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Comments on the Whittaker-Hill and the Generalized Spheroidal Wave Equations

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Abstract

One finds convergent solutions for the Whittaker-Hill equation destituted of free parameters by treating it as a particular case of the generalized spheroidal wave equation. Solutions in series of trigonometric functions result from the expansions in series of Gauss hypergeometric functions for the solutions of the GSWE. From these solutions, the four Arscott solutions are recovered when there are free parameters in the Whittaker-Hill equation.

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1 Introduction

The Whittaker-Hill equation (WHE) in Arscott's form [1] reads

$$\frac{d^2w}{du^2}+\left[\overline{\eta}-(p+1)\xi cos(2u)-\frac{\xi^2}{4}\sin^2(2u)\right]w=0, \qquad (1)$$

where $\overline{\eta}$, p and ξ are constants. This equation results from variable separation for the Helmholz equation in paraboloidal coordinates [1], as well as from variable separation for the Dirac equation in Minkowski spacetime when we use cylindrical curvilinear coordinates [2]. The Arscott solutions to Eq.(1) are given as series whose coefficients satisfy 3-term recurrence relations. The series convergence condition requires that a characteristic equation, relating its parameters, be valid and that is possible only when some parameter is at our disposal.

On the other hand, the Leaver version [3] for the generalized spheroidal wave equation (GSWE) is

$$x(x-x_0)\frac{d^2U}{dx^2}+(B_1+B_2x)\frac{dU}{dx}+\left[B_3+\omega^2x(x-x_0)-2\omega\eta(x-x_0)\right]U=0, \qquad (2)$$

and the coefficients for its usual series solutions also satisfy 3-term recurrence relations. Now, if we perform the Campbell change of variables [4]

$$x=x_0\cos^2(u), \quad w(u)=W(x) \qquad (3)$$

in Eq. (1), we find

$$x(x-x_0)\frac{d^2W}{dx^2}+(-\frac{x_0}{2}+x)\frac{dW}{dx}+$$
$$\left[\frac{(p+1)\xi-\overline{\eta}}{4}-\frac{\xi^2}{4x_0^2}x(x-x_0)+\frac{\xi(p+1)}{2x_0}(x-x_0)\right]W=0.$$

Comparing this equation with the Eq. (2) we see that the WHE is a GSWE with

$$B_1=-x_0/2, \quad B_2=1. \qquad (4)$$

and consequently, in addition to Arscott's solutions, we can also obtain solutions to Eq. (1) from those for Eq. (2). This becomes particularly important when we suppose that all the parameters in the WHE are fixed, that is, when Arscott's solutions do not converge. In effect, if x is real, the problem concerning the convergence is solved in the sense that, for the GSWE, convergent solutions are known for the case in which all the parameters are fixed. For $0\leq x\leq x_0$ ("angular" wave equation) the solutions can be expanded as a series of Gauss hypergeometric functions [5]. For $x_0\leq x<\infty$ ("radial" wave equation), there are the Leaver expansions in series of confluent hypergeometric functions and Coulomb wave functions [3]. In each of these solutions an arbitrary parameter, say ν, is introduced in order to satisfy the characteristic equation and the summation index n runs from $-\infty$ to ∞. Therefore our prescription to obtain solutions to the WHE is very simple: we rewrite it as a GSWE with $B_1 = -x_0/2$, $B_2 = 1$ and then particularize its solutions for the case in question. However, there are some more comments which we outline next.

The expansions in series of hypergeometric functions were first performed in order to find convergent solutions to an " angular" GSWE without free parameters. When we choose the phase parameter as equal to zero and take the summation index n to be $n \geq 0$, they become expansions of the Fackerell and Crossman type [6] for a GSWE with free parameters, that is, expansions in terms of Jacobi polynomials. Despite this, if we only restrict the values of n to $n \geq 0$, we find conditions under which each expansion in hypergeometric functions may provide two values for phase parameter, as we will see in Section 2.1. On the other hand, in Section 2.2, we will find that, for the WHE, the general expansions in terms of hypergeometric functions reduce to expansions in series of trigonometric functions with phase parameters and $-\infty < n < \infty$, generalizing, in this manner, the Arscott solutions for the case of no free parameters. Moreover, by setting $n \geq 0$, we will find, as a consequence of the results of Section 2.1, that in each of these solutions the phase parameter assumes the values 0 and 1/2, and hence we recover the four Arscott solutions. In Section 3 we present some additional comments.

2 On the Expansions in Hypergeometric Functions

In this section our analysis is based on the two expansions in terms of hypergeometric functions, U_1 and U_2, for the solutions to Eq. (2) and so we write them out. The first of them is

$$U_1 = e^{-i\omega x_0 y} \sum_{n=-\infty}^{\infty} b_n F(-n-\nu, n+\nu-+B_2-1, B_2+B_1/x_0; y), \tag{5}$$

$$y := (x_0 - x)/x_0 \tag{6}$$

(":=" means "equal by definition"). In the case where there are no free parameters in the differential equation, the recursive relations for the coefficients b_n are

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \tag{7}$$

where

$$\alpha_n = i\omega x_0 \frac{(n+\nu+1)(n+\nu-B_1/x_0)(n+\nu+B_2/2-i\eta)}{2(n+\nu+B_2/2)(n+\nu+B_2/2+1/2)}, \tag{8a}$$

$$\beta_n = -B_3 - (n+\nu)(n+\nu+B_2-1) -$$

$$\eta\omega x_0 \frac{(B_2+B_1/x_0)(B_2-2)+2(n+\nu)(n+\nu+B_2-1)}{2(n+\nu+B_2/2-1)(n+\nu+B_2/2)}, \tag{8b}$$

$$\gamma_n = -i\omega x_0 \frac{(n+\nu+B_2-2)(n+\nu+B_2+B_1/x_0-1)(n+\nu+B_2/2+i\eta-1)}{2(n+\nu+B_2/2-3/2)(n+\nu+B_2/2-1)}. \tag{8c}$$

The characteristic equation for ν is given by the sum of two infinite continued fractions, namely,

$$\beta_0 = \frac{\alpha_{-1}\gamma_0}{\beta_{-1}-}\frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-}\frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-}\ldots + \frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-}\frac{\alpha_2\gamma_3}{\beta_3-}\ldots \,. \tag{9}$$

Note that in Eq. (5) we are supposing that

$$B_2 + B_1/x_0 \neq 0, -1, -2, \ldots \,. \tag{10}$$

To write the second solution, we first define $B_2^{'}$ and $B_3^{'}$ through

$$B_2^{'} := 2 - B_2 - \frac{2B_1}{x_0}, \quad B_3^{'} := B_3 + \frac{B_1}{x_0}\left(\frac{B_1}{x_0} + B_2 - 1\right). \tag{11}$$

Then, U_2 is given by

$$U_2 = e^{-i\omega x_0 y} y^{1-B_2-B_1/x_0} \sum_{n=-\infty}^{\infty} b_n^{'} F(-n-\nu^{'}, n+\nu^{'}+B_2^{'}-1, B_2^{'}+B_1/x_0; y), \tag{12}$$

along with the the relations

$$\alpha_n^{'} b_{n+1}^{'} + \beta_n^{'} b_n^{'} + \gamma_n^{'} b_{n-1}^{'} = 0, \tag{13}$$

where $\alpha_n^{'}$, $\beta_n^{'}$ and $\gamma_n^{'}$ are obtained from α_n, β_n and γ_n, respectively, by the substitutions

$$B_2 \to B_2^{'}, \quad B_3 \to B_3^{'}, \quad \nu \to \nu^{'}. \tag{14}$$

There is also a characteristic equation similar to Eq. (9) and a condition analogous to Eq.(10). In what follows it will be important to note the the relations (7) and (13) hold only for $-\infty \leq n \leq \infty$ and have resulted from the equations

$$\left[\sum_{m=n-1}^{\infty} \alpha_m b_{m+1} + \sum_{m=n}^{\infty} \beta_m b_m + \sum_{m=n+1}^{\infty} \gamma_m b_{m-1}\right] \times$$
$$F(-m-\nu, m+\nu+B_2-1, B_2+B_1/x_0; y) = 0, \tag{15}$$

$$\left[\sum_{m=n-1}^{\infty} \alpha_m^{'} b_{m+1}^{'} + \sum_{m=n}^{\infty} \beta_m^{'} b_m^{'} + \sum_{m=n+1}^{\infty} \gamma_m^{'} b_{m-1}^{'}\right] \times$$
$$F(-m-\nu^{'}, m+\nu^{'}+B_2^{'}-1, B_2^{'}+B_1/x_0; y) = 0, \tag{16}$$

which we get when we insert Eqs. (5) and (12) into Eq. (2). In the sums above, we have explicitly displayed the changes of indices that we accomplished in order to factorize the hypergeometric functions.

2.1 Free Parameters in the GSWE: General Case

We now ask under what conditions the restriction of the sum index to $n \geq 0$ in the solutions U_1 and U_2 may lead to two possible values for the phase parameters ν and ν'. This restriction removes the freedom to choose ν and ν' and thus it is allowed only when there are free parameters in the differential equation. We define S_n first by

$$S_n := (\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1}) F(-n-\nu, n+\nu+B_2-1, B_2+B_1/x_0; y). \tag{17}$$

Then, for $n \geq 0$, Eq. (15) becomes

$$\sum_{n=1}^{\infty} S_n + \alpha_{-1} b_0 F(1-\nu, \nu+B_2-2, B_2+B_1/x_0; y) + \\ (\alpha_0 b_1 + \beta_0 b_0) F(-\nu, \nu+B_2-1, B_2+B_1/x_0; y) = 0, \tag{18}$$

where, from Eq. (8a),

$$\alpha_{-1} = i\omega x_0 \frac{\nu(\nu-1-B_1/x_0)(\nu+B_2/2-1-i\eta)}{2(\nu+B_2/2-1)(\nu+B_2/2-1/2)}. \tag{19}$$

For the solution U_1 we must consider three cases: $B_2 \neq 1$ or 2, $B_2 = 1$ and $B_2 = 2$. In each case Eq. (18) will be satisfied by choosing suitable values for ν and appropriate recurrence relations for b_n. We have also to find the characteristic equation for each case. Note that the results below will be applicable to concrete problems only if they satisfy the condition (10) and if α_n, β_n and γ_n turn out to be finite.

First case: $B_2 \neq 1$ or 2. In this case, Eq. (18) is satisfied by taking $\underline{\nu = 0}$ ($\alpha_{-1} = 0$) and

$$\begin{cases} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1) \end{cases} \tag{20}$$

with the characteristic equation given by

$$\beta_0 = \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots . \tag{21}$$

Another possibility occurs when B_2 and B_1/x_0 have fixed values and satisfy the conditions

$$\frac{B_1}{x_0} \neq \frac{3}{2} - \frac{B_2}{2}, \quad \frac{B_1}{x_0} \neq -\frac{B_2}{2}. \tag{22}$$

Then we can choose $\underline{\nu = 1 + B_1/x_0}$ along with recurrence relations and characteristic equation similar to Eqs. (20) and (21), respectively, since again we have $\alpha_{-1} = 0$. Teukolsky angular wave equation corresponds to the possibility $\nu = 0$ and we do not know any case satisfying the conditions (22).

Second case: $B_2 = 1$. For his case

$$\alpha_{-1} = i\omega x_0 \frac{(\nu-1-B_1/x_0)(\nu-1/2-i\eta)}{2(\nu-1/2)} \tag{23}$$

and Eq. (18) can be written as

$$\sum_{n=2}^{\infty} S_n + (\alpha_1 b_2 + \beta_1 b_1 + \gamma_1 b_0) F(-1-\nu, 1+\nu, 1+B_1/x_0; y) +$$

$$(\alpha_0 b_1 + \beta_0 b_0) F(-\nu, \nu, 1+B_1/x_0; y) + \alpha_{-1} b_0 F(1-\nu, \nu-1, 1+B_1/x_0; y) = 0 \qquad (24)$$

This equation is satisfied by $\underline{\nu = 0}$ and

$$\begin{cases} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_1 b_2 + \beta_1 b_1 + (\alpha_{-1} + \gamma_1) b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 2). \end{cases} \qquad (25)$$

The characteristic equation is obtained by noting that these recurrence relations are analogous to Eqs. (20) with $\alpha_{-1} + \gamma_1$ in the place of γ_1; then

$$\beta_0 = \frac{\alpha_0(\alpha_{-1} + \gamma_1)}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \ldots . \qquad (26)$$

Eq (24) can also be satisfied by $\underline{\nu = 1/2}$ and

$$\begin{cases} \alpha_0 b_1 + (\beta_0 + \alpha_{-1}) b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1), \end{cases} \qquad (27)$$

but only if $B_1/x_0 = -1/2$, on the contrary $\alpha_{-1} \to \infty$. The characteristic equation is obtained by changing β_0 for $\beta_0 + \alpha_{-1}$ in Eq. (21):

$$\beta_0 + \alpha_{-1} = \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \ldots . \qquad (28)$$

In the present case $\nu = 0$ does not require that $B_1/x_0 = -1/2$, but $\nu = 1/2$ does. Therefore the solution for $\nu = 1/2$ must be one of the solutions of a WHE endowed with free parameters: we will find in Section 2.2 that this solution in fact coincides with one of the Arscott solutions. The solution for $\nu = 0$ will coincide with another Arscott solution only when $B_1/x_0 = -1/2$.
Third case: $B_2 = 2$. Then

$$\alpha_{-1} = i\omega x_0 \frac{(\nu - 1 - B_1/x_0)(\nu - i\eta)}{2(\nu + 1/2)} \qquad (29)$$

and Eq. (18) can be written as

$$\sum_{n=1}^{\infty} S_n + (\alpha_0 b_1 + \beta_0 b_0) F(-\nu, \nu+1, 2+B_1/x_0; y) + \alpha_{-1} b_0 F(1-\nu, \nu, 2+B_1/x_0; y) = 0. (30)$$

This equation is satisfied by $\underline{\nu = 0}$, provided that

$$\begin{cases} \alpha_0 b_1 + (\beta_0 + \alpha_{-1}) b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1). \end{cases} \qquad (31)$$

The characteristic equation has the same form as Eq. (28). On the other hand, if the ratio B_1/x_0 is fixed and such that $B_1/x_0 \neq -3/2$, Eq. (30) is also satisfied by $\underline{\nu = 1 + B_1/x_0}$ ($\alpha_{-1} = 0$) and

$$\begin{cases} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0 \ (n \geq 1). \end{cases} \tag{32}$$

The characteristic equation now is formally identical to Eq. (21).

Therefore, we can conclude that the restriction of n to $n \geq 0$ always admits $\nu = 0$, but it must be noted that we have three different expressions for the characteristic equation: Eq. (21) if $B_2 \neq 1$ or 2, Eq. (26) if $B_2 = 1$ and Eq. (28) if $B_2 = 2$. For $\nu = 0$, U_1 represents one of the Fackerell-Crossman solutions [6] to an "angular" wave equation with free parameters only in the first case. The other two cases provide solutions to a GSWE with $B_2 = 1$ and $B_2 = 2$. As far as we are aware, expansions in terms of hypergeometric functions for these cases have not been discussed before in the literature. We have also found that another value for ν is possible only under special conditions as, for example, in the case of the Whittaker-Hill equation, since it belongs to the second case discussed above and corresponds to $\nu = 0$ and $\nu = 1/2$.

We will not repeat the analysis for the solution U_2, since it is similar to the preceding one, but taking into account the Eqs. (11-14) and (16). This means that we have to replace B_2 for B_2'. In particular, the WHE without free parameters will belong to the case $B_2' = 2$, due to Eqs. (4) and (11).

2.2 Solutions to Whitaker-Hill Equation and Arscott's Case

Now, in the first place we will show that, for the WHE deprived of free parameters, the solutions U_1 and U_2 reduce to two expansions in sines and cosines series; in the second place, using the results of Section 2.1, we will show that these expansions will formally provide the four Arscott's solutions for a WHE with free parameters. To obtain this result, we use Eqs. (3, 4, 6, 11) to get

$$y = \sin^2(u), \quad B_2' = 2, \quad B_3' = B_3 + 1/4. \tag{33}$$

Then, the hypergeometric functions in U_1 and U_2 can be written in terms of sine and cosine [7] according to

$$F(-n-\nu, n+\nu, 1/2; \sin^2 u) = \cos[(2n+2\nu)u], \tag{34}$$

$$F(-n-\nu', n+\nu'+1, 3/2; \sin^2 u) = \frac{\sin[(2n+2\nu'+1)u]}{(2n+2\nu'+1)\sin(u)} \tag{35}$$

Now, returning to Eqs. (5, 12), we get, up to a multiplicative constant,

$$U_1 = e^{i\omega\cos(2u)} \sum_{n=-\infty}^{\infty} b_n \cos[(2n+2\nu)u], \tag{36}$$

$$U_2 = e^{i\omega\cos(2u)} \sum_{n=-\infty}^{\infty} c_n \sin[(2n+2\nu'+1)], \tag{37}$$

$$c_n := b_n'/(2n+2\nu+1). \tag{38}$$

The recurrence relations to the coefficients b_n and c_n are analogous to Eqs. (7) and (13), that is,

$$\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \tag{39}$$
$$\overline{\alpha}_n c_{n+1} + \overline{\beta}_n c_n + \overline{\gamma}_n c_{n-1} = 0, \tag{40}$$

where now

$$\begin{aligned} \alpha_n &= \frac{1}{2}[\eta + i(n+\nu+1/2)]\omega x_0, \\ \beta_m &= -B_3 - (n+\nu)^2 - \eta\omega x_0, \\ \gamma_m &= \frac{1}{2}[\eta - i(n+\nu-1/2)]\omega x_0, \end{aligned} \tag{41}$$

and

$$\begin{aligned} \overline{\alpha}_n &= \frac{1}{2}[\eta + i(n+\nu^{'}+1)]\omega x_0, \\ \overline{\beta}_n &= -B_3 - \eta\omega x_0 - (n+\nu^{'}+1/2)^2, \\ \overline{\gamma}_n &= \frac{1}{2}[\eta - i(n+\nu^{'})]\omega x_0. \end{aligned} \tag{42}$$

If we suppose that there is no free parameter in the WHE, ν and $\nu^{'}$ must be determined from the characteristic equations and we have $-\infty < n < \infty$. Note, however, that we can take $\nu' = \mu - 1/2$ and then the characteristic equations for ν and μ become formally identical, since Eq. (42) and (43) also do.

The solutions we have just found are the generalization of Arscott's solutions for no free parameters in the Whittaker-Hill equation. Moreover they have the same form as the solutions in series of trigonometric functions for the Mathieu equation for the case in which there are no free parameters in it [4]. On the other hand, when we suppose that $n \geq 0$, as in Section 2.1, we find that ν and $\nu^{'}$ have both the values 0 and 1/2 and this provides the four Arscott solutions for the WHE – analogous to the four solutions for the Mathieu equation with free parameters. In effect, for U_1, that follows when we put $B_1 = -x_0/2$ in the second case discussed in Section 2.1. For U_2 we have not derived explicitly the results in Section 2.1, but they are obtained easily by noting that the recurrence relations (43) stem from

$$\left[\sum_{m=n-1}^{\infty} \overline{\alpha}_m c_{m+1} + \sum_{m=n}^{\infty} \overline{\beta}_m c_m + \sum_{m=n+1}^{\infty} \overline{\gamma}_m c_{m-1}\right](m+\nu^{'}+\frac{1}{2})\sin[(2m+2\nu^{'}+1)u] = 0, \tag{43}$$

when we consider $-\infty < n < \infty$. Then, for $n \geq 0$, we have

$$\sum_{m=1}^{\infty}\left(\overline{\alpha}_m c_{m+1} + \overline{\beta}_m c_m + \overline{\gamma}_m c_{m-1}\right)(m+\nu^{'}+\frac{1}{2})\sin[(2m+2\nu^{'}+1)u] +$$
$$(\nu^{'}+\frac{1}{2})(\overline{\alpha}_0 c_1 + \overline{\beta}_0 c_0)\sin[(2\nu^{'}+1)u] + (\nu^{'}-\frac{1}{2})\overline{\alpha}_{-1} c_0 \sin[(2\nu^{'}-1)u] = 0 \tag{44}$$

To satisfy this equation we have again to take $\nu' = 0$ and $\nu' = 1/2$. For $\underline{\nu' = 0}$, we see that

$$\begin{cases} \overline{\alpha}_0 c_1 + (\overline{\beta}_0 + \overline{\alpha}_{-1}) c_0 = 0, \\ \overline{\alpha}_n c_{n+1} + \overline{\beta}_n c_n + \overline{\gamma}_n c_{n-1} = 0, \ (n \geq 1) \end{cases} \tag{45}$$

with characteristic equation given by

$$\overline{\beta}_0 + \overline{\alpha}_{-1} = \frac{\overline{\alpha}_0 \overline{\gamma}_1}{\overline{\beta}_1 -} \frac{\overline{\alpha}_1 \overline{\gamma}_2}{\overline{\beta}_2 -} \frac{\overline{\alpha}_2 \overline{\gamma}_3}{\overline{\beta}_3 -} ..., \tag{46}$$

whereas, for $\underline{\nu' = 1/2}$, we have

$$\begin{cases} \overline{\alpha}_0 c_1 + \overline{\beta}_0 c_0 = 0, \\ \overline{\alpha}_n c_{n+1} + \overline{\beta}_n c_n + \overline{\gamma}_n c_{n-1} = 0, \ (n \geq 1) \end{cases} \tag{47}$$

with characteristic equation

$$\overline{\beta}_0 = \frac{\overline{\alpha}_0 \overline{\gamma}_1}{\overline{\beta}_1 -} \frac{\overline{\alpha}_1 \overline{\gamma}_2}{\overline{\beta}_2 -} \frac{\overline{\alpha}_2 \overline{\gamma}_3}{\overline{\beta}_3 -} ..., \tag{48}$$

Observe that these characteristic equations are formally identical with those of third case discussed in Section 2.1, as already mentioned at the end of that section.

3 Conclusions

First we have shown that the WHE is in fact a particular case of GSWE. Thus the representation for the solutions of the latter may be used for the former, even when there is no free parameter in the WHE. A peculiarity of the expansions in series of Gauss hypergeometric functions is the fact that, in the case of the WHE, they provide two expansions in series of sine and cosine similar to the Arscott solutions, but containing a phase parameter and with the summation index n running from $-\infty$ to $-\infty$. These solutions are also similar to the expansions in sine and cosine series for the Mathieu equation without free parameter. When n is restricted to $n \geq 0$, we recover the four Arscott's solutions to the WHE with free parameters, since such a restriction implies that the phase parameters assume the values 0 or 1/2.

The second and third case discussed in Section 2.1 are solutions to the special cases ($B_2 = 1$ and $B_2 = 2$, respectively) of an "angular" GSWE. They represent new solutions, not discussed in Ref. 5, since the the characteristic equations are different from those of the first case.

Finally we note that a WHE without free parameters is not a useless hypothesis. In effect, the time dependence of a massive scalar test field minimally and conformally coupled to the gravity of a Friedmann-Robertson- Walker (FRW) spacetime filled with dust is determined by a GSWE without free parameters [5]. If we examine the case of conformal coupling we see that we have a WHE. Thus, in the light of Section 2.2, it is not surprising that the hypergeometric functions (case of positive spatial curvature) have reduced to a sine or a cosine. On the other hand, Villalba and Percoco [8] have shown that the time dependence of a Dirac test field coupled to the gravity of FRW spacetimes filled with radiation is governed by a WHE as well. However, they did not notice that the imposition of regularity conditions on the spatial part of the spinor leads to fixed values for the separation constant which remains arbitrary in their work. Therefore, there is no free parameter and so the Arscott solutions do not hold for that case.

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