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ON BOUNDED SETS OF HOLOMORPHIC MAPPINGS

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1. INTRODUCTION

Let E and F be complex locally convex spaces, $U \subset E$ be open and non-void, and $\mathcal{H}(U;F)$ be the vector space of all holomorphic mappings from U into F. Motivated by uniform convergence of mappings, three natural topologies $\mathcal{T}_0$, $\mathcal{T}_\omega$ and $\mathcal{T}_\delta$ have been used on $\mathcal{H}(U;F)$; see references 5, 6, 7, and 8. Let $\mathcal{B}_i(U;F)$ be the collection of all $\mathcal{T}_i$-bounded subsets of $\mathcal{H}(U;F)$, where $i = 0$, ω, δ; and $\mathcal{B}_a(U;F)$ be the collection of all amply bounded subsets of $\mathcal{H}(U;F)$. Then $\mathcal{T}_0 \subset \mathcal{T}_\omega \subset \mathcal{T}_\delta$ and $\mathcal{B}_0(U;F) \supset \mathcal{B}_\omega(U;F) \supset$

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$\mathcal{B}_\delta(U;F) \supset \mathcal{B}_a(U;F)$. We are interested in the study of the following properties of E, where i, j = 0, ω, δ, a:

$A_{ij}(U;F)$: We have $\mathcal{B}_i(U;F) = \mathcal{B}_j(U;F)$.

$B_{ij}(U;F)$: A seminorm on $\mathcal{H}(U;F)$ is bounded on $\mathcal{B}_i(U;F)$ if and only if it is bounded on $\mathcal{B}_j(U;F)$.

$C(U;F)$: There is some locally convex topology on $\mathcal{H}(U;F)$ whose collection of bounded sets is $\mathcal{B}_a(U;F)$.

We can rephrase $B_{ij}(U;F)$ by saying that the bornological topologies associated to $\mathcal{B}_i(U;F)$ and $\mathcal{B}_j(U;F)$ are equal.

If one of the above properties is true for every F, or for every F and every U, we omit F, or F and U, from the notation and write $A_{ij}(U)$, $B_{ij}(U)$, $C(U)$, or A_{ij}, B_{ij}, C, respectively.

There are some routine facts about these properties, of which we point out the following: $A_{ii}(U;F)$ and $B_{ii}(U;F)$ are always true; $A_{ij}(U;F) = A_{ji}(U;F)$ and $B_{ij}(U;F) = B_{ji}(U;F)$; $A_{ij}(U;F) = B_{ij}(U;F)$ if $i \neq a$ and $j \neq a$; $A_{ia}(U;F) = B_{ia}(U;F) \cap C(U;F)$ if $i \neq a$.

This article is a continuation of ref. 2. Its main results give instances in which A_{oa} (Propositions 2, 3, 5, 6), $A_{\omega a}$ (Proposition 1) and $B_{oa}(E)$ (Proposition 4) hold. Some examples are given. We point out the relationship of E having A_{oa} to E being infrabarreled; it is phrased in such a way that E should be called holomorphically infrabarreled when E has A_{oa}. On the other hand, we also define E being holomorphically barreled, and point out some current examples having such a property.

For the notation and terminology we refer mostly to ref. 6, 8, 9; see also 10, 11.

2. THE WEAK CASE

PROPOSITION 1. A weak complex locally convex space E has $A_{\omega a}$.

PROOF. We first prove that E has $A_{\omega a}(E;F)$, where F is a complex normed space. Let I be an algebraic basis of the topological dual space E'. If $J \subset I$, we have the linear mapping

$$\pi_J : x \in E \mapsto (\phi(x))_{\phi \in J} \in C^J .$$

If $f \in \mathcal{H}(E;F)$, there are $J \subset I$ finite and $a \in \mathcal{H}(C^J;F)$ such that $f = a \circ \pi_J$. If a is effective, that is it depends effectively on all variables indexed by J (as we may assume by decreasing J), then J and a are determined by f; we then denote them by J_f and a_f. If $\alpha \in N^{(I)}$, we define $D^\alpha f(0) \in F$ as being equal to $D^\alpha a_f(0)$ if $s(\alpha) \subset J_f$, and to 0 if $s(\alpha) \not\subset J_f$, where $N^{(I)}$ is the additive group formed by every $\alpha : I \to N$ with finite support $s(\alpha)$, and $D^\alpha a_f$ is the α-th partial derivative of a_f. Let $b: N^{(I)} \to R$ be such that $b \geqslant 0$, $b(0) \neq 0$, and , for all finite $J \subset I$, the set formed by every α belonging to the support of b for which $s(\alpha) \subset J$ is finite, we set

$$p_b(f) = \sup\{b(\alpha) \cdot \| D^\alpha f(0) \| \; ; \; \alpha \in N^{(I)}\}.$$

This supremum is finite since, for every α, either $b(\alpha) = 0$, or $s(\alpha) \not\subset J_f$, or else $b(\alpha) \neq 0$ and $s(\alpha) \subset J_f$; in the first two cases $b(\alpha) \cdot \| D^\alpha f(0) \| = 0$, whereas the third case occurs only finitely many times. $p_b : f \mapsto p_b(f)$ is a seminorm on $\mathcal{H}(E;F)$. It is ported by $\{0\}$. In fact, let $\epsilon > 0$, $J \subset I$ be finite, and $V_{\epsilon J}$ be formed by every $x \in E$ such that $|\phi(x)| < \varepsilon$ for any $\phi \in J$. The family $(V_{\epsilon J})$ is a base of neighborhoods of 0 in E. Set

$$c_{\epsilon J} = \sup\{\alpha! \; b(\alpha) \varepsilon^{-|\alpha|} ; \; \alpha \in N^{(I)}, \; s(\alpha) \subset J\} ,$$

where

$$\alpha! = \Pi_{\phi\in I}\ \alpha(\phi)!,\ |\alpha| = \Sigma_{\phi\in I}\ \alpha(\phi);$$

clearly $0 < c_{\epsilon J} < +\infty$. The estimate

$$p_b(f) \leqslant c_{\epsilon J}\cdot \sup\{||\ f(x)||;\ x \in V_{\epsilon J}\}$$

holds true. Indeed, if $J_f \not\subset J$, then f is unbounded on $V_{\epsilon J}$ by Liouville theorem . If $J_f \subset J$, we distinguish two alternatives. By assuming $s(\alpha) \subset J_f$, hence $s(\alpha) \subset J$, Cauchy inequality gives us

$$||D^\alpha\ a_f(0)|| \leqslant \alpha!\ \epsilon^{-|\alpha|}\ \sup\{||a_f(z)||;\ z \in Q_{\epsilon J}\},$$

where $Q_{\epsilon J}$ is formed by every $z = (z_\phi)_{\phi\in J_f} \in C^{J_f}$ such that $|z_\phi| < \epsilon$ for any $\phi \in J_f$; hence

$$||D^\alpha f(0)|| \leqslant \alpha\ !\ \epsilon^{-|\alpha|}\ \sup\{||f(x)||;\ x \in V_{\epsilon J}\}\ .$$

By assuming $s(\alpha) \not\subset J_f$, then $D^\alpha f(0) = 0$. From these cases we conclude that the above estimate is true. Hence p_b is ported by $\{0\}$.

Let now $\mathcal{X} \subset \mathcal{H}(E;F)$ be $\mathcal{T}_\omega$- bounded. Then $\mathcal{X}$ depends on some finite $K \subset I$, that is $J_f \subset K$ for every $f \in \mathcal{X}$. Indeed, if $\mathcal{X}$ fails to do so, we argue inductively to find $f_m \in \mathcal{X}\,(m\in N)$ such that $J_{f_0} \neq \Phi$ and $J_{f_m} \not\subset K_m \equiv$ $\equiv J_{f_0} \cup \ldots \cup J_{f_{m-1}} (m\geqslant 1)$. Choose $\phi_0 \in J_{f_0}$ and $\phi_m \in J_{f_m}$, $\phi_m \notin K_m (m\geqslant 1)$; they are pairwise different. Since a_{f_m} is effective, choose $\alpha_m \in N^{(I)}$ so that $\phi_m \in s(\alpha_m) \subset J_{f_m}$ and $D^{\alpha_m} a_{f_m}(0) \neq 0$, that is $D^{\alpha_m} f_m(0) \neq 0$ $(m \in N)$. They are pairwise different since their supports are pairwise different; and, for all finite $J \subset I$, the set formed by every $m \in N$ for which $s(\alpha_m) \subset J$ is finite. Define b by letting

$$b(\alpha_m) = m\cdot ||D^{\alpha_m}\ f_m(0)||^{-1} (m \in N),$$

$b(0) = 1$ and $b(\alpha) = 0$ for every remaining $\alpha \in N^{(I)}$. Then $p_b(f_m) \geqslant m (m\in N)$, against the fact that $\mathfrak{X}$ is $\mathcal{C}_\omega$-bounded. There results that $\mathfrak{X}$ depends on some finite $K \subset I$. Since $\mathfrak{X}$ is $\mathcal{C}_\omega$-bounded, hence $\mathcal{C}_0$ - bounded, and C^K is locally compact, $\mathfrak{X}$ is locally bounded. Thus $A_{\omega a}(E;F)$ holds.

If $\xi \varepsilon E$, $\epsilon > 0$, $H \subset I$ is finite and $U \subset E$ is the band formed by every $x \varepsilon E$ such that $|\phi(x) - \phi(\xi)| < \epsilon$ for every $\phi \varepsilon H$, then $A_{\omega a}(U;F)$ holds too; the above proof applies except for minor notational changes having to do with the domain of definition of a_f for $f \varepsilon \mathcal{H}(U;F)$ and with replacing 0 by ξ. Since a non-void open $U \subset E$ is a union of bands $U_\lambda (\lambda \varepsilon \Lambda)$, we conclude that $A_{\omega a}(U;F)$ is true. Finally $A_{\omega a}(U;F)$ holds for every complex locally convex F once it does for every complex normed F. This proves $A_{\omega a}$. Q.E.D.

3. THE CARTESIAN PRODUCT CASE

PROPOSITION 2. A cartesian product $E = \Pi_{i\varepsilon I} E_i$ of semimetrizable complex locally convex spaces $E_i (i \varepsilon I)$ has A_{oa}.

PROOF. It is known that a semimetrizable complex locally convex space has A_{oa}. We first prove that E has $A_{oa}(E;F)$, where F is a complex normed space. Let $\mathfrak{X} \subset \mathcal{H}(E;F)$ be $\mathcal{C}_0$-bounded. Then $\mathfrak{X}$ depends on some finite $K \subset I$, that is $\mathfrak{X}$ is contained in the image of the mapping $\mathcal{H}(\Pi_{i\varepsilon K} E_i;F) \to \mathcal{H}(E;F)$ which results from composition by the projection $E \to \Pi_{i\varepsilon K} E_i$. To prove this it is enough to show that every sequence $f_m \varepsilon \mathfrak{X} (m \varepsilon \mathbb{N})$ depends on some finite subset of I. Since each $f \varepsilon \mathcal{H}(E;F)$ depends on a finite subset of I, there is a denumerable $J \subset I$ such that every f_m depends on a finite subset of J. Since $\Pi_{i\varepsilon J} E_i$ is semimetrizable, the sequence (f_m) is locally bounded. We can find a finite subset $K \subset J$ and a neighborhood V_i of 0 in E_i for every $i \in K$ such that, letting $V_i = E_i$ for the remaining $i \in I$, and $V = \Pi_{i\epsilon I} V_i$, we have

$$\sup\{||f_m(x)||; m \in N, x \in V\} < +\infty.$$

Since every f_m is bounded on V, it follows from Liouville theorem and uniqueness of holomorphic continuation that every f_m depends on K; hence (f_m) depends on K. Therefore $\mathfrak{X}$ depends on some finite $K \subset I$. Since $\mathfrak{X}$ is $\mathscr{C}_o$-bounded and $\Pi_{i\epsilon K}E_i$ is semimetrizable, $\mathfrak{X}$ is locally bounded. Thus $A_{oa}(E;F)$ holds.

If U is the band $\Pi_{i\epsilon I} U_i$, where $U_i \subset E_i$ is open and non-void for every $i \in I$ and $U_i = E_i$ except for finitely many i's, then $A_{oa}(U;F)$ holds too; the above proof applies with minor notational changes. Since a non-void open $U \subset E$ is a union of bands U_λ $(\lambda \in \Lambda)$, we conclude that $A_{oa}(U;F)$ is true. Finally, $A_{oa}(U;F)$ holds for every complex locally convex F once it does for every complex normed F. This proves A_{oa}. Q.E.D.

4. THE DENUMERABLE DIRECT SUM CASE

LEMMA 1. Let $E_m (m \in N)$ and F be complex locally convex spaces, $E = \Sigma_{m\epsilon N} E_m$ be the topological direct sum, $U \subset E$ be open and non-void, and $S_m = E_o+\ldots+E_m+ + 0 + \ldots$ $(m \in N)$. Then $\mathfrak{X} \subset \mathscr{H}(U;F)$ is amply bounded on U if, for every $\xi \in U$, there is a neighborhood V of ξ in U such that $\mathfrak{X}$ is bounded on $V \cap S_m$ for every $m \in N$.

PROOF. It is enough to prove the lemma when F is seminormed. We may assume that $0 \epsilon U$, and it suffices to show that $\mathfrak{X}$ is locally bounded at 0 on U.

There is a neighborhood V_m of 0 in $E_m (m \in N)$ such that, letting $V = \Sigma_{m \in N} V_m$, then $V \subset U$ and $\mathfrak{X}$ is bounded on $V \cap S_m$ for every $m \in N$. Now, $\mathfrak{X}$ is bounded on $V \cap S_0$. Thus the set of mappings $t \in V_0 \mapsto f(t, 0, \ldots) \in F$, for all $f \in \mathfrak{X}$, is bounded on V_0, hence equicontinuous at 0 on V_0. Set $W_0 = V_0$, and choose $M \in R$ so that

$$\sup\{||f(x_0, 0, \ldots)||;\ f \in \mathfrak{X},\ x_0 \in W_0\} < M . \tag{1}$$

Assume that, for some $m \in N$, we have defined a neighborhood $W_k \subset V_k$ of 0 in E_k for every $k = 0, \ldots, m$ so that

$$\sup\{||f(x_0,\ldots,x_m,0,\ldots)||;\ f \in \mathfrak{X},\ x_0 \in W_0,\ldots,x_m \in W_m\} < M; \tag{2}$$

this is indeed the case for $m = 0$, by (1). Now, $\mathfrak{X}$ is bounded on $V \cap S_{m+1}$. Thus the set of mappings $t \in V_{m+1} \mapsto f(x_0,\ldots, x_m, t, 0,\ldots) \in F$, for all $f \in \mathfrak{X}$, $x_0 \in W_0,\ldots, x_m \in W_m$, is bounded on V_{m+1}, hence equicontinuous at 0 on V_{m+1}. By (2), choose a neighborhood $W_{m+1} \subset V_{m+1}$ of 0 in E_{m+1} so that

$$\sup\{||f(x_0,\ldots,x_{m+1},0,\ldots)||;\ f \in \mathfrak{X},\ x_0 \in W_0,\ldots,x_{m+1} \in W_{m+1}\} < M$$

In this way, we get a sequence $W_m \subset V_m$ of neighborhoods of 0 in $E_m (m \in N)$ such that, letting $W = \Sigma_{m \in N} W_m \subset V$, we will have $||f(x)|| < M$ for every $f \in \mathfrak{X}$ and $x \in W$. Hence $\mathfrak{X}$ is locally bounded at 0 on U. Q.E.D.

A separable, or reflexive, complex normed space satisfies the following "boundedness hypothesis", see reference 3:

BH: A complex normed space either is finite dimensional, or else there is an entire complex function on it which is unbounded on some of its bounded subsets.

Actually, it is conjectured that BH holds true for every complex normed space.

PROPOSITION 3. Let $E = \Sigma_{m\in N} E_m$ be a topological direct sum of complex normed spaces $E_m (m\in N)$. For E to have A_{oa} it is sufficient that all E_m be finite dimensional, or else that all $E_m = 0$ except for finite many m's. Conversely, this condition is necessary for E to have $A_{oa}(E;C)$, provided all E_m satisfy the boundedness hypothesis.

PROOF. Set $S_m = E_o + \ldots + E_m + 0 + \ldots \subset E$ $(m \in N)$. Let us prove sufficiency. Assume first that all E_m are finite dimensional. Let $U \subset E$ be open and non-void, and F be a complex locally convex space. Each $\xi \in U$ has a neighborhood V in U such that $V \cap S_m$ is compact for every $m \in N$. If $\mathfrak{X} \subset \mathcal{H}(U;F)$ is $\mathcal{C}_o$-bounded, $\mathfrak{X}$ is bounded on $V \cap S_m$ for every $m \in N$. By Lemma 1, $\mathfrak{X}$ is amply bounded. Hence E has A_{oa}. Next, if all $E_m = 0$ except for finitely many m's, then E is normable, hence it has A_{oa}.

Let us prove necessity. We first remark that, if $f \in \mathcal{H}(E_o; C)$ is bounded on every bounded subset of E_o and $\phi_m \in (E_m)'$ $(m = 1, 2, \ldots)$, we have $f^* \in \mathcal{H}(E;C)$ defined by

$$f^*(x_o,\ldots,x_m,\ldots) = \Sigma_{m=1}^{\infty} f(mx_o)\phi_m(x_m) \text{ for } x_m \in E_m(m\in N).$$

In fact, let $g_k \in \mathcal{H}(E;C)$ be given by

$$g_k(x_o,\ldots,x_m,\ldots) = \Sigma_{m=1}^{k} f(mx_o)\ \phi_m(x_m) \quad \text{for } k = 1, 2, \ldots$$

Set $s(r) = \sup\{|f(x_o)|;\ ||x_o|| < r\}$ for $r\in R$, $r>0$. If $\rho = (\rho_1,\ldots\rho_m,\ldots)$, where $\rho_m \in R$, $\rho_m > 0$, is such that

$$\Sigma_{m=1}^{\infty} s(mr)\rho_m < +\infty,$$

define $V_{r\rho}$ as the open subset of all $(x_m)_{m\in N} \in E$ for which $||x_o|| < r$, $|\phi_m(x_m)| < \rho_m (m=1,2,\ldots)$. Then $g_k \to f^*$ as $k \to \infty$ uniformly on $V_{r\rho}$. Since these $V_{r\rho}$ cover E, then f^* is entire

Let E have $A_{oa}(E,C)$. We discard the case in which all $E_m = 0$ except for finitely many m's; without loss of generality, we may assume that all $E_m \neq 0$. Choose $\phi_m \varepsilon (E_m)'$, $\phi_m \neq 0$ $(m \varepsilon N)$. Let us prove that E_o is finite dimensional if it satisfeis BH. Assume that E_o is infinite dimensional. There is $f \varepsilon \mathcal{H}(E_o; C)$ which is unbounded on the closed unit ball B of E_o. Let f_n be the n-th partial sum of the Taylor series of f at 0 $(n \varepsilon N)$. Then $f_n \to f$ on E for $\mathcal{C}_o$ as $n \to \infty$, but $\{f_n; n \varepsilon N\}$ is not bounded on B. We may consider $f_n^{\star} \varepsilon \mathcal{H}(E;C)$ since f_n is a continuous on E_o, hence it is bounded on every bounded subset of E_o. Clearly $\{f_n^{\star}; n \varepsilon N\}$ is $\mathcal{C}_o$-bounded on E, since $\{f_n; n \varepsilon N\}$ is $\mathcal{C}_o$-bounded on E_o and every compact subset of E is contained in some S_m. However, $\{f_n^{\star}; n \varepsilon N\}$ is unbounded on every neighborhood of 0 in E. In fact, for every $\epsilon_m > 0$ $(m \varepsilon N)$, if we choose $m = 1, 2, \ldots$ so that $m\varepsilon_o \geqslant 1$, we see that

$$f_n^{\star}(x_o, 0, \ldots, Q, x_m, 0, \ldots) = f_n(mx_o)\phi_m(x_m)$$

is unbounded for $x_o \varepsilon E_o, ||x_o|| \leqslant \varepsilon_o$, $x_m \varepsilon E_m$, $|\phi_m(x_m)| \leqslant \epsilon_m$, $n \varepsilon N$. Thus E would not have $A_{oa}(E;C)$, against the assumption. It follows that E_o is finite dimensional; and likewise for all E_m. Q.E.D.

DEFINITION 1. A non-void open subset U of a complex locally convex space E is "Runge subset" of E if, for every complex locally convex space F and every $f \varepsilon \mathcal{H}(U;F)$, there are $f_m \varepsilon \mathcal{P}(E;F)$ and $\lambda_m \varepsilon C$ $(m \varepsilon N)$ such that $\lambda_m \to \infty$ as $m \to \infty$ and the sequence $\lambda_m(f - f_m|U)$ $(m \varepsilon N)$ is amply bounded on U.

LEMMA 2. Let U be a Runge subset of a complex locally convex space E which is a topological direct sum of its vector subspaces E_1 and E_2, and $\pi: E \to E_1$ the corresponding projection. If $U \cap E_1 = \pi(U)$, for every complex locally convex space F and every $f \in \mathcal{H}(U;F)$ such that $f|(U \cap E_1) = 0$, there are $f_m \in \mathcal{P}(E;F)$ and $\lambda_m \in C$ such that $f_m|E_1 = 0$ $(m \in N)$, $\lambda_m \to \infty$ as $m \to \infty$, and the sequence $\lambda_m(f - f_m|U)(m \in N)$ is amply bounded on U.

PROOF. Choose $f_m(m \in N)$ according to Definition 1. It suffices to replace f_m by $g_m = f_m - f_m \circ \pi \in \mathcal{P}(E;F)$, since $g_m|E_1 = 0$, and the sequence $(\lambda_m f_m) \circ \pi$ $(m \in N)$ is amply bounded on $\pi^{-1}(U \cap E_1)$, hence on U. Q.E.D.

REMARK 1. If U is ξ-balanced, where $\xi \in U$, then U is a Runge subset of E. In fact, if $f \in \mathcal{H}(U;F)$ and $f_m \in \mathcal{P}(E;F)$ is the m-th partial sum of the Taylor series of f at $\xi(m \in N)$, for every $t \in U$ and every continuous seminorm β on F there are $c \geqslant 0$, $0 < \Theta < 1$, and a neighborhood V of t in U such that $\beta[f(x) - f_m(x)] \leqslant c\Theta^m$ for $x \in V$, $m \in N$. If $\lambda_m \in C(m \in N)$ and $\lim.\sup|\lambda_m|^{1/m} \leqslant 1$ as $m \to \infty$, the sequence $\lambda_m(f - f_m|U)(m \in N)$ is then amply bounded. There remains to impose the additional condition $\lambda_m \to \infty$ as $m \to \infty$. In this case, if E_1 is a vector subspace of E and $f|(U \cap E_1) = 0$ then $f_m|E_1 = 0$ $(m \in N)$, so that we can discard Lemma 2, provided $\xi \in E_1$.

A subset X of a direct sum $E = \Sigma_{i \in I} E_i$ of vector spaces $E_i(i \in I)$ is "cylindrical" if X is the inverse image of its projection in $\Sigma_{i \in J} E_i$ for some finite $J \subset I$.

PROPOSITION 4. If $E = \Sigma_{m \in N} E_m$ is a topological direct sum of complex seminormed spaces $E_m(m \in N)$ and U is a cylindrical Runge subset of E, then E has $B_{oa}(U)$.

PROOF. We may assume that U is the inverse image in E of its projection in E_o; then U is the inverse image in E of its projection in $E_o \times \ldots \times E_m$ $(m \in N)$. Let F be a complex locally convex space. Take a seminorm p on

$\mathcal{H}(U;F)$ which is bounded on its amply bounded subsets. Set $S_m = E_0 + \ldots + E_m + 0 + \ldots \subset E$ $(m \in N)$. Let H_m be the vector subspace of all $f \in \mathcal{H}(U;F)$ such that $f|(U \cap S_m) = 0$ $(m \in N)$.

If $f \in H_m$, $\epsilon > 0$, there is $g \in \mathcal{P}(E;F)$ such that $g|S_m = 0$ and $p(f-g|U) < \epsilon$, hence $|p(f) - p(g|U)| < \epsilon$. In fact, by Lemma 2 there are $f_n \in \mathcal{P}(E;F)$ and $\lambda_n \in C$ such that $f_n|S_m = 0$ $(n \in N)$, $\lambda_n \to \infty$ as $n \to \infty$, and the sequence $\lambda_n(f-f_n|U)$ $(n \in N)$ is amply bounded. Hence $p\left[\lambda_n(f-f_n|U)\right]$ $(n \in N)$ is bounded. Then $p(f-f_n|U) \to 0$ as $n \to \infty$. It suffices to take $g = f_n$ for n large enough.

We claim that p vanishes on some H_m. Let us argue by contradiction. Assume that, for every $m \in N$, there is $f_m \in H_m$ such that $p(f_m) \neq 0$; we may assume that $p(f_m) > m$ and that f_m is the restriction to U of some element of $\mathcal{P}(E;F)$. Let $\mathcal{F} = \{f_m;\ m \in N\}$. Then $\mathcal{F}|(U \cap S_n)$ is finite since it consists of at most n+1 elements, for every $n \in N$. As an element of $\mathcal{P}(E;F)$ is bounded on every bounded subset of E, Lemma 1 implies that $\mathcal{F}$ is amply bounded on U. Thus p is bounded on $\mathcal{F}$, a contradiction.

Let m be such that p vanishes on H_m. To every $f \in \mathcal{H}(U;F)$ we associate $f_m \in \mathcal{H}(U;F)$ defined by $f_m(x_0,\ldots,x_m, x_{m+1},\ldots) = f(x_0,\ldots,x_m, 0,\ldots)$ if $(x_0,\ldots, x_m, x_{m+1},\ldots) \in U$. Then $f-f_m \in H_m$, $p(f-f_m) = 0$ and $p(f) = p(f_m)$. Let $\mathcal{X} \subset \mathcal{H}(U;F)$ be $\mathcal{T}_0$-bounded. Call $\mathcal{X}_m \subset \mathcal{H}(U;F)$ the image of $\mathcal{X}$ by the mapping $f \to f_m$. Since $\mathcal{X}|(U \cap S_m)$ is $\mathcal{T}_0$-bounded on $U \cap S_m$, it is amply bounded there because $E_0 \times \ldots \times E_m$ is seminormable. Hence $\mathcal{X}_m$ is amply bounded on U. It follows that p is bounded on $\mathcal{X}_m$, therefore on $\mathcal{X}$. Q.E.D.

Concerning Proposition 4, Dineen 4 has shown that E has $B_{o\delta}(E;C)$.

We conjecture that E in Proposition 4 has B_{oa}.

5. THE SILVA CASE

Let $E_m (m \in \mathbb{N})$ be complex locally convex spaces, E a complex vector space, $\rho_m: E_m \to E$ a linear mapping and $\sigma_m: E_m \to E_{m+1}$ a compact linear mapping such that $\rho_m = \rho_{m+1} \circ \sigma_m (m \in N)$. Assume that $E = \bigcup_{m \in N} \rho_m(E_m)$ and endow E with the inductive limit topology. Let $U \subset E$ be open, set $U_m = \rho_m^{-1}(U)$ $(m \in \mathbb{N})$, and assume U_0 non-void.

LEMMA 3. If F is a complex locally convex space, then $\mathcal{X} \subset \mathcal{H}(U;F)$ is amply bounded on U if and only if $\mathcal{X}_m = \mathcal{X} \circ \rho_m \subset \mathcal{H}(U_m; F)$ is amply bounded on U_m for every $m \in N$.

PROOF. Necessity being clear, let us prove sufficiency. It is enough to treat F as being seminormed. We may assume that $0 \in U$, and it suffices to show that $\mathcal{X}$ is locally bounded at 0 on U. Since $\mathcal{X}_0$ is locally bounded at 0 on U_0, choose a convex neighborhood V_0 of 0 in U_0 such that $\sigma_0(V_0)$ has a compact closure contained in U_1, hence $\rho_0(V_0) \subset U$, and

$$\sup\{||f[\rho_0(x)]||;\ f \in \mathcal{X},\ x \in V_0\} < M \tag{1}$$

for some $M \in R$. Assume that, for some $m \in N$, we have defined a convex neighborhood V_m of 0 in U_m such that $\sigma_m(V_m)$ has a compact closure contained in U_{m+1}, hence $\rho_m(V_m) \subset U$, and

$$\sup\{||f[\rho_m(x)]||;\ f \in \mathcal{X},\ x \in V_m\} < M; \tag{2}$$

this is indeed the case for $m = 0$, by (1). Since $\mathcal{X}_{m+1}$ is locally bounded at the closure of $\sigma_m(V_m)$ on U_{m+1}, hence equicontinuous there, use (2) to choose a convex neighborhood V_{m+1} of $\sigma_m(V_m)$ in U_{m+1} such that $\sigma_{m+1}(V_{m+1})$ has a compact closure contained in U_{m+2}, hence $\rho_{m+1}(V_{m+1}) \subset U$, and

$$\sup\{||f[\rho_{m+1}(x)]||;\ f \in \mathcal{X},\ x \in V_{m+1}\} < M .$$

We also have $\rho_m(V_m) \subset \rho_{m+1}(V_{m+1})$. Proceeding in this way and letting

$$V = \bigcup_{m\in N} \rho_m(V_m) ,$$

we get a neighborhood V of 0 in U such that $||f(x)|| < M$ for every $f \in \mathcal{X}$, and $x \in V$. Hence $\mathcal{X}$ is locally bounded at 0 on U.

Q.E.D.

A given E is said to be a Silva space if its topology may be defined as an inductive limit through suitable sequences (E_m), (ρ_m) and (σ_m) satisfying all the aforementioned conditions.

PROPOSITION 5. A complex Silva space E has A_{oa}.

PROOF. We may assume that E is separated. It is known that the topology of E may be defined as an inductive limit through suitable sequences (E_m), (ρ_m) and (σ_m) satisfying all the aforementioned conditions, where moreover each E_m is a Banach space. Let F be a complex locally convex space and $\mathcal{X} \subset \mathcal{H}(U;F)$ be $\mathcal{T}_o$-bounded, $U \subset E$ being open and non-void; we may assume that U_o is non-void. Since $f \in \mathcal{H}(U;F) \mapsto f \circ \rho_m \in \mathcal{H}(U_m;F)$ is $\mathcal{C}_o$-continuous, then $\mathcal{X} \circ \rho_m$ is $\mathcal{C}_o$-bounded, hence amply bounded for every $m \in N$. By Lemma 3, $\mathcal{X}$ is amply bounded.

Q.E.D.

REMARK 2. Let E be a Silva space. If F is a seminormed space, then $\mathcal{T}_o$ on $\mathcal{H}(U;F)$ is semimetrizable, hence bornological; it then follows from the fact that E has $A_{o\delta}$ (Proposition 5), that $\mathcal{C}_o = \mathcal{T}_\omega = \mathcal{C}_\delta$ on $\mathcal{H}(U;F)$. The same conclusion then results for every locally convex F.

6. THE BAIRE CASE

PROPOSITION 6. A Baire complex locally convex space E has A_{oa}.

PROOF. Let F be a complex seminormed space. We start with two classical remarks.

If X is a non-void Baire space, and $\mathcal{U}$ is a pointwise bounded set of continuous mappings from X into F, there is at least one point of X where $\mathcal{U}$ is locally bounded.

If $p: E \to F$ is an m-homogeneous polynomial ($m \in N$) and $a, b \in E$, then $\sup\{||p(\lambda a+b)||; \lambda \in C, |\lambda| \leqslant 1\} = \sup\{||p(a+\lambda b)||; \lambda \in C, |\lambda| \leqslant 1\}$; in fact by the maximum principle we may replace $|\lambda| \leqslant 1$ by $|\lambda| = 1$, and then equality is clear via $\lambda \to 1/\lambda$, by m-homogeneity. In particular $||p(b)|| \leqslant \sup\{||p(a+\lambda b)||; \lambda \in C, |\lambda| \leqslant 1\}$.

Now, let $\mathfrak{X} \subset \mathcal{H}(U;F)$ be $\mathcal{T}_0$-bounded, $U \subset E$ being open and non-void, and $\xi \in U$. Take a balanced open $V \subset E$ containing 0 such that $\xi + V \subset U$. By Cauchy integral the set

$$\mathcal{U} = \{(m!)^{-1}\, \hat{d}^m f(\xi);\ f \in \mathfrak{X},\ m \in N\}$$

is pointwise bounded on V because $\mathfrak{X}$ is bounded on every compact subset $\{\xi + \lambda x; \lambda \in C, |\lambda| \leqslant 1\}$ of U, where $x \in V$. By the first remark, since V is a Baire space, there is $a \in V$ where $\mathcal{U}$ is locally bounded; let W be a balanced neighborhood of 0 in V such that $a + W \subset V$ and $\mathcal{U}$ is bounded on $a + W$. By the second remark, $\mathcal{U}$ is bounded on W. By the Taylor series at ξ, $\mathfrak{X}$ is bounded on $\xi + \theta W$ if $0 < \theta < 1$. Hence $\mathfrak{X}$ is locally bounded. Thus $A_{oa}(U;F)$ holds for every U and every seminormed F, hence for every locally convex F. This proves A_{oa}. Q.E.D.

Proposition 6 was also found independently by Dineen 4.

7. SOME EXAMPLES

Example 1. Let E be an infinite dimensional complex vector space with the weak topology associated to a norm, and $\mathfrak{X}$ be the closed unit ball in E'. Then $\mathfrak{X}$ is $\mathscr{C}_o$-bounded, but it is not locally bounded ref.1;E does not have $A_{oa}(E;C)$. Actually $\mathfrak{X}$ is not $\mathscr{C}_\omega$-bounded ref.2; E does not have $A_{o\omega}$ (E;C). Q.E.D.

We shall see in Examples 2 and 3 that, if X and Y are complex locally convex spaces having A_{oa}, then E = XxY may fail to have $A_{oa}(E;C)$. However, it is easy to see that, if X has A_{ia} for some i = o, ω, δ and Y is finite dimensional, then E has A_{ia}.

Example 2. Let X be an infinite dimensional normed space satisfying BH (section 4), and $Y = C^{(N)}$ be an infinite denumerable direct sum of C. By sufficiency in Proposition 3, X and Y have A_{oa}. By necessity, there, E = XxY fails to have $A_{oa}(E;C)$. By Proposition 4, E has $B_{oa}(E)$. Now, $B_{oa}(E)$ implies $B_{o\delta}(E) = A_{o\delta}(E)$; hence E has $A_{o\delta}(E)$. Since $A_{oa}(E;C) = A_{\delta a}(E;C) \cap B_{oa}(E;C)$, we see that E fails to have $A_{\delta a}(E;C)$. Therefore $\mathcal{B}_o(E;C) = \mathcal{B}_\omega(E;C) = \mathcal{B}_\delta(E;C) \neq \mathcal{B}_a(E;C)$. Since $A_{oa}(E;C) = B_{oa}(E;C) \cap C(E;C)$, we see that E fails to have C(E;C). Q.E.D.

Example 3. Let $X = C^N$ and $Y = C^{(N)}$ be an infinite denumerable cartesian product and direct sum of C, respectively. By Proposition 2 and 3, X and Y have A_{oa}. Let $f_k \in \mathscr{H}(E;C)$ $(k \in N)$ be defined by $f_k(x,y) = x_k y_k$, where E = XxY, $x = (x_m)_{m\in N} \in X$ and $y = (y_m)_{m\in N} \in Y$. The sequence (f_k) is $\mathscr{C}_o$-bounded. However it is not locally bounded at 0. Hence E does not have $A_{oa}(E;C)$. Actually E does not have $A_{o\omega}(E;C)$ because we shall prove now that (f_k) is not $\mathscr{C}_\omega$-bounded. The linear mapping

$$r : g \in \mathcal{H}(E;C) \mapsto \hat{d}^2 g(0) \in \mathcal{P}(^2E;C)$$

is continuous if $\mathcal{H}(E;C)$ is given its topology $\mathcal{T}_\omega$ and $\mathcal{P}(^2E;C)$ is given its inductive limit topology. We have the natural continuous linear mapping $\mathcal{P}(^2E;C) \to \mathcal{P}(^2X;C) \times \mathcal{P}(^2Y;C) \times \mathcal{L}(X, Y, C)$, these spaces being given their inductive limit topologies. By projection, there results the natural continuous linear mapping

$$s: \mathcal{P}(^2E;C) \to \mathcal{L}(X,Y,C).$$

If $\phi \in \mathcal{L}(X,Y,C)$, then

$$\phi(x,y) = \Sigma_{m,n\in N}\, c_{mn}\, x_m y_n,$$

where $c_{mn} \in C(m, n \in N)$ and there is $\mu \in N$ such that $c_{mn} = 0$ for $m \geq \mu$ and all n. The linear form

$$t: \phi \in \mathcal{L}(X, Y;C) \mapsto \Sigma_{m\in N} m\, c_{mm} \in C$$

is continuous. Thus $u = t \circ s \circ r : \mathcal{H}(E;C) \to C$ is a continuous linear form. Since $u(f_k) = 2k$ $(k \in N)$, we see that (f_k) is not $\mathcal{T}_\omega$-bounded.

The proof of Proposition 2 shows that a cartesian product of complex locally convex spaces has A_{oa} if every denumerable cartesian subproduct has A_{oa}. This is false if we replace "denumerable" by "finite" as the following example shows.

Example 4. If we set $E_0 = C^{(N)}$, $E_m = C(m = 1, 2, \ldots)$, $E = \Pi_{m\in N} E_m$, it follows from Example 3 that E fails to have $A_{oa}(E;C)$, or even $A_{o\omega}(E;C)$, although every $E_0 \times \ldots \times E_m (m\in N)$ has A_{oa} by Proposition 3. Q.E.D.

The following example shows that Proposition 4 breaks down if the $E_m (m\in N)$ are only metrizable.

Example 5. If we set $E_0 = C^N$, $E_m = C(m = 1, 2, \ldots)$, $E = \Sigma_{m\in N} E_m$, it follows from Example 3 that E fails to have $A_{o\omega}(E;C) = B_{o\omega}(E;C)$; hence E fails to have $B_{oa}(E;C)$. Q.E.D.

The following example shows that we cannot drop compactness in Proposition 5; see Example 2.

Example 6. If E_0 is a complex infinite dimensional normed space satisfying BH (section 4), $E_m = C$ $(m = 1, 2, \ldots)$, $E = \Sigma_{m\in N} E_m$, necessity in Proposition 3 shows that E fails to have $A_{oa}(E;C)$. Q.E.D.

To following example shows that we cannot drop denumerability in Proposition 5, as well as in sufficiency in Proposition 3.

Example 7. If I is a set whose power is at least equal to that of the continuum, the direct sum $E = C^{(I)}$ of C indexed by I fails to have $A_{oa}(E;C)$. In fact, there is a homogeneous polynomial $p: E \to C$ of degree two which is not continuous. Let $c_{ij} \in C(i, j \in I)$ be such that $p(x) = \Sigma_{i,j\in I} c_{ij} x_i x_j$ for every $x = (x_i)_{i\in I} \in E$. If $J \subset I$ is finite, define $p_J \in \mathcal{P}(^2E;C)$ by $p_J(x) = \Sigma_{i,j \in J} c_{ij} x_i x_j$. The family (p_J) is $\mathcal{C}_0$-bounded. It is not locally bounded at 0 since $p_J \to p$ pointwisely. Thus E fails to have $A_{oa}(E;C)$. Q.E.D.

8. HOLOMORPHICALLY INFRABARRELED, OR BARRELED, SPACES

Classically, E is defined to be infrabarreled if, for every F and every $\mathcal{X} \subset \mathcal{L}(E;F)$, if $\mathcal{X}$ is bounded on any compact subset of E, then $\mathcal{X}$ must be amply bounded. This form of definition suggests up to say that E is holomorphically infrabarreled if, for every F and every $\mathcal{X} \subset \mathcal{H}(U;F)$, if $\mathcal{X}$ is bounded on any compact subset of U, then $\mathcal{X}$ must be amply bounded; in other words, E is called holomorphically infrabarreled if E has A_{oa}. It is then clear that a holomorphically infrabarreled space is infrabarreled.

Classically, E is defined to be barreled if, for every F and every $\mathcal{X} \subset \mathcal{L}(E;F)$, if $\mathcal{X}$ is bounded on any finite dimensional compact subset of E, then $\mathcal{X}$ must be amply bounded. This form of the definition suggests us to say that E is holomorphically barreled if, for every F and every $\mathcal{X} \subset \mathcal{H}(U;F)$, if $\mathcal{X}$ is bounded on any finite dimensional compact subset of U, then $\mathcal{X}$ must be amply bounded. It is then clear that a holomorphically barreled space is barreled. A superficial inspection of the proofs of Propositions 5, 6 and 2 shows us that the following complex locally convex spaces area actually not only holomorphically infrabarreled, as stated, but even holomorphically barreled: a Silva space, a Baire space, and a cartesian product of Frechet spaces.

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