CBPF-NF-016/00

Brillouin's Paradox – The Path-Integral Solution

Luiz C.L. Botelho

Centro Brasileiro de Pesquisas Físicas

Rua Dr. Xavier Sigaud, 150

22290-180 - Rio de Janeiro, RJ, Brazil

Permanent address:Departamento de Física

Universidade Federal Rural do Rio de Janeiro

Itaguaí, RJ 23851, Brazil

Abstract

We solve the famous Brillouin's paradox, Phys. Rev. 78, 627, (1950)).

Key-words: Path integral; Brillouin's paradox.

One of the most challenging paradox on physics out of thermal equilibrium is that related to the transient regime of the charge on a $p-n$ junction subject to a thermal random noise which obeys the following equation ([1] – pag. 251)

$$\frac{dQ(t)}{dt} = -A\left\{exp\left[\frac{eQ(t)}{kTc}\right] - 1\!\!1\right\} + \eta(t) \tag{1}$$

Here A is a positive constant; c the capacitance of the condensator, T the system temperature (Alkemades diode) and $\eta(t)$ denotes the white noise fluctuation stochastic process produced by the rectifier with correlation function of the form

$$\langle \eta(t)\eta(t')\rangle_\eta = (kT)\delta(t-t') \tag{2}$$

The Brillovin's paradox comes from the fact of considering directly the noise averaged motion equation at the steady value $\frac{dQ_\infty}{dt} \equiv 0$ (*see my paper ref. [2] on Ohm's law!*)

$$\begin{aligned} 0 &= -A\left\{\left\langle exp\left[\frac{eQ(t;[\eta]}{kcT}\right]\right\rangle_\infty - 1\right\} + \langle\eta\rangle \\ &= -A\left\{\frac{e}{kcT}\langle Q\rangle_\infty + \frac{1}{2}\left(\frac{e}{kcT}\right)^2\langle Q^2\rangle_\infty + \cdots\right\} + 0 \end{aligned} \tag{3}$$

which leads to the equilibrium dominant equation (note that the noise average factorization at the steady $t \to \infty$ limit!)

$$\langle Q\rangle_\infty = -\frac{e}{2kcT}\langle Q^2\rangle_\infty = -\frac{e}{2kcT}\langle Q\rangle_\infty^2 \tag{4}$$

with the unphysical *non-vanishing* solution ([1] – pag. 251)

$$\langle Q\rangle_\infty = -\frac{2kcT}{e} \neq 0 \tag{5}$$

This diode in thermal equilibrium has a non-zero charge and, thus, a voltage on the condenser!.

Let us in this short note solve the above paradox by following my previously exposed path-integral method to solve eq. (1) ([3]).

The generating functional of the stochastic process eq. (1)-eq. (2) may be represented by the following Wiener functional integral defined on the full range interval $[0,\infty] = U_{n=0}^{\infty}[0,n]$ (a σ-compact functional integral–ref. [4]).

$$Z[J(t)] = \frac{1}{Z(0)}\int d\mu^{Wiener}[Q]\ exp\left\{-\frac{A^2}{2kT}\int_0^\infty dt\left(e^{\frac{2eQ}{kcT}} - 2e^{\frac{eQ}{kcT}}\right)\right\}\ exp\left(i\int_0^\infty J(t)Q(t)\right) \tag{6}$$

where the (rigorously defined) Wiener measure over the continuous (but non differentiable!) paths $Q(t)$ is formally defined in terms of the Feynman measure

$$d\mu^{Wiener}[Q(t)] = \left(\prod_{0\le t\le\infty} (dQ(t))\right) exp\left\{-\frac{1}{2kT}\int_0^\infty dt\left(\frac{dQ}{dt}\right)^2(t)\right\} \tag{7}$$

At this point, we exploit the "translational-invariance" of the functional measure ([4]), to arrive at the following identity (the correct averaged motions equation or the Schwinger-Dyson equations)

$$\int d\mu^{Wiener}[Q(t)]\left\{\left[-\frac{d^2}{dt^2}Q + A^2\, e^{\frac{2eQ}{kTc}}\left(\frac{2e}{kcT}\right) - \frac{2A^2e}{kcT}e^{\frac{eQ}{kTc}}\right]\right\} \equiv 0 \tag{8}$$

which at the equilibrium regime leads to simple algebraic equation

$$\langle e^{\frac{2eQ}{kTc}}\rangle_\infty = \langle e^{\frac{eQ}{kTc}}\rangle_\infty \tag{9}$$

or in a equivalent way

$$1 + \frac{2e}{kcT}\langle Q\rangle_\infty + \cdots \frac{1}{n!}\left(\frac{2e}{kcT}\right)^n (\langle Q\rangle_\infty)^n = 1 + \frac{e}{kcT}\langle Q\rangle_\infty + \cdots \frac{1}{n!}\left(\frac{e}{kcT}\right)(\langle Q\rangle_\infty)^n$$

which leads by its turn to the *physical expected zero solution at the equilibrium*

$$\langle Q\rangle_\infty \equiv 0$$

Acknowledgements: Luiz C.L. Botelho is grateful to Helayël-Neto (DCP-CBPF) for warm hospitality.

References

[1] N.G. Van Kampen, "Stochastic Processes in Physics and Chemistry-North-Holland Co., Amsterdam, 1981.

[2] Luiz C.L. Botelho et al – Non-Linear Ohm's Law, CBPF-NF-004/00, January 2000.

[3] Luiz C.L. Botelho and E.P. da Silva, J. Phys. A: Math. Gen. 23, 1829 (1990).

[4] Luiz C.L. Botelho, "Domains of Bosonic Functional Integrals", CBPF-NF-052/98.
-R.H. Cameron and W.T. Martin; Duke Math. Journal, 14, 99, (1947).
-Xia Dao-Xing, "Measure and Integration Theory on Infinite-Dimensiional Spaces, (pag. 325, Corollary 5.3.8), Academic Press 1972.
-B. de Witt, Journal of Math. Phys. vol. 3, n. 6, 1073 (1962).