CBPF-NF-016/01

# On the Axisymmetric Lewis Metric

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**Abstract**

We obtain the general solution of the axisymmetric stationary vacuum spacetime of Lewis. After precising the fundamental hypothesis of Lewis, we demonstrate that the solution is related to an arbitrary harmonic function. Formally, these solutions are the same as for the corresponding cylindrically symmetric case, and can be classified in a similar way. Furthermore, the interpretation, in the cylindrically symmetric system, of the field equations as describing the motion of a classical particle in a central force field is still valid.

We conjecture that one of the solutions represent a distorted black hole.

**Key-words:** Theory of gravitation, Axisymmetric systems, Lewis metric.

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# I Introduction

Axially symmetric stationary vacuum spacetimes in Einstein's theory are important because they can describe the exterior fields of massive rotating astrophysical objects. Here we obtain the general solution of Lewis spacetime.

In a preceding paper [3] we have reexamined the vacuum solutions obtained by Lewis [1], and van Stockum [2], for a stationary cylindrically symmetric spacetime. Lewis established the existence of three classes of solutions in terms of four parameters. One of these classes appeared by the introduction of complex parameters. Through our approach the three classes arised without the need of complexification. Furthermore, we showed that the structure of the field equations can be associated to the motion of a classical particle in a central field. This association allowed a physical interpretation of the parameters, describing the Lewis spacetime, without the need of specifying a particular matter source of the field.

Here we precise the fundamental hypothesis which allows to define a Lewis metric. Then we follow some similar steps of the paper [3] to obtain the metric for an axially symmetric stationary vacuum spacetime of Lewis. We give a deduction of the metric for the cylindrically symmetric stationary spacetime which permits to extend it to the axisymmetric case. This extension is accomplished with the natural introduction of an arbitrary harmonic function of the cylindrical coordinates $r$ and $z$. Formally the solutions are similar to the cylindrical Lewis metric, but now being a function of this harmonic function. The classification and mechanical interpretation in the cylindrical case can be extended, also, to the axisymmetric case. For one of the solutions, we conjecture as representing a distorted black hole.

The paper is organized as follows. In section II we recall the system of equations to be solved for the axially symmetric stationary vacuum metrics. We introduce in section III the fundamental hypothesis to produce the Lewis metric, from which the linear dependence between the potentials is deduced. From a new presentation of the cylindrically symmetric solution, we deduce in section IV the axisymmetric solution of Lewis. In section V the solutions and classification are presented. We end with a short conclusion.

# II Field equations

The general line element for a stationary axisymmetric spacetime can be written like

$$ds^2 = -fdt^2 + 2kdtd\phi + e^{\mu}(dr^2 + dz^2) + ld\phi^2, \tag{1}$$

where $f, l, k$ and $\mu$ are all functions of the cylindrical coordinates $r$ and $z$. Defining for convenience,

$$f = rF(r,z), \quad l = rL(r,z), \quad k = rK(r,z), \tag{2}$$

we obtain from Einstein's vacuum field equations [1, 2],

$$\triangle F = -F\Omega, \tag{3}$$

$$\triangle L = -L\Omega, \tag{4}$$

$$\triangle K = -K\Omega, \tag{5}$$

$$\mu_r = -\frac{1}{2r}[1 + r^2(F_r L_r - F_z L_z + K_r^2 - K_z^2)], \tag{6}$$

$$\mu_z = -\frac{r}{2}(F_r L_z + F_z L_r + 2K_r K_z), \tag{7}$$

with

$$FL + K^2 = 1, \tag{8}$$

where the Laplacian $\triangle$ and $\Omega$ are defined by

$$\triangle F = F_{rr} + \frac{1}{r}F_r + F_{zz}, \tag{9}$$

$$\Omega = F_r L_r + K_r^2 + F_z L_z + K_z^2, \tag{10}$$

with the indexes standing for differentiation. The function $\mu$ is usually obtained by quadratures. Thus, we have only to determine $F, L$ and $K$.

# III Fundamental hypothesis for Lewis solution

In the cylindrically symmetric case, where in (1) $F$, $L$ and $K$, depend only on $r$, we have demonstrated a linear dependence between the potentials [3]. In the axially symmetric case, when $F$, $L$ and $K$ are functions of $r$ and $z$, such a general demonstration is no longer possible. Thus, we have to introduce some further hypothesis to solve the field equations. We assume that there exists a second relation, different from (8), between $F$, $L$ and $K$,

$$\Phi(F, L, K) = 0. \tag{11}$$

Then, from (8) and (11) we can obtain two general relations that can be expressed, for example, as

$$F = F(K), \quad L = L(K). \tag{12}$$

This hypothesis is equivalent to the Lewis' one ([1] p. ). From (12) we have the identities,

$$\nabla F \cdot \nabla L + (\nabla K)^2 \equiv (1 + F_K L_K)(\nabla K)^2, \tag{13}$$

$$\triangle F \equiv F_K \triangle K + F_{KK}(\nabla K)^2, \tag{14}$$

$$\triangle L \equiv L_K \triangle K + L_{KK}(\nabla K)^2, \tag{15}$$

where $\nabla$ is the gradient operator. With (12)-(15), we can rewrite the two first field equations (3) and (4) like

$$(1 + F_K L_K)(KF_K - F) = F_{KK}, \tag{16}$$

$$(1 + F_K L_K)(KL_K - L) = L_{KK}, \tag{17}$$

which is a system of two differential equations permitting to determine the functions (12), as we shall see (equations (36)). Hence, the only partial derivative equation to solve is the third field equation, (5), for the function $K(r, z)$,

$$\triangle K = -K(1 + F_K L_K)(K_r^2 + K_z^2). \tag{18}$$

A kinematical interpretation can be easily obtained from (16) and (17). Indeed, considering (16) multiplied by $L$ and (17) by $F$ and subtracting both equations, we obtain,

$$(1 + F_K L_K)K = \frac{(LF_K - FL_K)_K}{LF_K - FL_K}. \tag{19}$$

Without any loss of generality, we can make a change of unknown function by putting $K = K(\chi)$, where $\chi(r, z)$ is a new unknown function. We fix this change by the differential equation,

$$\frac{K_{\chi\chi}}{K_\chi} = -(1 + F_K L_K)KK_\chi. \tag{20}$$

Then, we can see from (18) that $\chi$ has to be a harmonic function. We shall study the consequences of this fact in the next section. Before, let us examine what (20) implies on the two first field equations. Substituting (19) into (20) and integrating we obtain

$$LF_\chi - FL_\chi = C_1, \tag{21}$$

where $C_1$ is an integration constant. In a similar way, but starting from (4) and (5) with $L = L(F)$ and $K = K(F)$, and considering (3) with $F(\chi)$; and repeating again from (3) and (5) with $F(L)$ and $K(L)$ and considering (4) with $L(\chi)$, we obtain

$$KF_\chi - FK_\chi = C_2, \tag{22}$$

$$LK_\chi - KL_\chi = C_3, \tag{23}$$

respectively, where $C_2$ and $C_3$ are also integration constants.

The equations (21)-(23) express the conservation of an *angular momentum* $\vec{C} = (C_1, C_2, C_3)$ in the *space* $(F, L, K)$, like in the cylindrical case [3]. But now it is $\chi$, instead of $r$, which plays the role of *time*. Besides, from (21)-(23), we can immediately deduce a linear relation between the potentials,

$$K = \alpha L + \beta F, \tag{24}$$

where $\alpha$ and $\beta$ are constants. Expression (24) is the relation (11) that we looked for. Hence, most of the interpretation in terms of a classical particle in a central field made in [3] holds again. In particular, the discussion about the nature of the conic, which is the intersection of the surfaces (8) and (24) in the $(F, L, K)$ space, followed in [3] for the cylindrical case, remains the same in the axisymmetric case.

Let us stress that all the results of this section can be obtained in the axisymmetric case only under the hypothesis (11), that we call the *fundamental hypothesis* for Lewis, while in the cylindrical case they were general, i.e. valid without any hypothesis. A well known counter example of an axisymmetric solution that does not satisfy this hypothesis is Kerr solution.

# IV $\chi(r, z)$ is a harmonic function

For the cylindrically symmetric case, we give now an integration method of the $K(r)$ equation slightly different from the one presented in [3]. By doing this, we want to enlight the common feature of the two types of Lewis solutions, cylindric and axisymmetric, namely the fact that they only depend on a harmonic function. However, this function is imposed in the cylindric case, whereas it is arbitrary in the axial case.

In the cylindrical case, (18) with (24) reduces to

$$K_{rr} + \frac{1}{r}K_r - \frac{\delta K K_r^2}{\Delta} = 0, \tag{25}$$

with

$$\Delta \equiv \delta K^2 - 4\alpha\beta, \quad \delta \equiv 1 + 4\alpha\beta. \tag{26}$$

Changing the unknown function $K = K(\chi)$ in (25) in such a way that

$$\frac{K_{\chi\chi}}{K_\chi} = \frac{\delta K K_\chi}{\Delta} \tag{27}$$

leads to

$$\frac{\chi_{rr}}{\chi_r} = -\frac{1}{r}. \tag{28}$$

Consequently, after integration of (28), we obtain

$$\chi = k_1 \ln\left(\frac{r}{r_0}\right), \tag{29}$$

where $k_1$ and $r_0$ are integration constants, and by integration of (27),

$$\int \frac{dK}{\sqrt{\Delta}} = k_1 \ln \frac{r}{r_0} + k_2, \tag{30}$$

where $k_2$ is an integration constant.

The study of the integral (30) leads to the cylindrical solutions of Lewis [3]. Let us note that all these solutions depend only on the solution of the differential equation (28), i.e.,

$$\triangle\chi = \chi_{rr} + \frac{1}{r}\chi_r = 0, \tag{31}$$

which means that $\chi$ is a harmonic function. In this special case of cylindrical symmetry, the differential equation (31) can be explicitly integrated, giving the only solution (29).

It is no longer the case in the more general axisymmetric situation, for which the corresponding equation (hereafter (34)) is a partial derivative equation, even though the line reasoning remains the same. Indeed, coming back to (18), it can be written as

$$\triangle K = f(K)(\nabla K)^2, \tag{32}$$

where

$$f(K) = -K(1 + F_k L_K). \tag{33}$$

The standard procedure of changing the unknown function $K = K(\chi)$ used in (20), gives now with (24),

$$K_{\chi\chi} + \frac{1}{\chi}K_\chi - \frac{\delta K K_\chi^2}{\triangle} = 0. \tag{34}$$

With (34), (32) reduces to

$$\triangle \chi = 0. \tag{35}$$

We have that (34) is (25) with $\chi$ in place of $r$, and $\chi(r, z)$ is an arbitrary harmonic function.

So, we can obtain the different classes of the Lewis solution by an analysis similar to the one used in the cylindric case [3].

# V Axisymmetric solutions of the three classes of Lewis

The solutions $K(\chi)$ of (34), expressed in terms of an arbitrary harmonic function $\chi(r, z)$ can be classified following the sign of $\delta$, defined in (26), like in the procedure used in the cylindrical case [3].

The corresponding functions $F(\chi)$ and $L(\chi)$ are deduced from the relations

$$F = \frac{K \mp \sqrt{\Delta}}{2\alpha}, \quad L = \frac{K \pm \sqrt{\Delta}}{2\beta}, \tag{36}$$

obtained from (8) and (24). So, we arrive to three classes, which are the following.

## V.1 Class I: $\delta > 0$

### V.1.1 $\alpha\beta > 0$

$$K = 2\left(\frac{\alpha\beta}{\delta}\right)^{1/2} \cosh\chi, \tag{37}$$

$$F = \left(\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \cosh\chi \mp \sinh\chi\right), \tag{38}$$

$$L = \left(\frac{\beta}{\alpha}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \cosh\chi \pm \sinh\chi\right). \tag{39}$$

### V.1.2 $\alpha\beta < 0$ with $-\alpha\beta < 1/4$

$$K = 2\left(-\frac{\alpha\beta}{\delta}\right)^{1/2} \sinh\chi, \tag{40}$$

$$F = \left(-\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \sinh\chi \mp \cosh\chi\right), \tag{41}$$

$$L = \left(-\frac{\beta}{\alpha}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \sinh\chi \pm \cosh\chi\right). \tag{42}$$

### V.1.3 $\alpha\beta = 0$

Here we use (8) and (24), instead of (26) and (37).

**Case** $\alpha = 0$ **and** $\beta \neq 0$

$$K = e^{\chi}, \tag{43}$$

$$F = \frac{1}{\beta} e^{\chi}, \tag{44}$$

$$L = \beta(e^{-\chi} - e^{\chi}). \tag{45}$$

**Case** $\alpha \neq 0$ **and** $\beta = 0$

$$K = e^{\chi}, \tag{46}$$

$$F = \alpha(e^{-\chi} - e^{\chi}), \tag{47}$$

$$L = \frac{1}{\alpha} e^{\chi}. \tag{48}$$

**Case** $\alpha = \beta = 0$

We use (3) obtaining the Weyl metric,

$$K = 0, \tag{49}$$

$$F = e^{\chi}, \tag{50}$$

$$L = e^{-\chi}. \tag{51}$$

This solution, without dragging, is an axisymmetric extension of the cylindrical Levi-Civita solution. The uniqueness of the Schwarzschild solution, for instance Chandrasekhar [4] and Novikov and Frolov [5] remarked, forbids the interpretaion of this static axisymmetric solution as representing a black hole. We conjecture, with them, that it represents a non isolated black hole distorted by an external distribution of mass.

## V.2 Class II: $\delta < 0$

We remark here, as we did in [3], that there is no need of introducing complex parameters in our approach, as it is usually done in the corresponding cylindrical case.

$$K = 2\left(\frac{\alpha\beta}{\delta}\right)^{1/2} \sin\chi, \tag{52}$$

$$F = \left(-\frac{\alpha}{\beta}\right)^{1/2} \left(\frac{1}{\sqrt{\delta}} \sin\chi \mp \cos\chi\right), \tag{53}$$

$$L = \left(-\frac{\beta}{\alpha}\right)^{1/2}. \tag{54}$$

### V.3 Class III: $\delta = 0$ or $\alpha\beta = -1/4$

$$K = \chi, \tag{55}$$

$$F = \frac{1}{2\beta}(\chi \mp 1), \tag{56}$$

$$L = \frac{1}{2\alpha}(\chi \pm 1). \tag{57}$$

## VI Conclusion

The general solution of the cylindrically symmetric stationary vacuum Einstein's field equations is the Lewis solution. It is no longer the case for the more general equations with axial symmetry. We precised here the fundamental hypothesis under which we can obtain the Lewis axisymmetric solution. This hypothesis allowed us to demonstrate a linear relation between the potentials. This fact implied that the field equations can be interpreted as describing the motion of a classical particle in a central force field, like in the cylindrical symmetric case [3]. The general solution for the Lewis axisymmetric vacuum spacetime that we obtained depends upon an arbitrary harmonic function, and its classification, in three different classes, is similar to the cylindrically symmetric case. This harmonic function plays the role of time in the motion of a classical particle interpretation.

## References

[1] Lewis T, *Proc. R. Soc.* **136**, 176 (1932)

[2] van Stockum W J, *Proc. R. Soc. Edinburgh* **57**, 135 (1937)

[3] Gariel J, Marcilhacy G and Santos N O, *J. Math. Phys.* **41**, 7535 (2000)

[4] Chandrasekhar S *The Mathematical Theory of Black Holes* (Oxford: Oxford University Press, 1983)

[5] Novikov I D and Frolov V P *Physics of Black Holes - Fundamental Theory of Physics* vol. 27 (Amsterdam: Kluwer Academic Publishers, 1989)