., $A = UP$. where U is a unitary matrix and P is a Hermitian matrix.

When a rigid body rotates, its length remains invariant. This effect is analogous to the invariance of the space-time interval ds under the Lorentz transformation L (Lorentz group) due to a rotation.

When a "rigid" body deforms, such effect is analogous to a deformation (e.g: stretching) of the spacetime interval ds that occurs close to the minimum speed (see Eqs. 30 and 31 for $v \to V$). Thus, at a first sight, the new transformation Λ could be written simply as a polar decomposition, so that $\Lambda = LD$, where L is a rotation matrix (Lorentz matrix) as a special case of the unitary matrix U and D is a deformation matrix (symmetric matrix) as a special case of the Hermitian matrix P, since Hermitian matrices can be understood as the complex extension of real symmetric matrices. However, we can verify that such analogy fails quantitatively when one tries to calculate an exact matrix $D_{(2x2)}$ that satisfies the linear decomposition $LD = \Lambda$ for any speed v (any energy scale), i.e., there is no $D_{(2x2)}$ that satisfies such decomposition, because it seems that we have a kind of non-linear or inseparable combination of rotation and deformation. So, in order to accomplish a stronger analogy with the polar decomposition, we need to make some mathematical approximations on the matrix Λ in such a way that we can be able to separate both effects of rotation and deformation. To do this, let us first write the matrix Λ in the following way:

$$\Lambda = \frac{\sqrt{1-\alpha^2}}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\frac{v(1-\alpha)}{c} \\ -\frac{v(1-\alpha)}{c} & 1 \end{pmatrix} \tag{18}$$

Instead of making $\alpha = 0$ (or $V = 0$) in order to recover the Lorentz matrix L, here we make an alternative approximation, namely $v >> V$, which means that, for higher energies, we recover practically the matrix L (rotation). On the other hand, for much lower energies, i.e., for $\alpha \approx 1$ (or $v \approx V$), we get

$$\Lambda_{(v\approx V)} \approx \sqrt{1-\alpha^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \theta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (19)$$

where we have considered the L'Hôpital's rule, by calculating $lim_{\alpha\to1}\frac{(1-\alpha)}{\sqrt{1-\alpha^2}} = lim_{\alpha\to1}\frac{\sqrt{1-\alpha^2}}{\alpha} = 0$. In other words, this means that $\sqrt{1-\alpha^2}|_{\alpha\approx1} >> (1-\alpha)|_{\alpha\approx1}$, so that we can neglect $(1-\alpha)$ with respect to $\sqrt{1-\alpha^2}$ in such an approximation ($v \approx V$), i.e, we make $\epsilon = (1-\alpha) = 0$ into Λ, keeping the factor $\theta = \sqrt{1-\alpha^2}$. We obtain $\det[\Lambda_{(v\approx V)}] = \theta^2 = (1-\alpha^2) = (1-V^2/v^2)$.

The inverse matrix $\Lambda_{(v\approx V)}^{-1}$ is

$$\Lambda_{(v\approx V)}^{-1} \approx \frac{1}{\sqrt{1-\alpha^2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \theta^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (20)$$

where $\det[\Lambda_{(v\approx V)}^{-1}] = \theta^{-2} = (1-\alpha^2)^{-1} = (1-V^2/v^2)^{-1}$.

Both the symmetric matrices in Eqs. (19) and (20) represent deformations, where, for instance, the matrix in Eq. (20) leads to a stretching of the space-time interval ds when v is closer to V. We realize that such deformations given only for much lower energies close to the background frame S_V, i.e., the matrices $\Lambda_{v\approx V} = \theta I$ and $\Lambda_{v\approx V}^{-1} = \theta^{-1}I$ do not belong to the structure of the Lie group connected to the indentity matrix I.

We conclude that the matrix Λ in Eq. (18) already contains effects of deformation ($ds' \neq ds$), which become completely evident for much lower energies ($v \approx V$), where $\det(\Lambda) \approx 0$, but, when the speed v increases drastically, i.e., $v >> V$, so, now, the rotations of Lorentz group are pratically recovered ($\det(\Lambda) \approx \det(L) = 1$) and, thus, we recover the invariance $ds' = ds$. With such approximations, the polar decomposition is practically valid by making $D = \Lambda_{(v\approx V)}$, so that we can verify the product $LD = L\Lambda_{(v\approx V)} = L(\theta I) \approx \Lambda$, where $\Lambda_{(v\approx V)}$ is a symmetric matrix, which is exactly the reason for the effects of deformation of ds close to V.

We finally conclude that the set formed by the matrices that appear above does not have a group structure or cannot be considered as a Poincaré's subgroup. This point must be discussed in depth.

Our next step will be to make an investigation of the main effect obtained directly from the violation of the rotation structure at much lower speeds ($v \approx V$). Such an effect should naturally lead to other deep implications, which will be pointed out, so that we will realize that the whole theory contains elements that are connected by a same mathematical and physical structure.

When we make a Lorentz transformation $L(v)$ from the frame $S(v = 0)$ to S' with speed v with regard to S, we have the well-known "boost". As the boosts represent rotations, the minimal boost is the identity matrix $L(v = 0) = L(0) = I$ connected to the rest state, such that $L(0)X = X$. However, as such minimal boost does not make sense in this spacetime with a minimum speed that prevents the rest state, we must stress that the component $\Lambda_{(v\approx V)}(=\theta I)$ in the new transformation ($\Lambda \approx L\Lambda_{(v\approx V)}$) leads to a non-existence of boosts only in the approximation for much lower energies ($v \to V$ or $\alpha \to 1$), due to the fact that we get $\theta = \sqrt{1-\alpha^2} << 1$. So, only for

higher energies ($v >> V$ or $\alpha \approx 0$), we get $\theta \approx 1$ and, thus, $\Lambda_{(v \approx V)} = \theta I \approx I$, recovering the regime where the boosts take place (Lorentz group).

In short, the effects of "boosts" are generally weakened in this spacetime, mainly in the regime when $v \approx V$, i.e., much closer to the background frame S_V. So, it is important to stress that, in such special regime, there are no boosts and, therefore, the transformation θI has another meaning since it does not lead to the change of reference frames. We will go deeper into this issue.

Actually, the symmetric matrix θI (Eq. 19) is the reason of breaking the structure of rotation group (boosts) and it should be interpreted just as a scale transformation (θ) that provides a variation of the usual space-time interval (ds) in function of speed v, especially when v is close to V. The deep physical implication directly related to the effect of variation of ds will be investigated below.

Since the matrix θI just deforms the interval ds, this transformation does not act for changing the reference frames. In view of this, we use the following notation to represent such a scale transformation, namely:

$$x^{*\mu} = \theta I x^{\mu}, \tag{21}$$

where $x^{*\mu}$ is the deformed vector, θ being a scale factor, since $\Lambda_{(v \approx V)} x^{\mu} = \theta x^{\mu} = x^{*\mu}$, so that we get

$$ds^{*2} = ds^2(v) = dx^{*\mu} dx^{*}_{\mu} = \theta^2 ds^2 = \det[\Lambda_{(v \approx V)}] ds^2, \tag{22}$$

where $ds^2 = dx^{\mu} dx_{\mu}$ is the usual squared space-time interval of SR and ds^{*2} is the deformed squared space-time interval due to new relativistic effects closer to V (no boosts).

As the usual interval ds does not remain invariant in this spacetime, specially when $v \approx V$; so according to Eq. (22), we realize that the deformed interval ds^* should be the new invariant interval under the change of reference frames in this flat spacetime with the presence of the background frame S_V, such that $ds^{*\prime} = ds^*$. In doing this, we introduce a new invariance of deformed intervals in SSR, namely:

$$ds^{*\prime 2} = ds^{*2} = g_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{23}$$

where $ds^{*2} = (1 - \alpha^2) dx^{\mu} dx_{\mu} = \theta^2 ds^2$ and $ds^{*\prime 2} = (1 - \alpha^2) dx^{\prime\mu} dx^{\prime}_{\mu} = \theta^2 ds^{\prime 2}$. Of course if we make $V \to 0$ or $\alpha \to 0$ in Eq. (23), we recover the invariance of the usual (non-deformed) ds of SR, i.e., $ds^{\prime 2} = ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$.

Indeed we realize that the deformed interval $ds^* = \theta ds = \sqrt{1 - V^2/v^2} ds$ remains finite (Eq. 23), since, in the limit of $\theta \to 0$ ($v \to V$), the usual interval ds undergoes a very large stretching, i.e., $ds \to \infty$.

From Eq. (23), we obtain

$$ds^{*\prime} = ds^* = \sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}} = \sqrt{c^2 dt^2 - dx^2}, \tag{24}$$

where, in this case, we have $dy = dz = 0$ and $ds^{*\prime} = \sqrt{(1-\alpha^2)dx'^\mu dx'_\mu} = \sqrt{(1-\alpha^2)(c^2 dt'^2 - dx'^2)}$. Now, if we make $dx' = 0$ (or $x' = 0$, i.e., at the origin of the reference frame S'), from Eq. (24) we obtain

$$ds^{*\prime} = cd\tau^* = \sqrt{1-\alpha^2}\,cd\tau = \sqrt{1-\alpha^2}\,cdt' = \sqrt{c^2 dt^2 - dx^2}, \qquad (25)$$

where we have considered $dt' = d\tau$ and so $dt'^* = d\tau^*$, with $d\tau^*(=ds^{*\prime}/c = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}/c)$ being the deformed proper time interval, where we have $d\tau^* = \theta d\tau = \sqrt{1-\alpha^2}\,d\tau$. This result has a deep physical implication that has origin in the breakdown of the structure of Lorentz group.

Now we are ready to investigate the physical implication from Eq. (25). So, by simply making $dx = vdt$ in Eq. (25) and performing the calculations, we finally obtain

$$d\tau\sqrt{1-\alpha^2} = dt\sqrt{1-\beta^2} \qquad (26)$$

and, then

$$\Delta\tau\sqrt{1 - \frac{V^2}{v^2}} = \Delta t\sqrt{1 - \frac{v^2}{c^2}}, \qquad (27)$$

where $\Delta\tau$ is the proper time interval and Δt is the improper one. Eq. (27) is the immediate physical implication of the violation of Lorentz group by means of the symmetric matrix θI (Eq. 19) that deforms the proper time, so that we can also write Eq. (27) as being $\sqrt{\det(\theta I)}\Delta\tau = \theta\Delta\tau = \Delta t\sqrt{1 - v^2/c^2}$, where $\theta = \sqrt{1 - V^2/v^2}$.

It is important to call attention to the fact that Eq. (27) shows us that the proper time interval $\Delta\tau$ depends on speed v and, thus, now it can also be deformed (dilated) like the improper time interval. So, we realize that Eq. (27) reveals a perfect symmetry in the sense that both intervals of time Δt and $\Delta\tau$ can dilate, namely Δt dilates for $v \to c$ and, on the other hand, $\Delta\tau$ dilates for $v \to V$. But, if we make $V \to 0$, we break down such new symmetry of SSR and so we recover the well-known time equation of SR, where only Δt dilates and $\Delta\tau$ remains invariant.

From Eq. (27) we notice that, if we make $v = v_0 = \sqrt{cV}$ (a geometric average between c and V), we find exactly the equality $\Delta\tau$ (at S')$= \Delta t$ (at S), namely this is a newtonian result where the time intervals are the same. Thus we conclude that v_0 represents a special intermediary speed in SSR ($V << v_0 << c$) such that, if:

(a) $v >> v_0$ (or $v \to c$), we get $\Delta\tau << \Delta t$. This is the well-known *improper time dilation*.

(b) $v << v_0$ (or $v \to V$), we get $\Delta\tau >> \Delta t$. Let us call such a new effect as *improper time contraction* or *dilation of the proper time interval $\Delta\tau$ with respect to the improper time interval Δt*. This new effect becomes more evident only for v (S') $\to V$ (S_V), so that, in this limit, we have $\Delta\tau \to \infty$ for a certain Δt fixed as being finite. In other words, this means that the proper time (S') can now elapse much faster than the improper one.

It is interesting to notice that we restore the newtonian regime when $V << v << c$, which represents a regime of intermediary speeds, so that we get the newtonian approximation from Eq. (27), i.e., $\Delta\tau \approx \Delta t$.

Squaring both members of Eq. (27) ($\Delta t = \Psi\Delta\tau = \theta\gamma\Delta\tau$) and manipulating the result, we write Eq. (27) as follows:

$$c^2\Delta\tau^2 = \frac{1}{(1 - \frac{V^2}{v^2})}\left[c^2\Delta t^2 - v^2\Delta t^2\right] \qquad (28)$$

By placing Eq. (28) in a differential form and manipulating it, we obtain

$$c^2\left(1 - \frac{V^2}{v^2}\right)\frac{d\tau^2}{dt^2} + v^2 = c^2 \qquad (29)$$

Eq. (29) shows us that both speeds related to the marching of time ("temporal-speed" $v_t = c\sqrt{1 - V^2/v^2}\,d\tau/dt$) and the spatial speed v form the vertical and horizontal legs of a rectangular triangle respectively (Fig. 3). The hypotenuse of the triangle is $c = (v_t^2 + v^2)^{1/2}$, which represents the spatio-temporal speed of any particle. If $V \to 0$ in Eq. (29), we recover the time equation in SR, i.e., $c^2(d\tau^2/dt^2) + v^2 = c^2$.

Looking at Fig. 3, now we see clearly three important cases, namely:

(a) If $v \approx c$, $v_t \approx 0$ (the marching of proper time in S' is much slower than in S), such that $\Delta t >> \Delta\tau$, with $\Psi \approx \gamma >> 1$, leading to the well-known dilation of the improper time.

(b) If $v = v_0 = \sqrt{cV}$, $v_t = \sqrt{c^2 - v_0^2}$, i.e., the marching of time in S' is faster, but it is still in an intermediary regime, such that $\Delta t = \Delta\tau$, with $\Psi = \Psi_0 = \Psi(v_0) = 1$ (newtonian regime).

(c) If $v \approx V(<< v_0)$, $v_t \approx \sqrt{c^2 - V^2} = c\sqrt{1 - V^2/c^2}$ (the marching of proper time is even faster), such that $\Delta t << \Delta\tau$, with $\Psi \approx \theta << 1$ (dilation of the proper time). To illustrate this new effect of proper time dilation, let us consider a
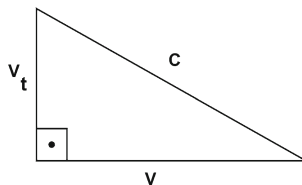

Fig. 3 We see that the horizontal leg represents the spatial-speed v, while the vertical leg represents the temporal-speed v_t (marching of time), where $v_t = \sqrt{c^2 - v^2} = c\sqrt{1 - v^2/c^2} = c\sqrt{1 - V^2/v^2}\,d\tau/dt$ (see Eq. 27), so that we always have $v^2 + v_t^2 = c^2$. In SR, when $v = 0$, the horizontal leg vanishes (no spatial speed) and so the vertical leg becomes maximum ($v_t = v_{tmax} = c$). However, now according to SSR, due to the existence of a minimum limit of spatial speed (V), we can never nullify the horizontal leg, so that the maximum temporal speed (maximum vertical leg) is $v_{tmax} = \sqrt{c^2 - V^2} = c\sqrt{1 - V^2/c^2} < c$. On the other hand v_t (the vertical leg) cannot be zero since $v = c$ is forbidden for massive particles. So we conclude that the rectangular triangle is always preserved since both temporal and spatial speeds cannot vanish and, thus, they always coexist. In this sense, we realize that there is a strong symmetry in SSR

box that contains an ideal gas with N particles in the frame S of a laboratory. Since the minimum speed V has a microscopic origin, then by considering an average speed per particle of the gas (atom or molecule), we should have such average speed v_{rms} (S') close to V (S_V) only when the temperature of the gas is $T \rightarrow 0K$ ($v_{rms} = \sqrt{\langle v \rangle_N^2} \rightarrow V$). Thus, an imaginary clock in thermal equilibrium with such ultra-cold system should measure a dilated time interval with respect to the time interval measured in the observer's clock (laboratory S), or in other words, we could say that ultra-cold systems "grow old" more rapidly, contrary to higher energies when one grows old more slowly. A great experimental effort should be made in order to detect the effect of proper time dilation, but, before this, we must search for the origin of the invariant minimum speed.

3.2.1 Flat space-time metric with the background frame S_V

From Eq. (23), we obtain

$$ds^2 = \frac{1}{\left(1 - \frac{V^2}{v^2}\right)} g_{\mu\nu} dx^\mu dx^\nu, \tag{30}$$

where we have $ds^{*2} = \theta^2 ds^2 = (1 - \alpha^2)ds^2 = (1 - V^2/v^2)ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (Eq. 23).

Eq. (30) is written as

$$ds^2 = \Theta g_{\mu\nu} dx^\mu dx^\nu, \tag{31}$$

where Θ is a function of speed v given with respect to the background frame S_V, namely:

$$\Theta = \Theta(v) = \frac{1}{\left(1 - \frac{V^2}{v^2}\right)}, \tag{32}$$

where we see that $\Theta = \theta^{-2}$.

The presence of the ultra-referential S_V deforms the Minkowsky metric (Eq. 31) and works like a uniform background field that fills the whole flat space-time as a perfect fluid, playing the role of a kind of de-Sitter (dS) space-time ($\Lambda > 0$) [18].

The function Θ can be understood as being a scale factor that increases for very large wavelengths (cosmological scales) governed by vacuum (dS), that is to say for much lower energies ($v \rightarrow V$) where we have $\Theta \rightarrow \infty$. Thus, the factor $\Theta(= \theta^{-2})$ breaks strongly the invariance of ds only for very large distances governed by vacuum of the ultra-referential S_V, leading to the cosmological anti-gravity governed by the tiny positive value of the cosmological constant (Sect. 6). In this regime of vacuum-S_V ($v \rightarrow V$ or $\Theta \rightarrow \infty$), the interval ds diverges.

On the other hand, we have $\Theta \rightarrow 1$ for smaller scales of length, namely for higher energies ($v >> V$), where dS space-time approximates to the Minkowski metric as a special case, restoring the Lorentz symmetry and the invariance of ds.

We realize that the presence of the background frame S_V deforms the metric $g_{\mu\nu}$ by means of the scale factor Θ, so that we define a deformed flat metric $G_{\mu\nu} = \Theta g_{\mu\nu}$ that remains a diagonal matrix, but now having Θ in its diagonal, namely: $G_{\mu\nu} = \Theta g_{\mu\nu} =$

$$\begin{pmatrix} \frac{1}{\left(1-\frac{V^2}{v^2}\right)} & 0 & 0 & 0 \\ 0 & -\frac{1}{\left(1-\frac{V^2}{v^2}\right)} & 0 & 0 \\ 0 & 0 & -\frac{1}{\left(1-\frac{V^2}{v^2}\right)} & 0 \\ 0 & 0 & 0 & -\frac{1}{\left(1-\frac{V^2}{v^2}\right)} \end{pmatrix}$$. Therefore, we simply write Eq. (31) as being

$ds^2 = G_{\mu\nu}dx^\mu dx^\nu$. If we make $v >> V$, this implies $\Theta \to 1$ and, thus, we recover the Minkowsky metric $g_{\mu\nu}$.

Now we are already able to conclude that there should be the same universal factor $\theta = \sqrt{\det(\theta I)}(< 1)$ that deforms all the invariant scalars of SR as, for instance, the space-time interval, i.e., $\theta ds = ds^*$, and the proper time interval, i.e., $\theta d\tau = d\tau^*$, so that $\Delta s^*(=\theta \Delta s)$ and $\Delta \tau^*(=\theta \Delta \tau)$ are the invariant intervals in SSR.

In Sect. 4, we will see that the mass, energy and momentum are also deformed by the same factor θ, i.e., $m_{(0,\alpha)} = \theta m_0$, $E = \theta mc^2 = \theta \gamma m_0 c^2 = \Psi m_0 c^2$ and $p = \theta \gamma m_0 v = \Psi m_0 v$. Thus, we already can conclude that all those invariant quantities of SR and others like the rest mass (Sect. 4) are abandoned in SSR, since they are modified by the factor θ due to the presence of the ultra-referential S_V connected to the own invariance of the minimum speed V.

In sum, we should understand that, as the invariance of c leads us to break down the newtonian invariance of the improper time interval ($\Delta t = \Delta \tau$), by introducing the dilation of the improper time interval ($\Delta t = \gamma \Delta \tau$), which still preserves the invariance of the proper time and the space-time interval, now with a further step towards a new invariance of a minimum speed V, we are led to break down such invariant quantities of SR, since the proper time interval can also dilate by means of the new factor θ (Eq. 27). So, the new invariant quantities in SSR are now the deformed intervals $\Delta \tau^*$ and Δs^*.

3.2.2 The alternative mathematical structure for the transformation Λ

Since the Lorentz matrix L represents a rotation in spacetime ($\det L = 1$), it is known that L can be alternatively written in the form, namely:

$$L = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}, \tag{33}$$

where $\cosh\phi = \gamma = 1/\sqrt{1-\beta^2}$, $\sinh\phi = \beta\gamma = \beta/\sqrt{1-\beta^2}$, $\tanh\phi = \beta$ and $\det L = (\cosh\phi)^2 - (\sinh\phi)^2 = 1$.

From Eq. (33), we obtain the following transformations:

$$x' = (\cosh\phi)x - (\sinh\phi)ct \equiv \gamma(x - vt) \tag{34}$$

and

$$ct' = (\cosh\phi)ct - (\sinh\phi)x \equiv \gamma(ct - \beta x) \tag{35}$$

Although we already know that the new matrix Λ is not a rotation matrix, even so we will use the above hyperbolic representation for computing Λ, which will be given in function of those hyperbolic functions and, after, we will interpret the results. In order to do that, we make a factoration of θ outside the matrix Λ, by writing it in the following way:

$$\Lambda = \theta \begin{pmatrix} \gamma & -\beta(1-\alpha)\gamma \\ -\beta(1-\alpha)\gamma & \gamma \end{pmatrix} = \theta \begin{pmatrix} \cosh\phi & -\sinh\phi(1-\alpha) \\ -\sinh\phi(1-\alpha) & \cosh\phi \end{pmatrix}, \tag{36}$$

where $\theta\gamma = \theta\cosh\phi = \Psi$, $\theta\beta\gamma = \theta\sinh\phi$ and $\det\Lambda = \theta^2[(\cosh\phi)^2 - (1-\alpha)^2(\sinh\phi)^2] \neq 1$.

From Eq. (36), we obtain the following new transformations:

$$x' = \theta\left[(\cosh\phi)X - (\sinh\phi)ct + (\sinh\phi)\frac{V}{v}ct\right] \equiv \Psi(X - vt + Vt) \tag{37}$$

and

$$ct' = \theta\left[(\cosh\phi)ct - (\sinh\phi)X + (\sinh\phi)\frac{V}{v}X\right] \equiv \Psi\left(ct - \frac{v}{c}X + \frac{V}{c}X\right) \tag{38}$$

We realize that, if we make $V = 0$, which implies $\theta = 1$, the rest state is recovered, so that the background frame $S_V(X, Y, Z)$ (Fig. 1) is eliminated and replaced by the galilean reference frame $S(x, y, z)$ at rest. Therefore, the new transformations in Eqs. (37) and (38) recover the well-known transformations of rotation given in Eqs. (34) and (35).

Since the angle ϕ in Eqs. (37) and (38) cannot be understood as being simply a rotation that preserves the norm of the usual 4-vector for the whole interval of speeds, i.e., $V < v < c$, let us deal with Eqs. (37) and (38) by considering basically two regimes, namely:

(a) For $v >> V$, such that $\frac{V}{v} \approx 0$ (this is not necessarily a relativistic regime $v \approx c$), we find $\theta \approx 1$ and we can also neglect the terms $(\sinh\phi)\frac{V}{v}ct$ (Eq. 37) and $(\sinh\phi)\frac{V}{v}X$ (Eq. 38) with respect to the others, so that we recover the rotation regime within a good approximation, i.e., Eqs. (34) and (35) begin to take place.

(b) For $v \approx V$, such that $\frac{V}{v} \approx 1$ (this is a very low energy $E \approx 0$, which is obtained by making $v \to V$ in Eq. (42), thus we can consider the following approximations: the second and third terms in the right member of both equations (Eqs. 37 and 38) are cancelled between themselves, so that we immediately obtain $x' = \theta(\cosh\phi)X$ and $ct' = \theta(\cosh\phi)ct$, with $\theta << 1$; however, as the rotations (boosts) do not exist close to the background frame S_V, we still should make $\phi \to$

0 (cosh $\phi \rightarrow 1$) in such approximations, such that we finally get $x^{*\mu} = \theta I x^{\mu}$, which is exactly Eq. (21), where we have replaced the index $'$ by the index $*$, since the boosts do not make sense in this new regime, where θ just plays the role of a scale factor that deforms the interval ds.

4 Energy and momentum with the presence of a minimum speed

Let us firstly define the 4-velocity in the presence of the background frame S_V connected to the invariant minimum speed V, as follows:

$$U^{\mu} = \left[\frac{v_{\alpha}\sqrt{1 - \frac{V^2}{v^2}}}{c\sqrt{1 - \frac{v^2}{c^2}}} , \frac{\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} \right] = \left[U^{\alpha}, U^4 \right] \tag{39}$$

where $\mu = 1, 2, 3, 4$ and $\alpha = 1, 2, 3$. If $V \rightarrow 0$, we recover the well-known 4-velocity of SR. From (39), it is interesting to observe that the 4-velocity of SSR vanishes in the limit of $v \rightarrow V$ (S_V), i.e., $U^{\mu} = (0, 0, 0, 0)$, whereas in SR, for $v = 0$ we find $U^{\mu} = (1, 0, 0, 0)$.

The 4-momentum is

$$p^{\mu} = m_0 c U^{\mu}, \tag{40}$$

being U^{μ} given in Eq. (39). So we find

$$p^{\mu} = \left[\frac{m_0 v_{\alpha}\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} , \frac{m_0 c\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} \right] = \left[p^{\alpha}, p^4 \right], \tag{41}$$

where $p^4 = E/c$, such that

$$E = cp^4 = mc^2 = m_0 c^2 \frac{\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{42}$$

where E is the total energy of the particle with speed v in relation to the background reference frame (ultra-referential S_V). From Eq. (42), we observe that, if $v \rightarrow c \Rightarrow E \rightarrow \infty$. If $v \rightarrow V \Rightarrow E \rightarrow 0$ and, if $v = v_0 = \sqrt{cV} \Rightarrow E = E_0 = m_0 c^2$ (proper energy in SSR), where we should stress that $m_0 c^2$ requires a non-zero motion $v(= v_0)$ in relation to S_V. Figure 4 shows us the graph for the energy E.

From Eq. (41) we also obtain the 3(spatial)-momentum, namely:

$$\vec{p} = m_0 \vec{v} \frac{\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{43}$$

where $\vec{v} = (v_1, v_2, v_3)$.

Fig. 4 $v_0 = \sqrt{cV}$ is a speed such that we get the proper energy of the particle ($E_0 = m_0c^2$) in SSR, where $\Psi_0 = \Psi(v_0) = 1$. For $v << v_0$ or closer to S_V ($v \to V$), a new relativistic correction on energy arises, so that $E \to 0$. On the other hand, for $v >> v_0$, being $v \to c$, so we find $E \to \infty$
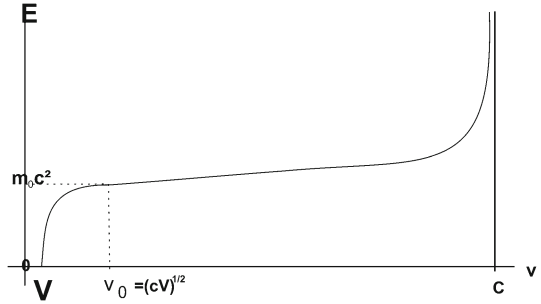
From Eq. (41), performing the quantity $p^\mu p_\mu$, we obtain the energy-momentum relation of SSR, as follows:

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2 \left(1 - \frac{V^2}{v^2}\right),\tag{44}$$

where $\vec{p}^2 = p_1^2 + p_2^2 + p_3^2$.

From Eq. (44), we obtain

$$E^2 = c^2 p^2 + m_0^2 c^4 \theta^2 = c^2 p^2 + m_0^2 c^4 \left(1 - \frac{V^2}{v^2}\right)\tag{45}$$

In SR theory, that is represented by the Lorentz group, some elements are preserved under rotations, as for instance, the 4-interval $ds^2 (=g_{\mu\nu}dx^\mu dx^\nu)$ and also the rest mass by means of the inner product $p^\mu p_\mu = m_0^2 c^2$, that is the dispersion relation, where the rest mass is conserved. This means that the rest condition and the rest mass are fundamental in SR, since they are independent of the state of motion, i.e., we have the well-known indistinguishability of motion and rest. However, the new dispersion relation given in Eq. (44) (or Eq. 45) shows us that the rest condition does not exist, since now the mass depends on its preferred state of motion v with respect to the background frame connected to an invariant minimum speed V, i.e., the ultra-referential S_V. This is the reason why we find the massive term as a function of α, i.e., we get $m_0^2 c^4 (1 - \alpha^2)$ in Eqs. (44) and (45).

In Eq. (44), when, $\alpha \to 1$ ($v \to V$), we find $p^\mu p_\mu \to 0$, however, we can never nullify $p^\mu p_\mu$, since the minimum speed V is unattainable (see Sect. 5).

In the present work, as we are focusing our attention on some dynamical implications of a minimum speed, let us leave a more detailed development of the physical consequences of SSR in terms of field-theory actions to be explored elsewhere. However, here it would be interesting to mention that the wave operator is covariant under the new transformations.

In order to obtain the new dispersion relation in an alternative and simple way, being consistent with the result obtained from the formalism of 4-momentum, we have to consider $p = \Psi m_0 v$, $E = \Psi m_0 c^2$ and $E_0 = m_0 c^2$, such that, we first calculate the quantity $c^2 p^2$ and we obtain

$$c^2 p^2 = \frac{m_0^2 c^2 (v^2 - V^2)}{\left(1 - \frac{v^2}{c^2}\right)} \tag{46}$$

On the other hand, we find

$$E^2 - E_0^2 = m^2 c^4 - m_0^2 c^4 = \frac{m_0^2 c^4}{\left(1 - \frac{v^2}{c^2}\right)} \left[\frac{v^2}{c^2} - \frac{V^2}{v^2}\right] \tag{47}$$

It is easy to verify that, if we make $V = 0$ in Eq. (46) and in Eq. (47), we recover the well-known dispersion relation of SR, i.e., $c^2 p^2 = E^2 - E_0^2 = m^2 c^4 - m_0^2 c^4$. However, according to Eqs. (46) and (47), we see that $(E^2 - m_0^2 c^4) \neq c^2 p^2$. So, in order to obtain a new energy-momentum relation (new dispersion relation) with the presence of the minimum speed V, we should compare Eq. (46) with Eq. (47) by introducing a certain correction function $A(v)$, so that now we write the following identity:

$$E^2 - m_0^2 c^4 = \frac{m_0^2 c^4}{\left(1 - \frac{v^2}{c^2}\right)} \left[\frac{v^2}{c^2} - \frac{V^2}{v^2}\right] \equiv \frac{m_0^2 c^2 (v^2 - V^2)}{\left(1 - \frac{v^2}{c^2}\right)} + A(v) = c^2 p^2 + A(v), \tag{48}$$

where $A(v)$ should be found in order to satisfy the identity in Eq. (48). After performing some calculations we find $A(v) = -m_0^2 \alpha^2 c^4 = -m_0^2 c^4 (V^2/v^2)$. Now by inserting $A(v)$ into Eq. (48), we finally obtain the relation, namely $E^2 - m_0^2 c^4 = c^2 p^2 + A(v) = c^2 p^2 - m_0^2 c^4 (V^2/v^2)$, from where we get

$$E^2 = c^2 p^2 + m_0^2 c^4 - m_0^2 c^4 \left(\frac{V^2}{v^2}\right) = c^2 p^2 + m_0^2 c^4 \left(1 - \frac{V^2}{v^2}\right), \tag{49}$$

that is the same relation in Eq. (45). Therefore, we can conclude that a certain massive term in this spacetime has always connection with the state of motion with respect to the preferred frame-S_V, where, according to Eq. (49), we can write the effective mass as $m_{(0,\alpha)} = \theta m_0 = m_0 \sqrt{1 - V^2/v^2}$, which does not represent a rest mass m_0, since $v > V$. In view of this, we can also write the total energy, as follows:

$$E = m_{(0,\alpha)} c^2 + K = \gamma m_{(0,\alpha)} c^2 = \frac{\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} m_0 c^2, \tag{50}$$

where K is the knetic energy and $E = \gamma m_{(0,\alpha)} c^2 = \gamma \theta m_0 c^2 = \Psi m_0 c^2$. So, from Eq. (50) we obtain K, namely:

$$K = m_{(0,\alpha)} c^2 (\gamma - 1) = m_0 c^2 \sqrt{1 - \frac{V^2}{v^2}} \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right), \tag{51}$$

where $K \to 0$ if $v \to V$. If $V \to 0$ in Eq. (51), we recover the relativistic knetic energy, i.e., $K = m_0 c^2 (\gamma - 1)$.

Making an expansion in Eq. (51) and consider the approximation $v << c$, we find

$$K = m_0 c^2 \sqrt{1 - \frac{V^2}{v^2}} \left(1 + \frac{v^2}{2c^2} + \cdots - 1 \right) \approx \frac{1}{2} \left(m_0 \sqrt{1 - \frac{V^2}{v^2}} \right) v^2 = \frac{1}{2} m_{(0,\alpha)} v^2, \tag{52}$$

where $m_{(0,\alpha)} = m_0 \theta(v) = m_0 \sqrt{1 - \frac{V^2}{v^2}}$.

Now, also making the approximation $v >> V$ ($\alpha \approx 0$) in Eq. (52), i.e., $m_{(0,\alpha)} \approx m_{(0,0)} = m_0$, we finally obtain the approximation $V << v << c$, so that we simply find

$$K \approx \frac{1}{2} m_0 v^2, \tag{53}$$

which is the newtonian knetic energy, that is recovered only for intermediary speeds in such a spacetime with an invariant minimum speed.

The de-Broglie wavelength of a particle is due to its motion v with respect to S_V, namely:

$$\lambda = \frac{h}{p} = \frac{h}{\Psi m_0 v} = \frac{h}{m_0 v} \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 - \frac{V^2}{v^2}}}, \tag{54}$$

from where we have used the momentum $p = \Psi m_0 v = \theta \gamma m_0 v$ given with respect to S_V (Eq. 43).

If $v \to c \Rightarrow \lambda \to 0$ (spatial contraction) and $p \to \infty$. If $v \to V(S_V) \Rightarrow \lambda \to \infty$ (spatial dilation by breaking down Lorentz symmetry), which means that we have very large wavelengths. This leads to $\Theta \to \infty$ (see Eqs. 30 and 31) and $p \to 0$, since we can alternatively write $p = \theta \gamma m_0 v = \Theta^{-1/2} \gamma m_0 v$, where $\Theta = \theta^{-2} = 1/(1 - \alpha^2)$.

4.1 Transformations of momentum-energy in the presence of the ultra-referential S_V

By considering the quadri-vector of momentum-energy given in Eq. (41), we have $p^\mu = [p^\alpha, E/c]$. Since we already have considered the motion in only one dimension (e.g: x), we obtain the vector $[p^1, E/c]$, where $p^1 = p_x$.

Now, as we want to investigate how p^μ transforms in such a spacetime with the presence of the ultra-referential S_V, we have to make those two transformations by using the matrix Λ (Eq. 3) and its inverse Λ^{-1} (Eq. 6). So, by first considering Λ, we rewrite

$$\Lambda = \begin{pmatrix} \Psi & -\beta(1-\alpha)\Psi \\ -\beta(1-\alpha)\Psi & \Psi \end{pmatrix}, \tag{55}$$

such that the direct matricial transformation $p'^{\nu} = \Lambda^{\nu}_{\mu} p^{\mu}$ ($S_V \to S'$) leads to the new momentum-energy transformations, as follows:

$$p'_x = \Psi \left[p_x - \frac{v(1-\alpha)E}{c^2} \right] = \Psi \left(p_x - \frac{vE}{c^2} + \frac{VE}{c^2} \right), \tag{56}$$

being $p'_y = p_y$ and $p'_z = p_z$.

$$E' = \Psi \left[E - v(1-\alpha)p_x \right] = \Psi \left(E - vp_x + Vp_x \right) \tag{57}$$

We know that the inverse matrix (Eq. 6) that transforms $S' \to S_V$ is

$$\Lambda^{-1} = \begin{pmatrix} \Psi' & \beta(1-\alpha)\Psi' \\ \beta(1-\alpha)\Psi' & \Psi' \end{pmatrix}, \tag{58}$$

where we find $\Psi' = \Psi^{-1}/[1-\beta^2(1-\alpha)^2]$. Thus, the inverse matricial transformation $p^{\nu} = \Lambda^{-1\nu}_{\mu} p'^{\mu}$ ($S' \to S_V$) leads to the following momentum-energy transformations, namely:

$$p_x = \Psi' \left[p'_x + \frac{v(1-\alpha)E'}{c^2} \right] = \Psi' \left(p'_x + \frac{vE'}{c^2} - \frac{VE'}{c^2} \right), \tag{59}$$

being $p_y = p'_y$ and $p_z = p'_z$.

$$E = \Psi' \left[E' + v(1-\alpha)p'_x \right] = \Psi' \left(E' + vp'_x - Vp'_x \right) \tag{60}$$

The Lorentz transformations of the energy-momentum p^{μ} are simply recovered if we make $V = 0$ (Fig. 5).

Fig. 5 The external and internal conical surfaces represent respectively the speed of light c and the unattainable minimum speed V, where V is represented by the *dashed line*, namely a definitely prohibited boundary for any particle. For a point P in the *world line* of a particle, in the interior of the two conical surfaces, we obtain a corresponding internal conical surface, such that we must have $V < v_p \le c$
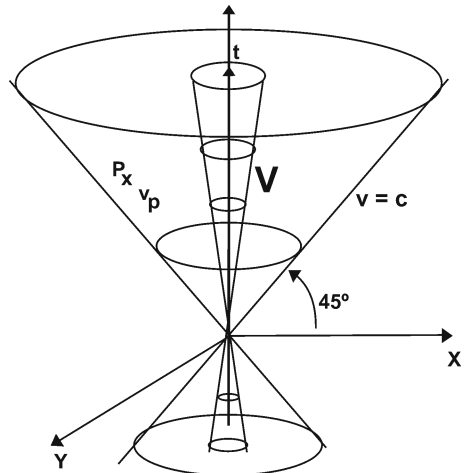
5 Power of an applied force: the energy barrier of a minimum speed connected to the vacuum energy

Let us consider a force applied to a particle, in the same direction of its motion. More general cases where the force is not necessarily parallel to velocity will be treated elsewhere. In our specific case ($\vec{F}||\vec{v}$), the relativistic power $P_{ow}(= vdp/dt)$ of SSR is given as follows:

$$P_{ow} = v\frac{d}{dt}\left[m_0 v\left(1 - \frac{V^2}{v^2}\right)^{\frac{1}{2}}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right], \tag{61}$$

where we have used the momentum p given in Eq. (43).

After performing the calculations in Eq. (61), we find

$$P_{ow} = \left[\frac{\left(1 - \frac{V^2}{v^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} + \frac{V^2}{v^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}\left(1 - \frac{V^2}{v^2}\right)^{\frac{1}{2}}}\right]\frac{dE_k}{dt}, \tag{62}$$

where $E_k = \frac{1}{2}m_0 v^2$.

If we make $V \to 0$ and $c \to \infty$ in Eq. (62), we simply recover the power obtained in newtonian mechanics, namely $P_{ow} = dE_k/dt$. Now, if we just consider $V \to 0$ in Eq. (62), we recover the well-known relativistic power of SR, namely $P_{ow} = (1 - v^2/c^2)^{-3/2}dE_k/dt$. We notice that such a relativistic power tends to infinite ($P_{ow} \to \infty$) in the limit $v \to c$. We explain this result as an effect of the drastic increase of an effective inertial mass close to c, namely $m_{eff} = m_0(1 - v^2/c^2)^{k''}$, where $k'' = -3/2$. We must stress that such an effective inertial mass is the response to an applied force parallel to the motion according to Newton second law, and it increases faster than the relativistic mass $m = m_r = m_0(1 - v^2/c^2)^{-1/2}$.

The effective inertial mass m_{eff} we have obtained is a longitudinal mass m_L, i.e., it is a response to the force applied in the direction of motion. In SR, for the case where the force is perpendicular to velocity, we can show that the transversal mass increases like the relativistic mass, i.e., $m = m_T = m_0(1 - v^2/c^2)^{-1/2}$, which differs from the longitudinal mass $m_L = m_0(1 - v^2/c^2)^{-3/2}$. So, in this sense, there is anisotropy of the effective inertial mass to be also investigated in more details by SSR in a further work.

The mysterious discrepancy between the relativistic mass m (m_r) and the longitudinal inertial mass m_L from Newton second law (Eq. 62) is a controversial issue [19–25]. Actually the newtonian notion about inertia as the resistance to acceleration (m_L) is not compatible with the relativistic dynamics (m_r) in the sense that we generally cannot consider $\vec{F} = m_r\vec{a}$. The dynamics of SSR aims to give us a new interpretation for the inertia of the newtonian point of view in order to make it compatible with the relativistic mass. This compatibility will be possible just due to the influence of the background field that couples to the particle and "dresses" its relativistic mass in order

to generate an effective (dressed) mass in accordance with the newtonian notion about inertia from Eqs. (61) and (62). This issue will be clarified in this section.

From Eq. (62), it is important to observe that, when we are closer to V, there emerges a completely new result (correction) for power, namely:

$$P_{ow} \approx \left(1 - \frac{V^2}{v^2}\right)^{-\frac{1}{2}} \frac{d}{dt}\left(\frac{1}{2}m_0 v^2\right), \tag{63}$$

given in the approximation $v \approx V$. So, we notice that $P_{ow} \to \infty$ when $v \approx V$. We can also make the limit $v \to V$ for the general case (Eq. 62) and so we obtain an infinite power ($P_{ow} \to \infty$). Such a new relativistic effect deserves the following very important comment: Although we are in the limit of very low energies close to V, where the energy of the particle (mc^2) tends to zero according to the approximation $E = mc^2 \approx m_0 c^2 (1 - V^2/v^2)^k$ with $k = 1/2$ (e.g.: make the approximation $v \approx V$ in Eq. (42), on the other hand the power given in Eq. (63) shows us that there is an effective inertial mass that increases to infinite in the limit $v \to V$, that is to say, from Eq. (63) we get the effective mass $m_{eff} \approx m_0(1 - V^2/v^2)^{k'}$, where $k' = -1/2$. Therefore, from a dynamical point of view, the negative exponent $k' (= -1/2)$ for the power at very low speeds (Eq. 63) is responsible for the inferior barrier of the minimum speed V, as well as the exponent $k'' = -3/2$ of the well-known relativistic power is responsible for the top barrier of the speed of light c according to Newton second law. Actually, due to the drastic increase of m_{eff} of a particle moving closer to S_V, leading to its strong coupling to the vacuum field in the background frame S_V, thus, in view of this, the dynamics of SSR states that it is impossible to decelerate a subatomic particle until reaching the rest.

In order to see clearly both exponents $k' = -1/2$ (inferior inertial barrier V) and $k'' = -3/2$ (top inertial barrier c), let us write the general formula of power (Eq. 62) in the following alternative way after some algebraic manipulations on it, namely:

$$P_{ow} = \left(1 - \frac{V^2}{v^2}\right)^{k'} \left(1 - \frac{v^2}{c^2}\right)^{k''} \left(1 - \frac{V^2}{c^2}\right) \frac{dE_k}{dt}, \tag{64}$$

where $k' = -1/2$ and $k'' = -3/2$. Now it is easy to see that, if $v \approx V$ or even $v << c$, Eq. (64) recovers the approximation in Eq. (63). As $V << c$, the ratio V^2/c^2 in Eq. (64) is a very small dimensionless constant. So it could be neglected.

From Eq. (64) we get the effective inertial mass m_{eff} of SSR, namely:

$$m_{eff} = m_0 \left(1 - \frac{V^2}{v^2}\right)^{-\frac{1}{2}} \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \left(1 - \frac{V^2}{c^2}\right) \tag{65}$$

We must stress that m_{eff} in Eq. (65) is a longitudinal mass m_L. The problem of mass anisotropy will be treated elsewhere, where we will intend to show that, just for the approximation $v \approx V$, the effective inertial mass becomes practically isotropic, that is to say $m_L \approx m_T \approx m_0 \left(1 - \frac{V^2}{v^2}\right)^{-1/2}$. This important result will show us the

isotropic aspect of the vacuum-S_V, so that the inferior barrier V has the same behavior of response ($k' = -1/2$) of a force applied at any direction in the space, namely for any angle between the applied force and the velocity of the particle.

We must point out the fact that m_{eff} has nothing to do with the "relativistic mass" (relativistic energy E in Eq. 42) in the sense that m_{eff} is dynamically responsible for both barriers V and c. The discrepancy between the "relativistic mass" (energy mc^2 of the particle) and such an effective inertial mass (m_{eff}) can be interpreted under SSR theory, as follows: m_{eff} is a dressed inertial mass given in response to the presence of the vacuum-S_V that works like a kind of "fluid" in which the particle m_0 is immersed, while the "relativistic mass" in SSR (Eq. 42) works like a bare inertial mass in the sense that it is not considered to be under the dynamical influence of the "fluid" connected to the vacuum-S_V. That is the reason why the exponent $k = 1/2$ in Eq. (42) cannot be used to explain the existence of an infinite barrier at V, namely the vacuum-S_V barrier is governed by the exponent $k' = -1/2$ as shown in Eqs. (63), (64) and (65), which prevents $v_*(= v - V) \leq 0$.

The difference betweeen the dressed (effective) mass and the relativistic (bare) mass, i.e., $m_{eff} - m$ represents an interactive increment of mass Δm_i that has purely origin from the vacuum energy of S_V, mamely:

$$\Delta m_i = m_0 \left[\frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V^2}{v^2}\right)^{\frac{1}{2}} \left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} - \frac{\left(1 - \frac{V^2}{v^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \right] \tag{66}$$

We have $\Delta m_i = m_{eff} - m$, being $m_{eff} = m_{dressed}$ given in Eq. (65) and m (m_r) given in Eq. (42).

From Eq. (66), we consider the following special cases:

(a) for $v \approx c$ we have

$$\Delta m_i \approx m_0 \left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} - \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \right] \tag{67}$$

As the effective inertial mass m_{eff} (m_L) increases much faster than the bare (relativistic) mass m (m_r) close to the speed c, there is an increment of inertial mass Δm_i that dresses m in direction of its motion and it tends to be infinite when $v \to c$, i.e., $\Delta m_i \to \infty$.

(b) for $V << v << c$ (newtonian or intermediary regime) we find $\Delta m_i \approx 0$, where we simply have m_{eff} ($m_{dressed}$)$\approx m \approx m_0$. This is the classical approximation.

(c) for $v \approx V$ (close to the vacuum-S_V regime), we have the following approximation:

$$\Delta m_i = (m_{dressed} - m) \approx m_{dressed} \approx \frac{m_0}{\sqrt{1 - \frac{V^2}{v^2}}}, \tag{68}$$
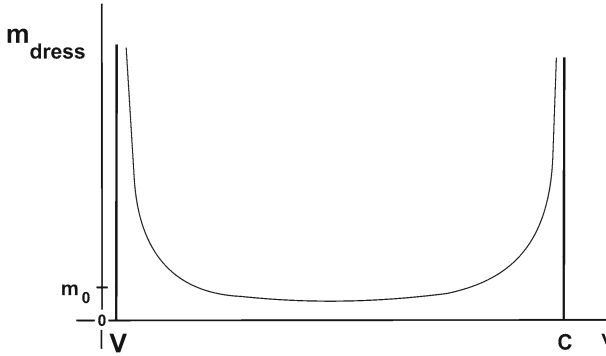
where $m \approx 0$ when $v \approx V$ (see Eq. 42).

Fig. 6 The graph shows us two infinite barriers at V and c, providing an aspect of symmetry of SSR. The first barrier (V) is exclusively due to the vacuum-S_V, being interpreted as a barrier of pure vacuum energy. In this regime we have the following approximations: $m_{eff} = m_{dressed} \approx \Delta m_i \approx m_0(1 - V^2/v^2)^{-1/2}$ and $m_r \approx m_0(1 - V^2/v^2)^{1/2}$ (see Fig. 4), so that $m_{dressed} \to \infty$ and $m = m_r = m_{bare} \to 0$ when $v \to V$. The second barrier (c) is a sum (mixture) of two contributions, namely the own bare (relativistic) mass m that increases with the factor $\gamma = (1 - v^2/c^2)^{-1/2}$ (see Fig. 4) plus the interactive increment Δm_i due to the vacuum energy-S_V, so that $m_{dressed} = m_L = m + \Delta m_i \approx m_0(1 - v^2/c^2)^{-3/2}$. This is a longitudinal effect. For the transversal effect, $\Delta m_i = 0$ since we get $m_T = m$. This result will be shown elsewhere

The approximation in Eq. (68) shows that the whole dressed mass has purely origin from the energy of vacuum-S_V, with $m_{dressed}$ being the pure increment Δm_i, since the bare (relativistic) mass m of the own particle almost vanishes in such a regime ($v \approx V$), and thus an inertial effect only due to the vacuum ("fluid")-S_V remains. We see that $\Delta m_i \to \infty$ when $v \to V$. In other words, we can interpret this infinite barrier of vacuum-S_V by considering the particle to be strongly coupled to the background field-S_V in all directions of the space. The isotropy of m_{eff} in this regime will be shown in detail elsewhere, being $m_{eff} = m_L = m_T \approx m_0(1 - V^2/v^2)^{-1/2}$. In such a regime, the particle practically loses its locality ("identity") in the sense that it is spread out isotropically in the whole space and it becomes strongly coupled to the vacuum field-S_V, leading to an infinite value of Δm_i. Such a divergence of the dressed mass has origin from the dilation factor $\Theta_v (\to \infty)$ for this regime when $v \approx V$, so that we can rewrite Eq. (68) in the following way: $\Delta m_i \approx m_{dressed} \approx m_0 \Theta(v)^{1/2}$. That is essentially the dynamical explanation why the particle cannot reach the rest in SSR theory so that the background frame of the vacuum-S_V becomes unattainable for any particle at quantum level. However, in the macroscopic (classical) level, the minimum speed V as well as the Planck constant $\hbar$ are negligible as a good approximation, such that the rest state is naturally recovered in spite of the subatomic particles that constitute a body at rest are always moving, since its temperature can never reach the absolute zero, as well as their constituent subatomic particles can never reach V.

Figure 6 shows the graph for the longitudinal effective inertial mass $m_{eff} = m_L$ ($m_{dressed}$) as a function of the speed v with respect to the ultra-referential S_V.

Now it is important to notice that a particle moving in one spatial dimension (x) goes only to right or to left, since the unattainable minimum limit of speed V prevents it to reach the rest in view of the rapid increase of its dressed mass (Eq. 68). So we cannot

stop the motion and return in the same spatial dimension x. On the other hand, in a complementary and symmetric way to V, the limit c, which represents the temporal aspect of the space-time, prevents to stop the marching of the time ($v_t = 0$), and so avoiding to come back to the past. In short, we perceive that the basic ingredient of the space-time structure in SSR, namely the $(1+1)D$ space-time, presents x and t in equal-footing in the sense that both of them are irreversible once the particle is moving only to right or to left. Such an equal-footing "xt" in SSR theory does not occurs in SR theory since we can stop the spatial motion in SR ($v_x = 0$) and after come back in x, but not in the time t. However, if we take into account more than one spatial dimension in SSR theory, at least two spatial dimensions (xy), thus the particle could return by moving in the additional dimension(s) y (and/or z). So, SSR theory is able to provide the reason why we must have more than one (1) spatial dimension for representing movement in reality $(3+1)D$, although we could have only one (1) spatial dimension just as a good practical approximation for the case of classical space-time as in SR theory (e.g.:a ball moving in a rectilinear path).

6 Cosmological implications

6.1 Energy-momentum tensor in the presence of the ultra-referential-S_V

Let us rewrite the 4-velocity (Eq. 39) in the following alternative way:

$$U^\mu = \left[\frac{\sqrt{1 - \frac{V^2}{v^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} , \frac{v_\alpha \sqrt{1 - \frac{V^2}{v^2}}}{c\sqrt{1 - \frac{v^2}{c^2}}} \right] = \left[U^0, U^\alpha \right], \tag{69}$$

where now we have $\mu = 0, 1, 2, 3$ and $\alpha = 1, 2, 3$. If $V \to 0$, we recover the 4-velocity of SR.

The well-known energy-momentum tensor to deal with perfect fluid is of the form

$$T^{\mu\nu} = (p + \epsilon)U^\mu U^\nu - pg^{\mu\nu}, \tag{70}$$

where U^μ is given in Eq. (69). p represents a pressure and ϵ an energy density.

From Eqs. (69) and (70), by performing the new component T^{00}, we obtain

$$T^{00} = \frac{\epsilon(1 - \frac{V^2}{v^2}) + p(\frac{v^2}{c^2} - \frac{V^2}{v^2})}{(1 - \frac{v^2}{c^2})} \tag{71}$$

If $V \to 0$, we recover the old component T^{00}.

Now, as we are interested only in obtaining T^{00} in absence of matter, i.e., the vacuum limit connected to the ultra-referential S_V, we perform the limit of Eq. (71) as follows:

$$lim_{v \to V} T^{00} = T^{00}_{vacuum} = \frac{p(\frac{V^2}{c^2} - 1)}{(1 - \frac{V^2}{c^2})} = -p. \tag{72}$$

From Eq. (71), we notice that the term $\epsilon \gamma^2 (1 - V^2/v^2)$ for representing matter naturally vanishes in the limit of vacuum-S_V ($v \to V$), and therefore just the contribution of vacuum prevails. As we always must have $T^{00} > 0$, we get $p < 0$ in Eq. (72). This implies a negative pressure for vacuum energy density of the ultra-referential S_V. So we verify that a negative pressure emerges naturally from such a new tensor in the limit of S_V.

We can obtain $T^{\mu\nu}_{vacuum}$ by calculating the limit of vacuum-S_V for Eq. (70), by considering Eq. (69), as follows:

$$T^{\mu\nu}_{vacuum} = lim_{v \to V} T^{\mu\nu} = -pg^{\mu\nu}, \tag{73}$$

where we conclude that $\epsilon = -p$. In Eq. (69), we see that the new 4-velocity vanishes in the limit of the vacuum-S_V ($v \to V$), namely $U^{\mu}_{vac.} = (0, 0)$. So, $T^{\mu\nu}_{vac.}$ is in fact a diagonalized tensor as we hope to be. The vacuum-S_V that is inherent to such a space-time with an invariant minimum speed works like a *sui generis* fluid in equilibrium with negative pressure, leading to a cosmological anti-gravity, i.e., the invariant minimum speed connected to a universal background field in the preferred frame S_V leads naturally to the well-known equation of state of the cosmological constant $p = w\epsilon$, with $w = -1$ [34–37].

6.2 The cosmological constant Λ and the vacuum energy density ρ

The well-known relation [34–37] between the cosmological constant Λ and the vacuum energy density $\rho_{(\Lambda)}$ is

$$\rho_{(\Lambda)} = \frac{\Lambda c^2}{8\pi G} \tag{74}$$

Let us consider a spherical universe with Hubble radius filled by a uniform vacuum energy density. On the surface of such a sphere (frontier of the observable universe), the bodies (galaxies) experience an accelerated expansion (anti-gravity) due to the whole "dark mass (energy)" of vacuum inside the sphere. So we could think that each galaxy is a proof body interacting with that big sphere like in the simple case of two bodies interaction. However, we need to show that there is an anti-gravitational interaction between the ordinary proof mass m and the big sphere with a "dark mass" of vacuum (M_{Λ}), but let us first start from the well-known simple model of a massive proof particle m_0 that escapes from a classical gravitational potential ϕ on the surface of a big sphere of matter, namely $E = m_0 c^2 (1 - v^2/c^2)^{-1/2} \equiv m_0 c^2 (1 + \phi/c^2)$, where E is its relativistic energy. Here the interval of escape velocity $0 \leq v < c$ is associated with the interval of potential $0 \leq \phi < \infty$, where we stipulate $\phi > 0$ to be the attractive (classical) gravitational potential.

Now we can show that the influence of the background field (vacuum energy inside the sphere) connected to the ultra-referential S_V (see Eq. 72) leads to a strong repulsive (negative) gravitational potential ($\phi << 0$) for very low energies ($E \to 0$). In order to see this non-classical aspect of gravitation [26], we use Eq. (42) just taking into account the new approximation given for very low energies ($v(\approx V) << c$), as follows:

$$E \approx \theta m_0 c^2 = m_0 c^2 \sqrt{1 - \frac{V^2}{v^2}} \equiv m_0 c^2 \left(1 + \frac{\phi}{c^2}\right), \tag{75}$$

where $\phi < 0$ (repulsive). For $v \to V$, this implies $E \to 0$, which leads to $\phi \to -c^2$. So, the non-classical (most repulsive) minimum potential $\phi(V)(= -c^2)$ connected to vacuum-S_V is responsible for the cosmological anti-gravity (see also Eqs. 72 and 73). We interpret this result assuming that only an exotic "particle" of the vacuum energy at S_V could escape from the anti-gravity ($\phi = -c^2$) generated by the vacuum energy inside the sphere (consider $v = V$ in Eq. 75). Therefore, ordinary bodies like galaxies and any matter on the surface of such a sphere cannot escape from its anti-gravity, being accelerated far away.

According to Eq. (75), we should note that such an exotic "particle" of vacuum (at S_V) has an infinite mass m since we should consider $v = V$ ($\theta = 0$) in order to have a finite value of E, other than the photon ($v = c$), that is a massless particle (see Eq. 42). So we conclude that an exotic "particle" of vacuum works like a counterparty of the photon, namely an infinitely massive boson.

We consider that the most negative (repulsive) potential ($\phi = -c^2$ for $v = V$, in Eq. 75) is related to the cosmological constant (vacuum energy density), since we have shown in Eqs. (72) and (73) that the background reference frame S_V plays the role of the vacuum energy density with a negative pressure, working like the cosmological constant Λ ($p = -\epsilon = -\rho_{(\Lambda)}$). So we write

$$\phi_\Lambda = \phi(\Lambda) = \phi(V) = -c^2 \tag{76}$$

Let us consider the simple model of spherical universe with a radius R_u, being filled by a uniform vacuum energy density $\rho_{(\Lambda)}$, so that the total vacuum energy inside the sphere $E_\Lambda = \rho_{(\Lambda)} V_u = -p V_u = M_\Lambda c^2$. V_u is its volume and M_Λ is the total dark mass associated with the dark energy for Λ (vacuum energy: $w = -1$ [34–37]). Therefore the repulsive gravitational potential on the surface of such a sphere is

$$\phi_\Lambda = -\frac{GM_\Lambda}{R_u} = -\frac{G\rho_{(\Lambda)} V_u}{R_u c^2} = \frac{4\pi G p R_u^2}{3c^2}, \tag{77}$$

where $p = -\rho_{(\Lambda)}$, with $w = -1$ [34–37].

By introducing Eq. (74) into Eq. (77), we find

$$\phi_\Lambda = \phi(\Lambda) = -\frac{\Lambda R_u^2}{6} \tag{78}$$

Finally, by comparing Eq. (78) with Eq. (76), we extract

$$\Lambda = \frac{6c^2}{R_u^2},\tag{79}$$

where $\Lambda S_u = 24\pi c^2$, $S_u = 4\pi R_u^2$.

And, by comparing Eq. (77) with Eq. (76), we have

$$\rho_{(\Lambda)} = -p = \frac{3c^4}{4\pi G R_u^2},\tag{80}$$

where $\rho_{(\Lambda)} S_u = 3c^4/G$. We can verify that Eqs. (80) and (79) satisfy Eq. (74).

In Eq. (79), Λ is a kind of *cosmological scalar field*, extending the old concept of Einstein about the cosmological constant for stationary universe. From Eq. (79), by considering the Hubble radius, with $R_u = R_{H_0} \sim 10^{26}$m, we obtain $\Lambda = \Lambda_0 \sim (10^{17}m^2s^{-2}/10^{52}m^2) \sim 10^{-35}s^{-2}$. To be more accurate, we know the age of the universe $T_0 = 13.7$ Gyr, being $R_{H_0} = cT_0 \approx 1.3 \times 10^{26}$m, which leads to $\Lambda_0 \approx 3 \times 10^{-35}s^{-2}$. It is interesting to notice that this tiny positive value is in agreement with the observational data [27–31]. The vacuum energy density [32,33] given in Eq. (80) for R_{H_0} is $\rho_{(\Lambda_0)} \approx 2 \times 10^{-29}g/cm^3$, which is also in agreement with observations. For the scale of the Planck length, where $R_u = l_P = (G\hbar/c^3)^{1/2}$, from Eq. (79) we find $\Lambda = \Lambda_P = 6c^5/G\hbar \sim 10^{87}s^{-2}$, and from Eq. (80), $\rho_{(\Lambda)} = \rho_{(\Lambda_P)} = T_{vac.P}^{00} = \Lambda_P c^2/8\pi G = 3c^7/4\pi G^2\hbar \sim 10^{113}J/m^3(= 3c^4/4\pi l_P^2 G \sim 10^{43}kgf/S_P \sim 10^{108}atm \sim 10^{93}g/cm^3)$. So, just at that past time, Λ_P or $\rho_{(\Lambda_P)}$ played the role of an inflationary vacuum field with 122 orders of magnitude [34–37] beyond the ones (Λ_0 and $\rho_{(\Lambda_0)}$) for the present time.

It must be stressed that our assumption for obtaining the tiny value of Λ_0 starts from first principles related to a new symmetry in spacetime, i.e., we have introduced the idea of a background reference frame S_V for representing the vacuum energy connected to an invariant minimum speed V, leading to the cosmological constant.

Here it should be also emphasized that both the cosmological constant and the minimum speed have non-zero values due to the same cause, which is essentially the existence of a fundamental state of vacuum with non-zero (very low) energy density, given by the potential $-c^2$ in Eq. (76). This means that the non-zero value of the cosmological constant $\Lambda_0(\sim 10^{-35}s^{-2})$ has origin in the fact that there must be also a non-zero value of a minimum speed V at subatomic level; however this does not mean that the value of V should be obtained directly as a function of Λ, since we just know that $V(> 0)$ and $\Lambda(> 0)$ are different aspects of the same reality, i.e., the existence of a non-null energy density of vacuum. In any way, a deeper investigation of the origin of the minimum speed V by obtaining its value should be important in order to clarify further this question.

7 Conclusion and prospects

The very high values obtained for the cosmological constant and the vacuum energy density by means of the quantum field theory for the quantum vacuum have a discrep-

ancy of about 120 orders of magnitude beyond their observational values. This puzzle is well-known as the *"Cosmological Constant Problem"*, [34–37]. The idea of an invariant minimum speed V connected to a background field for the ultra-referential S_V, within an extended structure of spacetime by breaking down Lorentz symmetry, has led to low values of the vacuum energy density and the cosmological constant in agreement with observational results of Perlmutter, Schmidt and Riess.

After investigating the origin of the minimum speed V and a possible connection between V and the zero-point energy of the quantum mechanics (the uncertainty principle), we should thoroughly explore many interesting consequences of SSR and its new dispersion relation in quantum field theories (QFT), since the existence of a mimimum speed for lower energies with the same status of the speed of light for higher energies leads to a new metric for describing such deformed (symmetric) spacetime, allowing us to build a modified QFT, where the Lorentz symmetry is broken down. This kind of metric $\Theta(v)g_{\mu\nu}$ in Eq. (31) is a special case of metric that has already been explored in the literature and it seems to lead to the Finsler's geometry, namely a Finslerian space with a metric that depends on the position and also the velocity, i.e., $G_{\mu\nu}(x, \dot{x})$ [38–40].

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