

Supersymmetric Field-Theoretic Models on a Supermanifold

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ABSTRACT

We propose an extension of some structural aspects that have successfully been applied in the development of the theory of quantum fields propagating on a general spacetime manifold so as to include superfield models on a supermanifold.

Keywords: Supermanifolds, Microlocal Analysis.

1 Introduction

There are topics in the physical literature which do not exhaust themselves, but always deserve new analyses. Amongst these, the program to a quantum gravity theory has a significant part, remaining an open problem of Physics and an active area of current research. In spite of the fact that many attempts have been made to include gravity in the quantization program, a satisfactory and definitive theory still does not exist. Many lines of research in quantum gravity developed over last decades, under different names, such as the Supergravity, Kaluza-Klein, String, Twistors, D-brane, Loop Quantum Gravity, Noncommutative Geometry and Topos theories, have elucidated the role of quantum gravity, without, however, providing conclusive results (see for instance [1] for a recent review of the status of quantum gravity). Whereas these good ideas stay only as good promises in the direction of a final theory of the quantum gravity, and since the relevant scale of the Standard Model, or any of its supersymmetric extensions, is much below the typical gravity scale, it seems appropriate to treat, in an intermediate step, some aspects of gravity in quantum field theory by considering the approach which describes the matter quantum fields under the influence of a gravitational background. This framework has a wide range of physical applicability, the most prominent being the gravitational effect of particle creation in the vicinity of black-holes, raised up for the first time by Hawking [2].

The study of quantum field theories on a general manifold has become an area of intensive research activity, and a substantial progress has been made on a variety of interesting problems. In particular, great strides have been made towards the understanding of the question of how the spectral condition can be defined. While the most of the Wightman axioms can be implemented on a curved spacetime, the spectral condition (which expresses the positivity of the energy) represents a serious conceptual problem. On a flat spacetime the Poincaré covariance, in particular the translations, guarantees the positivity of the spectrum, and fixes a unique vacuum state; but on a general curved spacetime, due the absence of a global Poincaré group, there does not exist a useful notion of a vacuum state. As a result, the concept of particles becomes ambiguous, and the problem of the physical interpretation becomes much more difficult. One possible resolution to this difficulty is to choose some quantities other than particles content to label quantum states. Such an advice was given by Wald [3] with the purpose of finding the expectation value of the energy-momentum tensor. For free fields, this approach leads to the concept of Hadamard states. The latter are thought to be good candidates for describing physical states, at least for free quantum field theories in curved spacetime, according to the work of DeWitt and Brehme [4] (see [5, 6, 7] for a general review and references). In a seminal

work, Radzikowski [8] showed that the global Hadamard condition can be locally characterized in terms of the wavefront set, and proved a conjecture by Kay [9] that a locally Hadamard quasi-free Klein-Gordon state on any globally hyperbolic curved spacetime must be globally Hadamard. His proof relies on a general *wavefront set spectrum condition* for the two-point distribution, which has made the connection with the spectral condition much more transparent (see also [10, 11]).

The wavefront set was introduced by the mathematicians Hörmander and Duistermaat around 1971 [12, 13] in their studies on the propagation of singularities of pseudodifferential operators, which rely on what is now known as a microlocal point of view. This subject is of growing importance, with a range of applications going beyond the original problems of linear partial equations. In particular, the link with quantum field theories on a curved spacetime is now firmly established, specially after Radzikowski's work. A considerable amount of recent papers devoted to this subject [10, 11, 14, 15, 16, 17, 18, 19, 20] emphasises the importance of the microlocal technique to solving some previously unsolved problems.

At the same time, it seems that not so much attention has been drawn to supersymmetric theories in this direction. Much of the progress made in understanding the physics of elementary particles has been achieved through a study of supersymmetry. The latter is a subject of considerable interest amongst physicists and mathematicians. It is not only it fascinating in its own right; in the 30 years that have passed since its proposal, supersymmetry has been studied intensively in the belief that such theories may play a part in a unified theory of the fundamental forces, and many issues are understood much better now. Although no clear signal has been observed up to now, supersymmetry is believed to be detectable, at least if certain minimal models of particle physics turn out to be realized in nature, and calculations and phenomenological analysis of supersymmetry models are well-justified in view of the forthcoming generation of machines, as the new super collider LHC being built at CERN, which is expected to operate in a few years time and will have probably enough high energy to reveal some of the predicted supersymmetry particles, such as neutralinos, sleptons and may be indirectly squarks. It also has proven to be a tool to link the quantum field theory and noncommutative geometry [21, 22]. Furthermore, in recent years the supersymmetry have been instrumental in uncovering non-perturbative aspects of quantum theories [23, 24]. All of this gives strong motivations for trying to get a deeper understanding of the structure and of the properties of supersymmetric field theories.

This work is inspired in the structurally significant, recent results on quantum fields propagating in a globally hyperbolic, curved spacetime, and represents a natural attempting to construct a generalization of some of the conventional mathematical structures used in quantum field theory, such as manifolds, so as to

include superfield models in supermanifolds (curved superspaces). These structural questions are not without physical interest and relevance! It is the purpose of the present paper to study how such a construction can be achieved.

The outline of the paper is as follows. We shall begin in Sec. 2 by describing some global properties of supermanifolds according to Rogers [25], and the problem of constructing their bodies in the sense of Catenacci *et al.* [31] and Bryant [32]. Then, by working with a class of G^∞ supermanifolds constructed by Bonora-Pastin-Tonin [28] (BPT-supermanifolds), we demonstrate that this class of supermanifolds satisfies the criterions which guarantee that a supermanifold admits a Hausdorff body manifold. In Sec. 3, superdistributions on superspace are defined. We derive some results not contained in [42]. In particular, we generalize straightforwardly the notion of distributions defined on a manifold to distributions defined on a supermanifold. In Sec. 4, we discuss the algebraic formalism so as to include supersymmetry on a supermanifold. The results from this section may be seen as a natural extension of the ‘‘Haag-Kastler-Dimock’’ axioms [45, 46] for local ‘‘observables’’ to supermanifolds. In Sec. 5, we summarize some basics on the description of Hadamard (super)states. The focus of the Sec. 6 will be on the extension of the Hörmander’s description of the singularity structure (wavefront set) of a distribution to include the supersymmetric case. This fills a gap in the literature between the usual textbook presentation of the singularity structure of superfunctions and the rigorous mathematical treatment based on microlocal analysis. In Sec. 7, we present the characterization of a type of microlocal spectral condition for a superstate ω^{susy} with m -point superdistribution ω_m^{susy} on a supermanifold, in terms of the wavefront set of superdistributions, which is equivalent to the requirement that all of the component fields satisfy the microlocal spectral conditions [11] on the body manifold. This is in accordance with the DeWitt’s remark [26] which asserts that in physical applications of supersymmetric quantum field theories, the spectral condition of the GNS-Hilbert superspace is restricted to the ordinary GNS-Hilbert space that sits inside the GNS-Hilbert superspace. Finally, the Sec. 8 contains our final considerations.

2 Notions of Supermanifolds

This section introduces some few basic fundamentals on the theory of supermanifolds. We follow here the work of Rogers [25] which is both general and mathematically rigorous. Rogers’ theory has an advantage, a supermanifold is an ordinary Banach manifold endowed with a Grassmann algebra structure, so that the topological constructions have their standard meanings. In this context see also the Refs. [26]-[36].

We start by introducing first some definitions and concepts of a Grassmann-

Banach algebra, i.e., a Grassmann algebra endowed with a Banach algebra structure. This leads to the key concept of supercommutative superalgebra.

DEFINITION 2.1. *An algebra is said to be a supercommutative superalgebra Λ – or a \mathbb{Z}_2 -graded commutative algebra – if Λ is the direct sum $\Lambda = \Lambda_0 \oplus \Lambda_1$ of two complementary subspaces such that $\mathbb{1} \in \Lambda_0$ and $\Lambda_0\Lambda_0 \subset \Lambda_0$, $\Lambda_0\Lambda_1 \subset \Lambda_1$, $\Lambda_1\Lambda_1 \subset \Lambda_0$. Moreover, for all homogeneous element^a x, y in Λ , $xy = (-1)^{|x||y|}yx$, where $|x| = 0$ if $x \in \Lambda_0$ and $|x| = 1$ if $x \in \Lambda_1$. In particular, it follows that the square of odd elements is zero.*

We shall assume that the superalgebra Λ is a Banach space with norm $\|\cdot\|$ satisfying the condition

$$\|xy\| \leq \|x\|\|y\|, \forall x, y \in \Lambda; \quad \|\mathbb{1}\| = 1.$$

Let L be a finite positive integer and \mathcal{G} denote a Grassmann algebra, such that \mathcal{G} can naturally be decomposed as the direct sum $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, where \mathcal{G}_0 consists of the even (commuting) elements and \mathcal{G}_1 consists of the odd (anti-commuting) elements in \mathcal{G} , respectively. Let M_L denote the set of sequences $\{(\mu_1, \dots, \mu_k) \mid 1 \leq k \leq L; \mu_i \in \mathbb{N}; 1 \leq \mu_1 < \dots < \mu_k \leq L\}$. Let Ω represent the empty sequence in M_L , and (j) denote the sequence with just one element j . A basis of \mathcal{G} is given by monomials of the form $\{\xi_\Omega, \xi^{\mu_1}\xi^{\mu_2}, \dots, \xi^{\mu_1}\xi^{\mu_2} \dots \xi^{\mu_k}\}$ for all $\mu \in M_L$, such that $\xi_\Omega = \mathbb{1}$ and $\xi^{(i)}\xi^{(j)} + \xi^{(j)}\xi^{(i)} = 0$ for $1 \leq i, j \leq L$. Furthermore, there is no other independent relations among the generators. By \mathcal{G}_L we denote the Grassmann algebra with L generators, where the even and the odd elements, respectively, take their values. L being assumed a finite integer (the number of generators L could be possibly infinite), it means that the sequence terminates at $\xi^1 \dots \xi^L$ and there are only 2^L distinct basis elements. An arbitrary element $q \in \mathcal{G}_L$ has the form

$$q = q_{\mathbf{b}} + \sum_{(\mu_1, \dots, \mu_k) \in M_L} q_{\mu_1, \dots, \mu_k} \xi^{\mu_1} \dots \xi^{\mu_k}, \quad (2.1)$$

where $q_{\mathbf{b}}, q_{\mu_1 \dots \mu_k}$ are real numbers. An even or odd element is specified by 2^{L-1} real parameters. The number $q_{\mathbf{b}}$ is called the body of q , while the remainder $q - q_{\mathbf{b}}$ is the soul of q , denoted $s(q)$. The element q is invertible if, and only if, its body is non-zero.

With reference to supersymmetric field theories, the commuting variable x has the form

$$x = x_{\mathbf{b}} + x_{ij}\xi^i\xi^j + x_{ijkl}\xi^i\xi^j\xi^k\xi^l + \dots, \quad (2.2)$$

^aElements from Λ_0 and Λ_1 are said to be homogeneous if they have a definite parity, i.e., an element $x \in \Lambda_0$ is said to have *even* parity, while an element $x \in \Lambda_1$ is said to have *odd* parity. Products of homogeneous elements of the same parity are even and of elements of different parities are odd.

where $x_{\mathbf{b}}, x_{ij}, x_{ijkl}, \dots$ are real variables. Similarly, the anticommuting variables (in the Weyl representation) θ and $\bar{\theta} = (\theta)^*$ have the form

$$\theta = \theta_i \xi^i + \theta_{ijk} \xi^i \xi^j \xi^k + \dots, \quad \bar{\theta} = \bar{\theta}_i \xi^i + \bar{\theta}_{ijk} \xi^i \xi^j \xi^k + \dots, \quad (2.3)$$

where $\theta_i, \theta_{ijk}, \dots$ are complex variables. The summation over repeated indices is to be understood unless otherwise stated.

Remark. As pointed out by Vladimirov-Volovich [37], from the physical point of view, superfields are not functions of $\theta_i, \theta_{ijk}, \dots$ and $x_{\mathbf{b}}, x_{ij}, x_{ijkl}, \dots$, but only depend on these variables through θ and x , as it occurs with ordinary complex analysis where analytic functions of the complex variables $z = x + iy$ are not arbitrary functions of the variables x and y , but functions that depend on x and y through z . \blacktriangle

The Grassmann algebra may be topologized. Consider the complete norm on \mathcal{G}_L defined by [38]:

$$\|q\|_p = \left(|q_{\mathbf{b}}|^p + \sum_{(\mu)=1}^L |q_{\mu_1 \dots \mu_k}|^p \right)^{1/p}. \quad (2.4)$$

A useful topology on \mathcal{G} is the topology induced by this norm. The norm $\|\cdot\|_1$ is called the Rogers norm and $\mathcal{G}_L(1)$ the Rogers algebra [25]. The Grassmann algebra \mathcal{G} equipped with the norm (2.4) becomes a Banach space. In fact \mathcal{G} becomes a Banach algebra, i.e., $\|\mathbb{1}\| = 1$ and $\|qq'\| \leq \|q\| \|q'\|$ for all $q, q' \in \mathcal{G}$.

DEFINITION 2.2. *A Grassmann-Banach algebra is a Grassmann algebra endowed with a Banach algebra structure.*

A superspace must be constructed using as a building block a Grassmann-Banach algebra \mathcal{G}_L and not only a Grassmann algebra.

DEFINITION 2.3. *Let $\mathcal{G}_L = \mathcal{G}_{L,0} \oplus \mathcal{G}_{L,1}$ be a Grassmann-Banach algebra. Then the (m, n) -dimensional superspace is the topological space $\mathcal{G}_L^{m,n} = \mathcal{G}_{L,0}^m \times \mathcal{G}_{L,1}^n$, which generalizes the space \mathbb{R}^m , consisting of the Cartesian product of m copies of the even part of \mathcal{G}_L and n copies of the odd part.*

For an (m, n) -dimensional superspace, a typical element of this set used in physics is denoted by $(z) = (z_1, \dots, z_{m+n}) = (x_1, \dots, x_m, \theta_1, \dots, \theta_{n/2}, \bar{\theta}_1, \dots, \bar{\theta}_{n/2})$. For instance, for the $(4, 4)$ -dimensional Minkowski superspace, which is the space of e.g. $N = 1$ Wess-Zumino model formulated in superfield language and modelled as $\mathcal{G}_L^{4,4} = \mathcal{G}_{L,0}^4 \times \mathcal{G}_{L,1}^4$, $(z) = (x_1, \dots, x_4, \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2)$. The norm on $\mathcal{G}_L^{4,4}$ is defined by $\|z\| = \sum_{i=1}^4 \|x_i\| + \sum_{j=1}^2 \|\theta_j\| + \sum_{k=1}^2 \|\bar{\theta}_k\|$. The topology on $\mathcal{G}_L^{4,4}$ is the topology induced by this norm – which is also the product topology.

In supersymmetric quantum field theory, superfields are functions in superspace usually given by their (terminating) standard expansions in powers of the odd coordinates

$$F(x, \theta, \bar{\theta}) = \sum_{(\gamma)=0}^{\Gamma} f_{(\gamma)}(x)(\theta)^{(\gamma)} , \quad (2.5)$$

where $(\theta)^{(\gamma)}$ comprises all monomials in the anticommuting variables θ and $\bar{\theta}$ (belonging to odd part of a Grassmann-Banach algebra) of degree $|\gamma|$; $f_{(\gamma)}(x)$ is called a component field, whose Lorentz properties are determined by those of $F(x, \theta, \bar{\theta})$ and by the power (γ) of (θ) . The following notation, extended to more than one θ variable, is used (2.5): $(\theta) = (\theta_1, \bar{\theta}_1, \dots, \theta_n, \bar{\theta}_n)$, and (γ) is a multi-index $(\gamma_1, \bar{\gamma}_1, \dots, \gamma_n, \bar{\gamma}_n)$ with $|\gamma| = \sum_{r=1}^n (\gamma_r + \bar{\gamma}_r)$ and $(\theta)^{(\gamma)} = \prod_{r=1}^n \theta_r^{\gamma_r} \bar{\theta}_r^{\bar{\gamma}_r}$. In Eq.(2.5), for a (4,4)-dimensional superspace, $\Gamma = (2, 2)$.

Rogers [25] considered superfields in $\mathcal{G}_L^{m,n}$ as G^∞ superfunctions,^b i.e., functions whose coefficients $f_{(\gamma)}(x)$ of their expansions are smooth functions of \mathbb{R}^m into \mathcal{G}_L , extended from \mathbb{R}^m to all of $\mathcal{G}_L^{m,0}$ by z -continuation [25], which maps functions of real variables into functions of variables in $\mathcal{G}_L^{m,0}$.

DEFINITION 2.4. *Let U be an open set in $\mathcal{G}_L^{m,0}$ and let $\epsilon : \mathcal{G}_L^{m,0} \rightarrow \mathbb{R}^m$ be the body projection which associates to each m -tuple $(x_1, \dots, x_m) \in \mathcal{G}_L^{m,0}$*

an m -tuple $(\epsilon(x_1), \dots, \epsilon(x_m)) \in \mathbb{R}^m$. Let V be an open set in \mathbb{R}^m with $V = \epsilon(U)$. We get through z -continuation – or “Grassmann analytic continuation” – of a function $f \in C^\infty(V, \mathcal{G}_L)$ a function $z(f) \in G^\infty(U, \mathcal{G}_L)$, which admits an expansion in powers of the soul of x

$$z(f)(x_1, \dots, x_m) = \sum_{i_1=\dots=i_m=0}^L \frac{1}{i_1! \dots i_m!} [\partial_1^{i_1} \dots \partial_m^{i_m}] f(\epsilon(x)) s(x_1)^{i_1} \dots s(x_m)^{i_m} ,$$

where $s(x_i) = (x_i - \epsilon(x_i))$ and $\epsilon(x_i) = (x_i)_{\mathbf{b}}$.

One should keep always in mind that the continuation involves only the even variables $z : C^\infty(\epsilon(U)) \rightarrow G^\infty(U)$, and that $z(f)(x_1, \dots, x_m)$ is a supersmooth function if their components are smooth for soulless values of x . This justifies the formal manipulations in the physics literature, where superfields are manipulated as if their even arguments were ordinary numbers [34]: a supersmooth function is completely determined when its components are known on the body of superspace.

According to Definition 2.4, the superfield $F(x, \theta, \bar{\theta}) \in G^\infty(U, \mathcal{G}_L)$ admits an expansion

$$F(x, \theta, \bar{\theta}) = \sum_{(\gamma)=0}^{\Gamma} z(f_{(\gamma)})(x)(\theta)^{(\gamma)} ,$$

^bWe use the prefix “super” for entities involving odd Grassmann variables.

but here with suitable $f_{(\gamma)} \in C^\infty(\epsilon(U), \mathcal{G}_L)$.

Now, we are going to consider some helpful aspects about supermanifolds, based on the work of Rogers [25], replacing the simple superspace $\mathcal{G}_L^{m,n}$ by a more general supermanifold. Rogers used the concept of G^∞ superfunctions to define the concept of G^∞ supermanifolds (which can be considered as Banach real manifolds C^∞ modelled on $\mathcal{G}_L^{m,n}$ of $\dim N = 2^{L-1}(m+n)$), with a structure allowing for the definitions of neighbouring points and continuous superfunctions. An (m, n) -dimensional G^∞ supermanifold generalizes the concept of an m -dimensional C^∞ manifold: just as a manifold is a Hausdorff topological space such that every point has a neighbourhood homeomorphic to \mathbb{R}^m and has local coordinates $(x_1(p), \dots, x_m(p))$ in \mathbb{R}^m , a supermanifold is a topological space which locally looks like $\mathcal{G}_L^{m,n}$ (but not necessarily in its global extent) and has local coordinates $(x_1(p), \dots, x_m(p), \theta_1(p), \dots, \theta_n(p))$ in $\mathcal{G}_L^{m,n}$, and whose transition functions fulfill a suitable supersmoothness condition.

DEFINITION 2.5. *A supermanifold is in general a paracompact Hausdorff topological space \mathcal{M} , together with an atlas of charts $\{(X_\alpha, k_\alpha) \mid \alpha \in I\}$, over a Grassmann-Banach algebra \mathcal{G}_L , where the X_α cover \mathcal{M} and each coordinate function k_α is a homeomorphic local maps from X_α onto an open subset $\tilde{X}_\alpha \subset \mathcal{G}_L^{m,n}$, also Hausdorff.*

The existence of infinitely differentiable coordinates systems makes the supermanifold differentiable. The differentiable structure in this topological space is due to G^r ($r = p$ or $p = \infty$) structure of transition functions, $k_\beta \circ k_\alpha^{-1}$, between overlapping coordinate patches, $k_\alpha(X_\alpha \cap X_\beta)$ and $k_\beta(X_\alpha \cap X_\beta)$, required to be supersmooth morphisms for any $\alpha, \beta \in I$. The local coordinates are:

$$\begin{aligned} u_i &= p_i \circ k_\alpha \longmapsto (i = 1, \dots, m) , \\ v_j &= p_{j+m} \circ k_\alpha \longmapsto (j = 1, \dots, n) . \end{aligned}$$

In this sense $\mathcal{G}_L^{m,n}$ is an example of G^∞ supermanifold, unlike of the coarse topology in the DeWitt sense [26] whose structure cannot be even a metric one.

DEFINITION 2.6. *Let \tilde{X}_α be an open in $\mathcal{G}_L^{m,n}$ and $f : \tilde{X}_\alpha \rightarrow \mathcal{G}_L$, then:*

(a) *f is called G^0 in \tilde{X}_α if f is continuous in \tilde{X}_α .*

(b) *f is called G^1 in \tilde{X}_α if exist $m+n$ functions $G_k f : \tilde{X}_\alpha \rightarrow \mathcal{G}_L$, $k = 1, \dots, m+n$ and functions $\eta : \mathcal{G}_L^{m,n} \rightarrow \mathcal{G}_L$ such that:*

$$\begin{aligned} f(a+h, b+k) &= f(a, b) + \sum_{i=1}^m h_i \{G_i f(a, b)\} + \sum_{j=1}^n k_j \{G_{j+m} f(a, b)\} + \\ &+ \|h, k\| \eta(h, k) , \end{aligned}$$

and $\eta(h, k) \rightarrow 0$ when $\|h, k\| \rightarrow 0$. In this sense, $G_i f \rightarrow f'_i$.

We can generalize to G^p , with finite p in the following: f is G^p in \tilde{X}_α if is possible choose $G_k f$ which are G^{p-1} with $f \in G^1$ em \tilde{X}_α . If it is true to all p , f is called G^∞ . In fact, any function which is absolutely convergent (power series) is G^∞ on \tilde{X}_α , in other words:

$$f(z) = \sum_{k_1 \dots k_{m+n}=0}^{\infty} a_{k_1 \dots k_{m+n}} z_1^{k_1} \dots z_{m+n}^{k_{m+n}} ,$$

$$f : \tilde{X}_\alpha \rightarrow \mathcal{G}_L, \quad \tilde{X}_\alpha \subset \mathcal{G}_L^{m,n} \quad \text{and} \quad a_{k_1 \dots k_{m+n}} \in \mathcal{G}_L .$$

Another important fact is the C^∞ structure:

$$[D^p f(z)][\ell^1, \ell^2, \dots, \ell^p] = \sum_{k_1 \dots k_p=1}^{m+n} l_{k_1}^1 \dots l_{k_p}^p (G_{k_p} G_{k_{p-1}} \dots G_{k_1} f)(z) ,$$

for all $z \in \tilde{X}_\alpha$ open in $\mathcal{G}_L^{m,n}$ and $l_{k_1}^1 \dots l_{k_p}^p \in (\mathcal{G}_L^{m,n})^p$. The latter denotes a product space of p copies of $\mathcal{G}_L^{m,n}$. In this way the p derivative of $f \in \mathcal{L}[(\mathcal{G}_L^{m,n})^p, \mathcal{G}_L]$ are elements of continuous p -linear maps of $(\mathcal{G}_L^{m,n})^p$ into \mathcal{G}_L . This formalism is interesting and agrees to the Hörmander's one [39] (pg.11), where $f^{(p)} \in L^p(X_\alpha, X_\beta)$, are elements of continuous p -linear forms from X_α to X_β .

Remark. The discussion of differentiability by Jadczyk-Pilch [27] is simpler than the one given by Rogers [25]. In particular, knowing already that a function f is a C^∞ map between Banach spaces, it is needed only to look at its first derivative to know whether f is supersmooth or not, while according to Rogers an investigation of all derivatives is necessary. However, the concept of supersmoothness by Jadczyk-Pilch, and the concept of G^∞ differentiability by Rogers are equivalent. \blacktriangle

2.1 The Body of a Supermanifold

Now that the general idea of structure on a supermanifold has been introduced, it is time to restrict our attention to the case of fundamental interest: the problem of constructing the body of a G^∞ supermanifold which serves as the physical spacetime. Roughly speaking, the body of a supermanifold \mathcal{M} is an ordinary C^∞ spacetime manifold \mathcal{M}_0 obtained from \mathcal{M} getting rid of all the soul coordinates. Because of its extreme generality, Rogers' theory includes many topologically exotic supermanifolds which are not physically useful, admitting the possibility of nontrivial topology in the anticommuting directions and classes of supermanifolds without a body manifold. But, intuition suggests that only a bodied G^∞ supermanifold can be physically relevant!

The question of the existence of the body of a supermanifold was clarified in the papers by R. Catenacci *et al.* [31] and P. Bryant [32]. Their approach is independent

of the atlas used, and it is based on the fact that any G^∞ supermanifold \mathcal{M} admits a foliation \mathfrak{F} . This type of structure is defined and related to the natural notions of quotient and substructure on a supermanifold. As with many important concepts in mathematics, there are several equivalent ways of defining the notion of a foliation. The simplest and most geometric is the following. Let \mathcal{M} be an (m, n) -dimensional supermanifold of class G^p , $0 \leq p \leq k$.

DEFINITION 2.7. *A foliation of class G^r , $0 \leq r \leq p$, and of codimension m , is a decomposition of \mathcal{M} into disjoint connected subsets $\{\mathfrak{L}_\alpha\}_{\alpha \in A}$, called the leaves of the foliation, such that each point of \mathcal{M} has a neighbourhood U and a system of G^r coordinates $(x, \theta) : U \rightarrow \mathfrak{G}_{L,0}^m \times \mathfrak{G}_{L,1}^n$ such that for each leaf \mathfrak{L}_α , the components of $U \cap \mathfrak{L}_\alpha$ are described by surfaces on which all the body coordinates $\epsilon(x_1), \dots, \epsilon(x_m)$ are constant. We denote the foliation by $\mathfrak{F} = \{\mathfrak{L}_\alpha\}_{\alpha \in A}$.*

The coordinates referred in the Definition 2.7 are said to be distinguished by the foliation \mathfrak{F} . Under certain regularity conditions on \mathfrak{F} , the quotient space \mathcal{M}/\mathfrak{F} can be given the structure of an ordinary m -dimensional differentiable manifold \mathcal{M}_0 , which is called the body manifold of \mathcal{M} (for details see [31]). A G^∞ supermanifold whose \mathfrak{F} foliation is regular is called regular itself. On regular supermanifolds the following theorem holds:

THEOREM 2.8 (Catenacci-Reina-Teofilatto Theorem). *Let \mathcal{M} be a regular G^∞ supermanifold. Then its body \mathcal{M}_0 is a C^∞ manifold.* \square

However, according to P. Bryant [32], the necessity of regularity of the soul foliation in the sense of Catenacci-Reina-Teofilatto is not sufficient to guarantee that a supermanifold admits a body manifold. He derived necessary and sufficient conditions, namely that leaves should be closed and do not accumulate, for the existence of a Hausdorff body manifold.

THEOREM 2.9 (Bryant Theorem 2.5). *Suppose that \mathcal{M} is a supermanifold. In order that \mathcal{M} admits a body manifold, it is necessary and sufficient that the leaves of the soul foliations are closed in \mathcal{M} and do not accumulate.* \square

For our purposes, it will be sufficient to consider the class of G^∞ supermanifolds constructed by Bonora-Pasti-Tonin [28] (we shall call BPT-supermanifolds for brevity), which has important applications in theoretical physics and fulfills Theorems 2.8 and 2.9, as we shall verify presently. These supermanifolds consist of the Grassmann extensions of *any* ordinary C^∞ spacetime manifold. From a given m -dimensional physical spacetime, one constructs first a $(m, 0)$ -dimensional supermanifold, and the (m, n) -dimensional supermanifold by taking the direct product

with $\mathcal{G}_L^{0,n}$. This construction is the closest to the physicist's intuitive view of super-space as a manifold with some anticommuting coordinates, with the odd Grassmann variables being topologically trivial.

For the convenience of the reader, we recall now the construction of Bonora-Pasti-Tonin [28]. Let $\{(U_\alpha, \psi_\alpha) \mid \alpha \in I\}$ be an atlas for \mathcal{M}_0 . For each $\alpha \in I$ consider the subset X_α of the Cartesian product $U_\alpha \times \mathcal{G}_L^{m,0}$ defined by

$$X_\alpha = \{(x, \bar{x}) \mid x \in U_\alpha, \bar{x} \in \mathcal{G}_L^{m,0}, \text{ and } \epsilon(\bar{x}) = \psi_\alpha(x)\}, \quad (2.6)$$

and define $k_\alpha : X_\alpha \rightarrow \mathcal{G}_L^{m,0}$ by $k_\alpha(x, \bar{x}) = \bar{x}$ for $(x, \bar{x}) \in X_\alpha$. k_α is a homeomorphism and its image is an open subset of $\mathcal{G}_L^{m,0}$.

An important property of the z -continuation is the composition of functions. Let U be an open set in \mathbb{R}^m , and let the map $f : \mathbb{R}^m \rightarrow \mathcal{G}_L^{k,0}$ be represented by the set of C^∞ functions $\{f_i(x_1, \dots, x_m), i = 1, \dots, k\}$. Define $z(f)$ as the set of functions $\{z(f_i)\}$. Let V be an open set in \mathbb{R}^n , and consider the maps $f : U \rightarrow V$ and $g : V' \rightarrow \mathcal{G}_L^{k,0}$, respectively, where $V' \subseteq V$, and both f, g are C^∞ functions. Then

$$z(g \circ f) = z(g) \circ z(f). \quad (2.7)$$

Now consider the disjoint union $M = \bigcup_{\alpha \in I} X_\alpha$. Two points of M are equivalent if and only if $(x, \bar{x}) \sim (x', \bar{x}')$, such that $(x, \bar{x}) \in X_\alpha$ and $(x', \bar{x}') \in X_\beta$ and $x = x'$, $\bar{x}' = z(\psi_\beta \circ \psi_\alpha^{-1})(\bar{x})$. Of course M is a Hausdorff space. Then consider the space \mathcal{M}_G equal to the space M modulo the equivalence relation above. The k_α 's provide \mathcal{M}_G with a G^∞ differentiability structure, so that \mathcal{M}_G is a $G^\infty(m, 0)$ supermanifold. Let $\pi_G : \mathcal{M}_G \rightarrow \mathcal{M}_0$ be a continuous and open projection. Locally $\pi_G|_{X_\alpha}(x, \bar{x}) = x$ for $(x, \bar{x}) \in X_\alpha$. Since \mathcal{M}_G is a *regular* supermanifold, we find straightforwardly that $\pi_G \circ k_\alpha^{-1} = \psi_\alpha^{-1} \circ \epsilon$ for $\bar{x} \in k_\alpha(X_\alpha)$. This can be expressed by the commutative diagram:

$$\begin{array}{ccc} X_\alpha & \xleftarrow{k_\alpha^{-1}} & \mathcal{G}_L^{m,0} \\ \pi_G \downarrow & & \downarrow \epsilon \\ U_\alpha & \xleftarrow{\psi_\alpha^{-1}} & \mathbb{R}^m \end{array}$$

Finally, we construct the (m, n) -dimensional supermanifold \mathcal{M} by taking the direct product of \mathcal{M}_G with $\mathcal{G}_L^{0,n}$. The projection $\pi_S : \mathcal{M} \rightarrow \mathcal{M}_0$ is the composite map $\pi_G \circ \gamma$, where $\gamma : \mathcal{M} \rightarrow \mathcal{M}_G$ is the projection onto the first factor. The map γ is G^∞ , unlike π_G which is a C^∞ function but not a G^∞ .

COROLLARY 2.10. *Let \mathcal{M} be a BPT-supermanifold. Then the leaves of the soul foliation are regular, closed in \mathcal{M} and do not accumulate.*

Proof. First of all, it is worthwhile noticing that, according to the construction of Bonora-Pasti-Tonin, two points of a BPT-supermanifold are in the same leaf if,

and only if, they are equivalents in the sense defined in their paper. Then the soul foliation can be defined by $\mathcal{M}/\sim \stackrel{\text{def}}{=} \mathcal{M}/\mathfrak{F}$. Once verified the corollary, we see that a BPT-supermanifold possesses an ordinary body manifold defined by soul foliation $\mathcal{M}_0 \stackrel{\text{def}}{=} \mathcal{M}/\mathfrak{F}$, where \mathcal{M}_0 denotes the body manifold.

In order to show that the leaves of a BPT-supermanifold are closed, the following considerations are needed: we say that the soul foliation of a BPT-supermanifold is a Hausdorff space, and that the structure of their supermanifold is regular. This can be verified through the following theorem by Bryant [32] (Theorem 3.2): Suppose that \mathcal{M} is a supermanifold of dimension (m, n) and $\Gamma = \{U_i, \phi_i\}$ is a good atlas; then the following conditions are equivalent: (i) $\Gamma = \{U_i, \phi_i\}$ is a regular superstructure on \mathcal{M} , (ii) when s and t lie in U_i , $s \approx t$ implies $s \sim t$ and (iii) the body map $\epsilon : \mathcal{M} \rightarrow \mathcal{M}/\mathfrak{F}$ is locally modelled on $\epsilon_0 : B^{m,n} \rightarrow \mathbb{R}^m$ in the sense that exist homeomorphisms $\bar{\phi}_i : \epsilon U_i \rightarrow \epsilon_0 \phi_i U_i$ such that $\bar{\phi}_i \circ \epsilon|_{U_i} = \epsilon_0 \circ \phi_i$. When these conditions are satisfied, \mathcal{M}/\mathfrak{F} is Hausdorff and is a smooth manifold of dimension m with charts $\{\epsilon U_i, \bar{\phi}_i\}$. For the case of the equivalence relation ($s \sim t$) of a BPT-supermanifold, we see that it must be \approx in the Bryant sense because embodies \sim and is transitive. Then \approx implies \sim on the same charts. This means that the conditions of the Theorem 3.2 by Bryant must be properties of the BPT foliation, and hence is Hausdorff and regular. Now, the fact that the leaves of a BPT-supermanifold are closed is clear: each point ($\epsilon(s)$) of \mathcal{M}/\sim is closed, given that the BPT-supermanifolds is a Hausdorff space, and the inverse application theorem guarantees that a leaf is necessarily closed, since being F the leaf in \mathcal{M} , $F = \epsilon^{-1}\epsilon(s)$ where ϵ^{-1} is a continuous map.

Finally, we shall verify that the leaves of a BPT-supermanifod do not accumulate. First, we shall suppose that the leaves of soul foliation accumulate^c in a given pair of points, eg $s_+, s_- \in \mathcal{M}$. Note that as \mathcal{M}/\mathfrak{F} is Hausdorff, given two points $x \in \mathcal{M}/\mathfrak{F}$ and $y \in \mathcal{M}/\mathfrak{F}$ with $x \neq y$, we can separate them by disjoint open sets. Choice, for example, $\epsilon s_+ = x$ and $\epsilon s_- = y$, where $\epsilon : \mathcal{M} \rightarrow \mathcal{M}/\mathfrak{F}$. Then, we also can choose $s_+ \in F' \cup \Sigma_+$ (a transverse submanifold) and $s_- \in F' \cup \Sigma_-$ (another transverse submanifold). If this is true, s_+, s_- must be in the same leaf, by indicating that $\epsilon s_+ = \epsilon s_-$ contradicting the statement which a soul foliation is Hausdorff. Hence, the leaves do not accumulate. In order to complete the prove, we examine the condition $\epsilon s_+ = \epsilon s_-$. Due the possibility of choosing arbitrary transverse submanifolds, we select $\Sigma(s)$ and $\Sigma(t)$ through the some disjoint neighbourhoods of s and t resp. such that does not exist a U_i which intersects $\Sigma(s)$ and $\Sigma(t)$. But $\epsilon s_+ = \epsilon s_-$ implies that s and t are in the same chart U_i , so the leaves do not accumulate since $\Sigma(s) \cup \Sigma(t) = \emptyset$. \square

^cThe leaves of a soul foliation accumulate if there exist two points s_+, s_- and a sequence $F^{(n)}$ such that for arbitrary transverse submanifolds Σ_+ through s_+ and Σ_- through s_- , there is some $F^{(n_0)}$, with $F^{(n_0)} \cup \Sigma_+ \neq \emptyset$ and $F^{(n_0)} \cup \Sigma_- \neq \emptyset$.

The existence of a body manifold places us in a position to consider physically interpretable field theories on supermanifolds. In order to establish applicability in a physical system, we need to impose some restrictions regarding to the body manifold \mathcal{M}_0 , associated with the supermanifold \mathcal{M} . Apart from another aspects, the causality principle plays a crucial role in our construction. Therefore, we restrict our body manifold, (\mathcal{M}_0, g_0) , to be globally hyperbolic Lorentz manifold, by consisting of a 4-dimensional smooth manifold \mathcal{M}_0 (any dimension would be possible) that can be smoothly foliated by a family of acausal Cauchy surfaces [6] and a smooth metric g_0 with signature $(+, -, -, -)$. This means that the body manifold must be topologically equivalent to the Cartesian product of \mathbb{R} and a smooth spacelike hypersurface Σ (a Cauchy surface). Σ intersects any endless timelike curve at most once. A 4-dimensional globally hyperbolic Lorentz manifold is orientable and time orientable, i.e., at each $x \in \mathcal{M}_0$ we may designate a future and past light cone continuously. Moreover, \mathcal{M}_0 is assumed to have a spin structure, so that one can consider spinors defined on it.^d

Remark. As it has been emphasized in [10], a natural background geometry that admits a supersymmetric extension of its isometry group can only be of the Anti-De-Sitter (AdS) type. In other words, the global supersymmetry should not be compatible with most spacetimes, an exception being the AdS space. This requirement seems to be an extremely restrictive condition, since the AdS space has problems with closed time-like curves, apparently violating causality and leading to problems during quantization. Namely, boundary conditions at infinity are needed. Nevertheless, one should remind that this result refers to extended supergravity theories with gauged $SO(N)$ internal symmetry [41]; this is not, however, our case in this paper. Furthermore, this result can mainly be justified by the heuristic form of introducing the superspace (which may be bypassed taking into account the Rogers' theory of a global supermanifold). As stressed by Bruzzo [36], "...the usual ways of dealing with superspace field theories are highly unsatisfactory from a mathematical point of view. The superspace is defined formally, and, for instance, general coordinate transformations are mathematically not well defined. As a consequence, there is now room for studying global topological properties of superspace." As it shall be tackled further on, Section 4, the mathematical structure of the supermanifolds chosen here leads to a natural formulation of superdiffeomorphisms, G^∞ , from (\mathcal{M}, g) to (\mathcal{M}', g') , from the z -continuation of ordinary diffeomorphisms, so that these structures become, projectively, well-defined isometries whenever $\mathcal{M}' = \mathcal{M}$ and restricted

^dIt can be shown that a 4-dimensional globally hyperbolic Lorentz manifold admits a spin structure [40]. In fact, Geroch [40] pointed out that a *noncompact, parallelizable* 4-dimensional manifold admits a spin structure. Geroch's parallelizability criterion applies to a 4-dimensional globally hyperbolic Lorentz manifold.

to the ordinary body manifold. ▲

3 Superdistributions

In this section, as a natural next step, we extend the definition of the objects most widely used in physics: distributions. We define superdistributions on supermanifolds over the Grassmann-Banach algebra \mathcal{G}_L , as continuous linear mappings to \mathcal{G}_L from the test function space of G^∞ superfunctions with compact support. We derive some results not contained in [42].

3.1 Distributions on a Manifold

To prepare for the extension of the theory of distributions to supermanifolds, we first consider their definition on manifolds. Following [39], the spacetime manifold \mathcal{M}_0 (here \mathcal{M}_0 denotes an ordinary manifold obtained from a supermanifold \mathcal{M} by throwing away all the soul coordinates) is a Hausdorff space covered by charts (X_α, k_α) , where the open sets X_α are homeomorphic neighbourhoods to open sets in \mathbb{R}^n . A C^∞ structure on \mathcal{M}_0 is a family $\mathcal{F} = \{(X_\alpha, k_\alpha) \mid \alpha \in I\}$, called an atlas, of homeomorphisms k_α , called coordinate functions, of open sets $X_\alpha \subset \mathcal{M}_0$ on open sets $\tilde{X}_\alpha \subset \mathbb{R}^n$, such that (i) if $k_\alpha, k_\beta \in \mathcal{F}$, then the map $k_\beta \circ k_\alpha^{-1} : k_\alpha(X_\alpha \cap X_\beta) \rightarrow k_\beta(X_\alpha \cap X_\beta)$ is infinitely differentiable, (ii) $\mathcal{M}_0 = \bigcup_{\alpha \in I} X_\alpha$. Let $f \in C_0^\infty(\mathbb{R}^n)$ denotes the set of C^∞ functions of compact support on $\tilde{X}_\alpha \subset \mathbb{R}^n$. Then, we can represent each f by functions \bar{f} of compact support on \mathcal{M}_0 by $f = \bar{f} \circ k_\alpha^{-1}$, for each k_α , where $\bar{f} \in C_0^\infty(\mathcal{M}_0)$. Elements of $\mathcal{D}'(\mathcal{M}_0)$, the topological dual of $C_0^\infty(\mathcal{M}_0)$, are distributions u on \mathcal{M}_0 , by which we mean collections $\{u_{k_\alpha}\}_{k_\alpha \in \mathcal{F}}$ of distributions $u_{k_\alpha} \in \mathcal{D}'(\tilde{X}_\alpha)$ such that u is uniquely determined by the u_{k_α} and relations $u = u_{k_\alpha} \circ k_\alpha$. Moreover, since for any other coordinate system one has $u = u_{k_\beta} \circ k_\beta$ in $(X_\alpha \cap X_\beta)$, it follows that $u_{k_\beta} = (k_\alpha \circ k_\beta^{-1})^* u_{k_\alpha} = u_{k_\alpha} \circ (k_\alpha \circ k_\beta^{-1})$ in $(X_\alpha \cap X_\beta)$.

3.2 Distributions on the Flat Superspace

With the purpose of defining superdistributions on supermanifolds, we must first consider superdistributions on an open set $U \subset \mathcal{G}_L^{m,n}$, where $\mathcal{G}_L^{m,n}$ denotes the flat superspace. We begin by introducing the concept of superdistributions as the dual space of supersmooth functions in $\mathcal{G}_L^{m,0}$, with compact support, equipped with an appropriate topology, called *test superfunctions*. This can be done relatively straightforward in analogy to the notion of distributions as the dual space to the space $C_0^\infty(U)$ of functions on an open set $U \subset \mathbb{R}^m$ which have compact support, since the spaces $\mathcal{G}_L^{m,0}$ and $\mathcal{G}_L^{m,n}$ are regarded as ordinary vector spaces of $2^{L-1}(m)$ and

$2^{L-1}(m+n)$ dimensions, respectively, over the real numbers.

Let $\Omega \subset \mathbb{R}^m$ be an open set. $\Omega = \epsilon(U)$ regarded as a subset of $\mathcal{G}_L^{m,0}$, it is identified with the body of some domain in superspace. Let $C_0^\infty(\Omega, \mathcal{G}_L)$ be the space of \mathcal{G}_L -valued smooth functions with compact support in \mathcal{G}_L . Every function $f \in C_0^\infty(\Omega, \mathcal{G}_L)$ can be expanded in terms of the basis elements of \mathcal{G}_L as:

$$f(x) = \sum_{(\mu_1, \dots, \mu_k) \in M_L^0} f_{\mu_1, \dots, \mu_k}(x) \xi^{\mu_1} \cdots \xi^{\mu_k}, \quad (3.1)$$

where $M_L^0 \stackrel{\text{def}}{=} \{(\mu_1, \dots, \mu_k) \mid 0 \leq k \leq L; \mu_i \in \mathbb{N}; 1 \leq \mu_1 < \cdots < \mu_k \leq L\}$ and $f_{\mu_1, \dots, \mu_k}(x)$ is in the space $C_0^\infty(\Omega)$ of real-valued smooth functions on Ω with compact support. Thus, it follows that the space $C_0^\infty(\Omega, \mathcal{G}_L)$ is isomorphic to the space $C_0^\infty(\Omega) \otimes \mathcal{G}_L$ [42]. In accordance with the Definition 2.4, the smooth functions of $C_0^\infty(\Omega, \mathcal{G}_L)$ can be extended from $\Omega \subset \mathbb{R}^m$ to $U \subset \mathcal{G}_L^{m,0}$ by Taylor expansion.

In order to define superdistributions, we need to give a suitable topological structure to the space $G_0^\infty(U, \mathcal{G}_L)$ of \mathcal{G}_L -valued superfunctions on an open set $U \subset \mathcal{G}_L^{m,0}$ which have compact support. According to a proposition by Rogers, every G^∞ superfunction on a compact set $U \subset \mathcal{G}_L^{m,0}$ can be considered as a real-valued C^∞ function on $U \subset \mathbb{R}^N$, where $N = 2^{L-1}(m)$, regarding $\mathcal{G}_L^{m,0}$ and \mathcal{G}_L as Banach spaces. In fact, the identification of $\mathcal{G}_L^{m,0}$ with $\mathbb{R}^{2^{L-1}(m)}$ is possible [31]. We have here an example of functoriality. Indeed, let X and Y denote a G^∞ supermanifold and a Banach manifold C^∞ , respectively. Then with each supermanifold X we associate a Banach manifold Y , via a *covariant* functorial relation $\lambda : X \rightarrow Y$, and with each G^∞ map ϕ defined on X , a C^∞ map $\lambda(\phi)$ defined on Y [31].

Following, we shall first consider only the subset C_K^∞ of $C_0^\infty(U \subset \mathbb{R}^N)$ which consists of functions with support in a fixed compact set K . Since by construction C_K^∞ is a Banach space, the functions C_K^∞ have a natural topology given by the finite family of norms

$$\|\phi\|_{K,m} = \sup_{\substack{|p| \leq m \\ x \in K}} |D^p \phi(x)|, \quad D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}}, \quad (3.2)$$

where $p = (p_1, p_2, \dots, p_m)$ is a m -tuple of non-negative integers, and $|p| = p_1 + p_2 + \dots + p_m$ defines the order of the derivative. Next, let U be considered as a union of compact sets K_i which form an increasing family $\{K_i\}_{i=1}^\infty$, such that K_i is contained in the interior of K_{i+1} . That such family exist follows from the Lemma 10.1 of [43]. Therefore, we think of $C_0^\infty(U \subset \mathbb{R}^N)$ as $\bigcup_i C_{K_i}^\infty(U \subset \mathbb{R}^N)$. We take the topology of $C_0^\infty(U \subset \mathbb{R}^N)$ to be given by the strict inductive limit topology of the sequence $\{C_{K_i}^\infty(U \subset \mathbb{R}^N)\}$. Of another way, we may define convergence in $C_0^\infty(U \subset \mathbb{R}^N)$ of a sequence of functions $\{\phi_k\}$ to mean that for each k , one has $\text{supp } \phi_k \subset K \subset U \subset \mathbb{R}^N$ such that for a function $\phi \in C_0^\infty(U \subset \mathbb{R}^N)$ we have $\|\phi - \phi_k\|_{K,m} \rightarrow 0$ as $k \rightarrow \infty$. This

notion of convergence generates a topology which makes $C_0^\infty(U \subset \mathbb{R}^N)$, certainly, a topological vector space.

Now, let F and E be spaces of smooth functions with compact support defined on $U \subset \mathcal{G}_L^{m,0}$ and $U \subset \mathbb{R}^N$, respectively. If $\lambda : E \rightarrow F$ is a *contravariant* functor which associates with each smooth function of compact support in E , a smooth function of compact support in F , then we have a map

$$\|\phi\|_{K,m} \longrightarrow \|\lambda(\phi)\|_{K,m} , \quad (3.3)$$

providing $G_0^\infty(U, \mathcal{G}_L)$ with a limit topology induced by a finite family of norms.

We now take a result by Jadczyk-Pilch [27], later refined by Hoyos et al [29], which establishes as a natural domain of definition for supersmooth functions a set of the form $\epsilon^{-1}(\Omega)$, where Ω is open in \mathbb{R}^m . Let $\epsilon^{-1}(\Omega)$ be the domain of definition for a superfunction $f \in G_0^\infty(\epsilon^{-1}(\Omega), \mathcal{G}_L)$, where $\epsilon^{-1}(\Omega)$ is an open subset in $\mathcal{G}_L^{m,0}$ and Ω is an open subset in \mathbb{R}^m , and let $\tilde{\phi} \in C_0^\infty(\Omega, \mathcal{G}_L)$ denotes the restriction of ϕ to $\Omega \subset \mathbb{R}^m \subset \mathcal{G}_L^{m,0}$. Then, it follows that $(\partial_1^{p_1} \cdots \partial_m^{p_m} \phi)^\sim = \partial_1^{p_1} \cdots \partial_m^{p_m} \tilde{\phi}$, where the derivatives on the right-hand side are with respect to m real variables. Now, suppose $\Omega = \bigcup_i \tilde{K}_i$ where each \tilde{K}_i is open and has compact closure in \tilde{K}_{i+1} . It follows that $C_0^\infty(\Omega, \mathcal{G}_L) = \bigcup_i C_{\tilde{K}_i}^\infty(\Omega, \mathcal{G}_L)$. Then, one can give $C_0^\infty(\Omega, \mathcal{G}_L)$ a limit topology induced by finite family of norms [42]

$$\|\tilde{\phi}\|_{\tilde{K},m} = \sup_{\substack{|p| \leq m \\ x \in \tilde{K}}} |D^p \tilde{\phi}(x)| = \sup_{\substack{|p| \leq m \\ x \in \tilde{K}}} \left\{ \sum_{(\mu_1, \dots, \mu_k) \in M_L^0} |D^p \tilde{\phi}_{\mu_1, \dots, \mu_k}(x)| \right\} . \quad (3.4)$$

Finally, a suitable topological structure to the space $G_0^\infty(U, \mathcal{G}_L)$ of \mathcal{G}_L -valued superfunctions on an open set $U \subset \mathcal{G}_L^{m,n}$ which have compact support, it is obtained immediately by the natural identification of $\mathcal{G}_L^{m,n}$ with $\mathbb{R}^{2^{L-1}(m+n)}$ and by the obvious extension of the construction above, which allows us define a limit topology induced to the space $G_0^\infty(U, \mathcal{G}_L)$ by finite family of norms,

$$\|\lambda(\phi)\|_{K,m+n} = \sup_{\substack{|p| \leq m+n \\ z \in K}} |D^p(\lambda(\phi))(z)| , \quad D^p = \frac{\partial^{|q|+|r|}}{\partial x_1^{q_1} \cdots \partial x_m^{q_m} \partial \theta_1^{r_1} \cdots \partial \theta_n^{r_n}} . \quad (3.5)$$

The derivatives $\partial^{|q|}/\partial x_1^{q_1} \cdots \partial x_m^{q_m}$ commute while the derivatives $\partial^{|r|}/\partial \theta_1^{r_1} \cdots \partial \theta_n^{r_n}$ anticommute, and $|p| = |q| + |r| = \sum_{i=1}^m q_i + \sum_{j=1}^n r_j$ defines the total order of the derivative, with $r_j = 0, 1$.

We are now ready to define a superdistribution in an open subset U of $\mathcal{G}_L^{m,n}$. The set of all superdistributions in U will be denoted by $\mathfrak{D}'(U)$. A superdistribution is a continuous linear functional $u : G_0^\infty(U) \rightarrow \mathcal{G}_L$, where $G_0^\infty(U)$ denotes the test superfunction space of $G^\infty(U)$ superfunctions with compact support in $K \subset U$. The continuity of u on $G_0^\infty(U)$ is equivalent to its boundedness on a neighbourhood of

zero, i.e., the set of numbers $u(\phi)$ is bounded for all $\phi \in G_0^\infty(U)$. The last statement translates directly into:

PROPOSITION 3.1. *A superdistribution u in $U \in \mathcal{G}_L^{m,n}$ is a continuous linear functional on $G_0^\infty(U)$ if and only if to every compact set $K \subset U$, there exists a constant C and $(m+n)$ such that*

$$|u(\phi)| \leq C \sup_{\substack{|p| \leq m+n \\ z \in K}} |D^p(\phi)(z)|, \quad \phi \in G_0^\infty(K).$$

Proof. First, it is worth keeping in mind that \mathcal{G}_L can be identified with $\mathbb{R}^{2^{L-1}}$ [31]. In fact, a number system assuming values in some Grassmann algebra with L generators is specified by 2^{L-1} real parameters. Let F and E be spaces of smooth functions with compact support defined on $K \subset U \subset \mathcal{G}_L^{m,n}$ and $K \subset U \subset \mathbb{R}^{2^{L-1}(m+n)}$, respectively. If we have a functorial relation $\lambda : F \rightarrow E$ and a linear functional $\tilde{u} : E \rightarrow \mathbb{R}^{2^{L-1}}$, we can compose λ with \tilde{u} to obtain the pullback of \tilde{u} by λ , i.e., $u = \lambda^* \tilde{u} = \tilde{u} \circ \lambda$, and hence a linear functional $\lambda^* \tilde{u} : F \rightarrow \mathbb{R}^{2^{L-1}}$. Then, the statement follows if \tilde{u} is continuous on E . But this clear from the Proposition 21.1 of [43], which can be applied *verbatim* for a functional \tilde{u} on E . \square

3.3 Distributions on a Supermanifold

Next we will obtain an extension of basic results about superdistributions on the flat superspace in the case of general supermanifolds.

DEFINITION 3.2. *Let \mathcal{M} a G^∞ supermanifold. For every coordinate system $p_i \circ k_\alpha$ in \mathcal{M} one has a distribution $u_{k_\alpha} \in \mathcal{D}'(\tilde{X}_\alpha)$ where \tilde{X}_α is an open from $\mathcal{G}_L^{m,n}$ such that*

$$u_{k_\beta} = \{(p_i \circ k_\alpha) \circ (k_\beta^{-1} \circ p_i^{-1})\}^* u_{k_\alpha}, \quad (i = 1, \dots, m+n), \quad (3.6)$$

in $k_\beta(X_\alpha \cap X_\beta)$, where p_i is a projection into each copies (i) from $\mathcal{G}^{m,n}$, such that $x_i = p_i \circ k_\alpha$ and $y_j = p_{j+m} \circ k_\alpha$, with $(i = 1, \dots, m; j = 1, \dots, n)$. We call the system u_{k_α} a distribution u in \mathcal{M} . The set of every distribution in \mathcal{M} is denoted by $\mathcal{D}'(\mathcal{M})$.

THEOREM 3.3. *Let $\tilde{X}_\alpha, \alpha \in I$, be an arbitrary family of open sets in $\mathcal{G}_L^{m,n}$, and set $\tilde{X} = \bigcup_{\alpha \in I} \tilde{X}_\alpha$. If $u_\alpha \in \mathcal{D}'(\tilde{X}_\alpha)$ and $u_\alpha = u_\beta$ in $(\tilde{X}_\alpha \cap \tilde{X}_\beta)$ for all $\alpha, \beta \in I$, then there exist one and only one $u \in \mathcal{D}'(\tilde{X})$ such that u_α is the restriction of u to \tilde{X}_α for every α .*

To prove this theorem, it is interesting to state the following results:

LEMMA 3.4. *Let $\tilde{X}_1, \dots, \tilde{X}_k$ be open sets in $\mathcal{G}_L^{m,n}$ and let $\phi \in G_0^\infty(\bigcup_1^k \tilde{X}_\alpha)$. Then one can find $\phi_\alpha \in G_0^\infty(\tilde{X}_\alpha), \alpha = 1, \dots, k$, such that $\phi = \sum_1^k \phi_\alpha$ and if $\phi \geq 0$ can take all $\phi_\alpha \geq 0$.*

Proof. We can choose compact sets K_1, \dots, K_k with $K_\alpha \subset \tilde{X}_\alpha$, so that the supp $\phi \subset \bigcup_1^k K_\alpha$. (every point in supp ϕ has a compact neighbourhood contained in some \tilde{X}_α , a finite number of such neighbourhoods can be chosen which cover all of supp ϕ . The union of those which belong to X_α is a compact set $K_\alpha \subset \tilde{X}_\alpha$. Now, if \tilde{X} is an open set in $\mathcal{G}_L^{m,n}$ and K is a compact subset, then one can find $\phi \in G_0^\infty(\tilde{X})$ with $0 \leq \phi \leq 1$ so that $\phi = 1$ in a neighbourhood of K . So, we can choose $\psi_\alpha \in G_0^\infty(\tilde{X}_\alpha)$ with $0 \leq \psi_\alpha \leq 1$ and $\psi_\alpha = 1$ in K_α , then the functions:

$$\phi_1 = \phi\psi_1, \phi_2 = \phi\psi_2(1 - \psi_1), \dots, \phi_k = \phi\psi_k(1 - \psi_1) \dots (1 - \psi_{k-1}) .$$

have the required properties since

$$\sum_1^k \phi_\alpha - \phi = -\phi \prod_1^k (1 - \psi_\alpha) = 0 ,$$

because either ϕ or some $1 - \psi_\alpha$ is zero at any point. \square

COROLLARY 3.5. *Let $\tilde{X}_1, \dots, \tilde{X}_k$ be open sets in $\mathcal{G}_L^{m,n}$ and K a compact subset $\subset \tilde{X}_\alpha$. Then one can find $\phi_\alpha \in G_0^\infty(\tilde{X}_\alpha)$ so that $\phi_\alpha \geq 0$ and $\sum_1^k \phi_\alpha \leq 1$ with equality in a neighbourhood of K . \square*

Proof of the Theorem 3.3. If u is a distribution, then:

$$u(\phi) = \sum u_\alpha(\phi_\alpha) , \quad \text{if } \phi = \sum \phi_\alpha \quad (\text{where } \phi_\alpha \in G_0^\infty(\tilde{X}_\alpha)) ,$$

and the sum is finite. By the Lemma 3.4, every $\phi \in G_0^\infty(\tilde{X})$ can be written as such a sum. If $\sum \phi_\alpha = 0 \Rightarrow \sum u_\alpha(\phi_\alpha) = 0$, then we conclude that $\sum u_\alpha(\phi_\alpha)$ is independent of how we choose the sum. Let $K = \bigcup \text{supp } \phi$ compact set $K \subset \tilde{X}$ and using the corollary 3.5, we can choose $\psi_\beta \in G_0^\infty(\tilde{X}_\beta)$ such that $\sum \psi_\beta = 1$ in K and the sum is finite. Then $\psi_\beta \phi_\alpha \in G_0^\infty(\tilde{X}_\alpha \cap \tilde{X}_\beta)$ so $u_\alpha(\psi_\beta \phi_\alpha) = u_\beta(\psi_\beta \phi_\alpha)$. Hence

$$\sum u_\alpha(\phi_\alpha) = \sum \sum u_\alpha(\phi_\alpha \psi_\beta) = \sum \sum u_\beta(\phi_\alpha \psi_\beta) = \sum u_\beta(\psi_\beta \sum \phi_\alpha) = 0 .$$

We have showed that if $\sum \phi_\alpha = 0 \Rightarrow \sum u_\alpha(\phi_\alpha)$ is zero, then u is unique. In order to show that u is distribution, choose a compact set $K \subset \tilde{X}$ and a function $\psi_\beta \in G_0^\infty(\tilde{X}_\beta)$ with $\sum \psi_\beta = 1$ in K and finite sum. If $\phi \in G_0^\infty(K)$ we have $\phi = \sum \phi \psi_\beta$ with $\phi \psi_\beta \in G_0^\infty(\tilde{X}_\beta)$ so that the first equation this proof gives

$$u(\phi) = \sum u_\beta(\phi \psi_\beta)$$

but, if u_β is a distribution, then:

$$|u_\beta(\phi \psi_\beta)| \leq C \sup_{\substack{|p| \leq m+n \\ z \in K}} |D^p(\phi \psi_\beta)(z)| , \quad \phi \psi_\beta \in G_0^\infty(\tilde{X}_\beta)$$

where $\sup D^p \phi$ can be estimated in terms of ϕ , and so we conclude that

$$|u(\phi)| \leq C \sup_{\substack{|p| \leq m+n \\ z \in K}} |D^p \phi(z)|, \quad \phi \in G_0^\infty(K).$$

This completes our proof. \square

THEOREM 3.6. *Let \mathcal{F} an atlas for \mathcal{M} . If for every $p_i \circ k \in \mathcal{F}$ one has a distribution $u_k \in \mathcal{D}'(\tilde{X}_k)$ and the above definition is true when $p_i \circ k$ and $p'_i \circ k'$ belongs to \mathcal{F} , then there is one, and only one, distribution $u \in \mathcal{D}'(\mathcal{M})$ such that $u \circ (k^{-1} \circ p_i^{-1}) = u_k$ for every $p_i \circ k \in \mathcal{F}$.*

Proof. Let $\psi \in G^\infty$ be a coordinate system in \mathcal{M} . The Theorem 3.3 states that there exists one, and only one, distribution $U_\psi \in \mathcal{D}'(\tilde{X}_\psi)$ in such a way for every $p_i \circ k$, $U_\psi = ((p_i \circ k) \circ \psi^{-1})^* u_k$ in $\psi(X_\psi \cap X_k) \subset \tilde{X}_\psi$. If $\psi \in \mathcal{F} \rightarrow U_\psi = u_\psi$, we can choose $p_i \circ k = \psi$. Now, one defines u as a distribution, since U_ψ satisfies (3.6) for both coordinate systems $p_i \circ k$ and $p'_i \circ k'$. \square

4 Algebraic Framework on a Supermanifold

In the usual treatment of quantum field theory in flat spacetime, the existence of a unitary representation of the restricted Poincaré group, \mathcal{P}_+^\uparrow , with generators P_μ fulfilling the spectral condition $\text{sp}P_\mu \subset \overline{V}_+$, is very essential. This unitary operator plays a key role in picking out a preferred vacuum state, i.e., a state which is invariant under all translations. We choose a complete system of physical states, with positive energies, just when it is possible to define this vacuum state and consequently the Fock Space, \mathcal{F} . One then defines observables as operators on \mathcal{F} which act upon the states. However, the characterization of the vacuum involves global aspects, and in the case of a curved spacetime it is not evident how to select a distinguished state. As already mentioned in the Introduction, due the absence of a *global* Poincaré group there is no analogous selection criterium on a curved spacetime: no vacuum state can be used as reference. To understand the significance of this point under another point of view, we take into account that, initially, a theory defined on a globally hyperbolic Lorentz manifold could be reduced to the tangent space at a given point, one neglecting the gravitational effects. One finds that the tangent space theory reduces to a free quantum field theory in a Minkowski space which has local translation invariance and a distinguished invariant state could be established by a *local* unitary mapping. Nevertheless, this unitary operator depends on the region and there exists no unitary operator which does the mapping for all open regions simultaneously. Therefore, the problem of how to characterize the physical states arises. For the discussion of this problem on a general manifold, the setting of the

so-called algebraic approach to quantum field theory (see [44, 6, 7]) is particularly appropriate, because it treats all states on equal footing, specially that states arising of unitarily inequivalent representations.

The algebraic approach involves the theory of $*$ -algebras and their states and Hilbert space representations. In this framework the basic objects are the algebras generated by observables localized in a given spacetime region. Fields are not mentioned in this setting and are regarded as a type of coordinates of the algebras. The basic assumption is that *all physical* information must already be encoded in the structure of the local observables. Haag and Kastler introduced a mathematical structure for the set of observables of a physical system by proposing the now so-called Haag-Kastler axioms [45] for nets of C^* algebras, later generalized by Dimock [46] for local observables to globally hyperbolic manifolds. Recently, a new approach to the model independent description of quantum field theories has been introduced Brunetti-Fredenhagen-Verch [47], which incorporates in a local sense the principle of general covariance of general relativity, thus giving rise to the concept of a locally covariant quantum field theory. The usual Haag-Kastler-Dimock framework can be re-gained from this new approach as a special case.

In this section, we intend to discuss the algebraic formalism so as to include supersymmetry on a supermanifold. A straight formulation on a supermanifold can be performed over the algebraic approach easily, since the construction of the algebra *does not depend “a priori” of the manifold*. Let us describe a physical theory in a general supermanifold from an extended formulation of the ordinary theory in curved spacetime. An observable algebra can be generated from $\Phi_{\text{sd}}(f_{\text{sf}})$, where Φ_{sd} are superdistributions (superfields) and f_{sf} test superfunctions. A complete superalgebra, like above, is represented by $\mathfrak{A}_{\text{sa}} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}_{\text{sa}}(\mathcal{O})}$, where \mathfrak{A}_{sa} denotes the superalgebra, with $\mathcal{O} \subset \mathcal{M}$ denoting a bounded open region on a supermanifold \mathcal{M} . We shall assume we have assigned to every bounded open region \mathcal{O} in \mathcal{M} the following properties:

- P.1 All $\mathfrak{A}_{\text{sa}}(\mathcal{O})$ are $*$ -superalgebras containing a common unit element, where it is assumed that the following condition of isotony holds:

$$\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathfrak{A}_{\text{sa}}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}_{\text{sa}}(\mathcal{O}_2) .$$

This condition expresses the fact that the set (which we call in an improper way) of supersymmetric “observables” increases with the size of the localization region.^e

- P.2 We define the essential notion of locality so that the restriction of a compact region $\mathcal{O} \in \mathcal{M}$ to a compact region of the body of the supermanifold, $\mathcal{O}_{\mathbf{b}} \in \mathcal{M}_{\mathbf{b}}$, is

^eCertainly the set of physically interesting observables are obtained taking the body.

causally separated from another compact region $\mathcal{O}'_{\mathbf{b}} \in \mathcal{M}_0$. This implies in the spacelike commutativity, $[\mathfrak{A}_{\text{sa}}(\mathcal{O}), \mathfrak{A}_{\text{sa}}(\mathcal{O}')] = 0$. We see that this requirement is important, because only with this restriction we can work with causality: the notion of a suitable proper time curve which intersects the Cauchy surface in a global hyperbolic spacetime makes sense only on the body manifold. So, there we can establish an evolution of Cauchy surfaces to give us a criterion to define a Hadamard form to the vacuum state. A superdistribution on a supermanifold as a two-point function shows us that the causality is well-defined in this context. Therefore, we now state: if $\mathcal{O}_{\mathbf{b}}$ is causally dependent on $\mathcal{O}'_{\mathbf{b}}$, then $\mathfrak{A}_{\text{sa}}(\mathcal{O}) \subset \mathfrak{A}_{\text{sa}}(\mathcal{O}')$.

P.3 Following Dimock [46], we require that there be an $\mathfrak{A}_{\text{sa}}(\mathcal{O})$ for each supermanifold \mathcal{M} equipped with some supermetric g , which generalizes the Lorentz metric, in a diffeomorphic class. Let $k : \mathcal{M}_0 \rightarrow \mathcal{M}'_0$ be a C^∞ diffeomorphism on the body manifold, such that $k^*(g'_0) = g_0$, where g_0 is a metric of signature $(+, -, -, -)$ of the body manifold. Then $z(k) : \mathcal{M} \rightarrow \mathcal{M}'$ is a G^∞ superdiffeomorphism $z(k)$ from (\mathcal{M}, g) to (\mathcal{M}', g') such that $z(k)^*(g') = g$, and there is an isomorphism $\alpha_{z(k)} : \mathfrak{A}_{\text{sa}} \rightarrow \widehat{\mathfrak{A}}_{\text{sa}}$ such that $\alpha_{z(k)}[\mathfrak{A}_{\text{sa}}(\mathcal{O})] = \widehat{\mathfrak{A}}_{\text{sa}}(z(k)(\mathcal{O}))$. One can also show that $z(\text{id}_{\mathcal{M}_0}) = \text{id}_{\mathcal{M}}$, where $\text{id}_{\mathcal{M}_0}$ ($\text{id}_{\mathcal{M}}$) are the identity functions on \mathcal{M}_0 (\mathcal{M}), respectively. Hence, $\alpha_{z(\text{id}_{\mathcal{M}_0})} = \alpha_{(\text{id}_{\mathcal{M}})}$ and, by eq.(2.7), we have $\alpha_{z(k_1)} \circ \alpha_{z(k_2)} = \alpha_{z(k_1 \circ k_2)}$.

It is interesting, in a particular way, choose a suitable $*$ -algebra for a formulation of quantum fields in connection to the Gårding-Wightman approach [48].^f In quantum field theory, it is natural to work with tensor product over test functions, since is usual the presence of more than one field. Therefore, we introduce a tensor algebra of smooth superfunctions of compact support over $\mathcal{O} \in \mathcal{M}$, where \mathcal{O} is an open region in a supermanifold. Let f_m be a test superfunction in $\mathfrak{D}_m(\mathcal{O})$, so that $F = \bigoplus_{m \in \mathbb{N}} f_m(z_1, \dots, z_m) \in \mathfrak{A}_{\text{sa}}(\mathcal{O})$, where here $z_i = (x_i, \theta_i, \bar{\theta}_i)$ denotes the supercoordinates. In a same way we take $\omega_m(z, \dots, z_m) \in \mathfrak{D}'_m(\mathcal{O})$, here \mathfrak{D}'_m is the dual space of \mathfrak{D}_m consisting of m -point superdistributions $\omega = \{\omega_m\}_{m \in \mathbb{N}}$, such that ω_m belongs to the dual algebra denoted by $\mathfrak{A}'_{\text{sa}}(\mathcal{O})$. As we are working on involutive superalgebras, let us define the operation of involution $(*)$ by $f_m^*(z_1, \dots, z_m) = \overline{f_m(z_m, \dots, z_1)}$, where $f_m^* = \overline{f_m}$ denotes the complex conjugation.

A superstate ω in this class of algebra is a normalized positive linear functional $\omega : \mathfrak{A}_{\text{sa}}(\mathcal{O}) \rightarrow \mathcal{G}_L$, with $\omega(F^*F) \geq 0$ for all $F \in \mathfrak{A}_{\text{sa}}(\mathcal{O})$. The normalization means that $\omega^0 = 1$. This net of algebra is the Borchers-Uhlmann one [52]. Such an algebra

^fSome work concerning the axiomatic supersymmetric quantum field theory is contained in [49, 50], where it is shown that the standard Wightman axioms of a relativistic quantum field theory can be modified so as to allow supersymmetry.

does not contain any specific dynamical information, which can be obtained by specifying a vacuum state on it. Once the vacuum state has been specified, through the GNS construction which fixes a Hilbert superspace and a vacuum vector, one can extract from the corresponding time-ordered, advanced or retarded superfunctions the desired information.

A superstate is said to satisfy the essential property of *local commutativity* if and only if for all $m \geq 2$ and all $1 \leq i \leq m - 1$ we have

$$\omega_m(f_1 \otimes \cdots \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_m) = \omega_m(f_1 \otimes \cdots \otimes f_{i+1} \otimes f_i \otimes \cdots \otimes f_m) ,$$

for all $f_i \in G_0^\infty(\mathcal{O})$, such that the restriction of each f_i on compact regions of the body of supermanifold implies that the $\text{supp } f_i|_{\mathcal{O}_b}$ and $\text{supp } f_{i+1}|_{\mathcal{O}_b}$ are spacelike separated. Furthermore, a superstate ω is “quasi-free” if the one-point superdistribution and all the truncated m -point superdistributions for $m \neq 2$ vanish, i.e., all m -point superdistributions are obtained from the two-point superdistribution via relation:

$$\begin{aligned} \omega_{2m+1}(f_1 \otimes \cdots \otimes f_m) &= 0 \quad \text{for } m \geq 0 , \\ \omega_{2m}(f_1 \otimes \cdots \otimes f_m) &= \sum_{\substack{i_1 < \cdots < i_{2m} \\ i_k < j_k \\ i_1, \dots, j_{2m} \text{ distinct}}} \omega_2(f_{i_1} \otimes f_{j_1}) \omega_2(f_{i_2} \otimes f_{j_2}) \cdots \omega_2(f_{i_{2m}} \otimes f_{j_{2m}}) , \end{aligned}$$

for $m \geq 1$.

It is a well-know result that the physical model can be described by the GNS construction, showing us how the Hilbert space is constructed and defining what are the operators (just the algebra representation) acting in this space. According to conventional prescription, for getting the Hilbert space we choose the quotient between the observable algebra and the ideal \mathcal{N}_ω (to guarantee the scalar product existence). In this stage the problem of several inequivalent representation persists. In flat superspaces, the super-Poincaré invariance of the vacuum state picks out the correct representation [49]. In general supermanifolds the case is more delicated; we will look for (super)Hadamard structures. This is motivated by the ordinary general manifold case. At last, we choose an acceptable Hilbert superspace from the algebraic properties via GNS construction by the following identification:

$$\omega_m(f_1 \otimes \cdots \otimes f_m) = \langle \Omega_\omega, \pi_\omega(f_1) \cdots \pi_\omega(f_m) \Omega_\omega \rangle ,$$

where here Ω_ω is a distinguished vector in Hilbert superspace, and π_ω is the representation of the elements $F \in \mathfrak{A}_{\text{sa}}(\mathcal{O})$ which play the role of self-adjoint linear operator acting in the Hilbert superspace over test superfunctions. In addition, we use the physical requirements on the body manifold in order to define whole set of

superstates which are supposed to be distinguished by a certain generalized form of the spectral condition [11].

Remark. The main features of Hilbert superspaces relevant for our purposes are summarized as follows: (i) when the Grassmann algebra \mathcal{G}_L is endowed with the Rogers norm, every Hilbert superspace is of the form $\mathcal{H} = \mathcal{H} \otimes \mathcal{G}_L$, where \mathcal{H} is an ordinary Hilbert space,[§] (ii) the \mathcal{G}_L -valued inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{G}_L$ respects the body operation $\langle x_{\mathbf{b}}, y_{\mathbf{b}} \rangle = \langle x, y \rangle_{\mathbf{b}}$ and $\langle x, x \rangle_{\mathbf{b}} \geq 0$ for all $x \in \mathcal{H}$, so that $x \in \mathcal{H}$ has nonvanishing body if and only if $\langle x, x \rangle_{\mathbf{b}} > 0$. For generalizations of some basic results of the theory of Hilbert space to Hilbert superspaces we refer to the recent paper [38] and references therein. \blacktriangle

5 Hadamard (Super)states

As already emphasized, the Hadamard state condition provides a framework in which we may improve our understanding to the problem concerning the determination of physically acceptable states. The motivation for we adopt the Hadamard structure of the vacuum state in curved spacetime quantum field theory is quite simple. In general, as we lost the possibility of pick out a good representation for the model due the fact that now we have not more an invariant structure over the action of an isometry group (in the flat case, the global Poincaré group), we must get another condition of choose. Since we are able to describe some aspects of a manifold observing the evolution of Cauchy surface (CS) coming from of asymptotic flat space, a new kind of invariance becomes natural, and this invariance arises from the preservation of some particular structure while the CS geometry is changing in determinated manifolds.

In particular, for states whose expectation values of the energy-momentum tensor operator can be defined by using the point separation prescription for renormalization, Fulling *et al.* [53] showed that if such states have a singularity structure of the Hadamard form in an open neighbourhood of a Cauchy surface, then they have their forms preserved independently of the Cauchy evolution. In this case, the states are said to have the Hadamard form if they can be expressed as

$$\Delta_{\text{Had}}(x_1, x_2) = \frac{U(x_1, x_2)}{\sigma(x_1, x_2)} + V(x_1, x_2) \ln|\sigma(x_1, x_2)| + W(x_1, x_2) ,$$

where $\sigma(x_1, x_2)$ is one-half of the square of the geodesic distance between x_1 to x_2 . In flat spacetime or in the $x_1 \rightarrow x_2$ limit in curved spacetime, $\sigma = \frac{1}{2}(x_1 - x_2)^2$. It is clear of this that $\text{sing supp } \Delta_{\text{Had}} = \{(x_1, x_2) \mid \sigma = \frac{1}{2}(x_1 - x_2)^2 = 0\}$ (we recall that

[§]The existence of a subspace \mathcal{H} of \mathcal{H} called a base Hilbert space is important in physical applications [51].

the singular support of a distribution $u \in \mathcal{D}'(X)$ is the smallest closed subset Y of X such that $u|_{X \setminus Y}$ is of class C^∞). U, V and W are regular functions for all choices of x_1 and x_2 . The functions U and V are geometrical quantities independent of the quantum state, and only W carries information about the state. Therefore, for free quantum field models in ordinary globally hyperbolic manifolds, the Hadamard form plays an important role: it is a strong candidate to describe an acceptable physical representation.

The search for the Hadamard form in the superspace case is simple, since the latter is, in general, obtainable by applying the function $\delta^2(\bar{\theta} - \bar{\theta}')$ (or $\delta^2(\theta - \theta')$) and an exponential structure $e^{E(\partial_x, \theta, \bar{\theta})}$ to the ordinary Hadamard form Δ_{Had} (see Proposition 7.3 below and [54, 55] for details), such that the singularity structure region is not affected.^h This issue is recaptured in Section 6. Since we can deal with a supermanifold which has a body manifold being a globally hyperbolic one (to guarantee this we just report to the construction of Bonora-Pasti-Tonin [28]), it is important to establish that only *projectively* superHadamard structures make sense. The obvious explanation for this statement is that the structure must cover the global time notion, and consequently the argument of causality, but over a supermanifold the notion of causal curves are not well defined unless projectively. The tool to extend the Hadamard structure to the supersymmetric environment arises from the fact that the existence and uniqueness of the Grassmannian continuation (z -continuation) for C^∞ functions is checked. By a body projection, we always get the ordinary Hadamard structure such that the latter must be invariant by CS evolution on the body manifold. This is a consistent result, since we will show in the next section, through an alternative and equivalent characterization of the Hadamard condition due Radzikowski [8] which involves the notion of the wavefront set of a superdistribution, that the structure of singularity is not changed and is condensed in the ordinary region of any Green superfunction, corroborating to the fact that only on the body of a supermanifold the causality makes sense.

6 Microlocal Analysis in Superspace

Important progress in understanding the significance of the Hadamard form relates it to Hörmander's concept of wavefront sets and microlocal analysis [8], in a particular way by the wavefront set of their two-point functions. It satisfies the Hadamard condition if its wavefront set contains only positive frequencies propagating forward in time and negative frequencies backward in time.

^hSee, for example, the textbook of Piguet and Sibold [56] where a comprehensive account on the renormalization of supersymmetric theories through the "algebraic" renormalization approach can be found.

The focus in this section will be on the extension of the Hörmander's description of the singularity structure (wavefront set) of a distribution to include the super-symmetric case. The well-known result that the singularities of a superdistribution may be expressed in a very simple way through the ordinary distribution is proved by functional analytical methods, in particular the methods of microlocal analysis formulated in superspace language.

6.1 Standard Facts on Microlocal Analysis

The study of singularities of solutions of differential equations is simplified and the results are improved by taking what is now known as microlocal analysis. This leads to the definition of the wavefront set, denoted (WF) , of a distribution, a refined description of the singularity spectrum. Similar notion was developed in other versions by Sato [57], Iagolnitzer [58] and Sjöstrand [59]. The definition, as known nowadays, is due to Hörmander. He used this terminology due to an existing analogy between his studies on the “propagation” of singularities and the classical construction of propagating waves by Huyghens.

The key point of the microlocal analysis is the transference of the study of singularities of distributions from the configuration space only to the rather phase space, by exploring in frequency space the decay properties of a distribution at infinity and the smoothness properties of its Fourier transform. For a distribution u we introduce its wavefront set $WF(u)$ as a subset in phase space $\mathbb{R}^n \times \mathbb{R}^n$.ⁱ We shall be thinking of points (x, k) in phase space as specifying those singular directions k of a “bad” behaviour of the Fourier transform \hat{u} at infinity that are responsible for the non-smoothness of u at the point x in position space. So we shall usually want $k \neq 0$. A relevant point is that $WF(u)$ is independent of the coordinate system chosen, and it can be described locally.

As it is well-known [60, 39], a distribution of compact support, $u \in \mathcal{E}'(\mathbb{R}^n)$, is a smooth function if, and only if, its Fourier transform, \hat{u} , rapidly decreases at infinity (i.e., as long as $\text{supp } u$ does not touch the singularity points). By a fast decay at infinity, one must understand that for all positive integer N exists a constant C_N , which depends on N , such that

$$|\hat{u}(k)| \leq (1 + |k|)^{-N} C_N, \quad \forall N \in \mathbb{N}; k \in \mathbb{R}^n. \quad (6.1)$$

If, however, $u \in \mathcal{E}'(\mathbb{R}^n)$ is not smooth, then the directions along which \hat{u} does not fall off sufficiently fast may be adopted to characterize the singularities of u .

For distributions does not necessarily of compact support, still we can verify if its Fourier transform rapidly decreases in a given region V through the technique of

ⁱThe functorially correct definition of phase space is $\mathbb{R}^n \times (\mathbb{R}^n)^*$. We shall here ignore any attempt to distinguish between \mathbb{R}^n and $(\mathbb{R}^n)^*$.

localization. More precisely, if $V \subset X \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(X)$, we can restrict u to a distribution $u|_V$ in V by setting $u|_V(\phi) = u(\phi)$, where ϕ is a smooth function with support contained in a region V , with $\phi(x) \neq 0$, for all $x \in V$. The distribution ϕu can then be seen as a distribution of compact support on \mathbb{R}^n . Its Fourier transform will be defined as a distribution on \mathbb{R}^n , and must satisfy, in absence of singularities in $V \in \mathbb{R}^n$, the property (6.1). From this point of view, all development is local in the sense that only the behaviour of the distribution on the arbitrarily small neighbourhood of the singular point, in the configuration space, is relevant.

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution and $\phi \in C_0^\infty(V)$ a smooth function with support $V \subset \mathbb{R}^n$. Then, ϕu has compact support. The Fourier transform of ϕu produces a smooth function in frequency space.

LEMMA 6.1. *Consider $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C_0^\infty(V)$. Then $\widehat{\phi u}(k) = u(\phi e^{-ikx})$. Moreover, the restriction of u to $V \subset \mathbb{R}^n$ is smooth on V if, and only if, for every $\phi \in C_0^\infty(V)$ and each positive integer N there exist a constant $C(\phi, N)$, which depends on N and ϕ , such that $|\widehat{\phi u}(k)| \leq (1 + |k|)^{-N} C(\phi, N)$, for all $N \in \mathbb{N}$ and $k \in \mathbb{R}^n$. \square*

If $u \in \mathcal{D}'(\mathbb{R}^n)$ is singular in x , and $\phi \in C_0^\infty(V)$ is $\phi(x) \neq 0$; then ϕu is also singular in x and has compact support. However, in some directions in k -space $\widehat{\phi u}$ until will be asymptotically limited. This is called the set of *regular directions* of u .

DEFINITION 6.2. *Let $u(x)$ be an arbitrary distribution, not necessarily of compact support, on an open set $X \subset \mathbb{R}^n$. Then, the set of pairs composed by singular points x in configuration space and by its associated nonzero singular directions k in Fourier space*

$$WF(u) = \{(x, k) \in X \times (\mathbb{R}^n \setminus \{0\}) \mid k \in \Sigma_x(u)\} , \quad (6.2)$$

is called **wavefront set** of u . $\Sigma_x(u)$ is defined to be the complement in $\mathbb{R}^n \setminus \{0\}$ of the set of all $k \in \mathbb{R}^n \setminus \{0\}$ for which there is an open conic neighbourhood M of k such that $\widehat{\phi u}$ rapidly decreases in M , for $|k| \rightarrow \infty$.

Remarks. We will now collect some basic properties of the wavefront set:

1. The $WF(u)$ is conic in the sense that it remains invariant under the action of dilatations, i.e., when we multiply the second variable by a positive scalar. This means that if $(x, k) \in WF(u)$ then $(x, \lambda k) \in WF(u)$ for all $\lambda > 0$.
2. From the definition of $WF(u)$, it follows that the projection onto the first variable, $\pi_1(WF(u)) \rightarrow x$, consists of those points that have no neighbourhood wherein u is a smooth function, and the projection onto the second variable, $\pi_2(WF(u)) \rightarrow \Sigma_x(u)$, is the cone around k attached to a such point denoting the set of high-frequency directions responsible for the appearance of a singularity at this point.

3. The wavefront set of a smooth function is the empty set.
4. For all smooth function ϕ with compact support $WF(\phi u) \subset WF(u)$.
5. For any partial linear differential operator P , with C^∞ coefficients, we have

$$WF(Pu) \subseteq WF(u) .$$

6. If u and v are two distributions belonging to $\mathcal{D}'(\mathbb{R}^n)$, with wavefront sets $WF(u)$ and $WF(v)$, respectively; then the wavefront set of $(u + v) \in \mathcal{D}'(\mathbb{R}^n)$ is contained in $WF(u) \cup WF(v)$.
7. If U, V are open set of \mathbb{R}^n , $u \in \mathcal{D}'(V)$, and $\chi : U \rightarrow V$ a diffeomorphism such that $\chi^*u \in \mathcal{D}'(U)$ is the distribution pulled back by χ , then $WF(\chi^*u) = \chi^*WF(u)$. ▲

One further result, which we merely state, is needed to complete this briefing on microlocal analysis.

THEOREM 6.3 (Wavefront set of pushforwards of a distribution). *Let $f : X \rightarrow Y$ be a submersion, and let $u \in \mathcal{E}'(X)$. Then*

$$WF(f_*u) \subset \{(f(x), \eta) \mid x \in X, (x, {}^t f'_x \eta) \in WF(u) \text{ or } {}^t f'_x \eta = 0\} ,$$

where ${}^t f'_x$ denotes the transpose matrix of the Jacobian matrix f'_x of f . □

6.2 Wavefront set of a Superdistribution

It is already well-known that the singularity structure of Feynman (or more precisely Wightman) superfunctions is completely associated with the “bosonic” sector of the superspace. Although claims exist that the result is completely obvious, we do not think that a clear proof is available in the literature, to the best of our knowledge. In fact, there is a certain gap in the scientific literature between the usual textbook presentation of the singularity structure of superfunctions and the very mathematical treatment based on microlocal analysis. The purpose of the present subsection is to fill this gap. As expected, our result confirms that the decay properties of an ordinary distribution hold also to the case of a superdistribution, i.e., no new singularity appear by taking into account the structure of the superspace.

LEMMA 6.4. *Let $X \subset \mathcal{G}_L^{m,0}$ be an open set, and u be a superdistribution on X taking values in \mathcal{G}_L , i.e., a linear functional $u : G_0^\infty(X) \rightarrow \mathcal{G}_L$. Let ϕ be a supersmooth function with compact support $K \subset X$. Then ϕu is also supersmooth on K , if its*

components $(\phi u)(\epsilon(x))$ are smooth on a compact set $K' \subset \Omega$, where Ω is the body of superspace. Therefore, the following estimate holds:

$$\left| \widehat{\phi u}(k) \right| \leq (1 + |k_{\mathbf{b}}|)^{-N} C(N, \phi) .$$

Indication of Proof. A schematic proof may be constructed along the lines suggested by DeWitt [26]: from Definition 2.4 follows that functions of x are in one-to-one correspondence with functions of $x_{\mathbf{b}}$; this implies that in working with integrals over $\mathcal{G}_L^{m,0}$ one may for many purposes proceed as if one were working over the body of superspace, $\Omega = \{(x, 0, 0) \in X \mid \epsilon(x) \in \mathbb{R}^m\}$. Because $\phi u(x)$ vanishes at infinity, independently of their souls, the contour in $\mathcal{G}_{L,0}^m$ may be displaced to coincide with Ω , without affecting the value of the integral. So, the theory of the Fourier transforms remains unchanged in form. For the sake of simplicity, we take the case for which $s(x) = (x - \epsilon(x))$ is a smooth singled-valued function of $\epsilon(x) = x_{\mathbf{b}}$ and $L = 2$ is the number of generators of $\mathcal{G}_2^{1,0}$. This implies

$$\begin{aligned} \widehat{\phi u}(k) &= \int dx e^{ikx} \phi u(x) \\ &= \int dx_{\mathbf{b}} e^{ik_{\mathbf{b}}x_{\mathbf{b}}} (\phi u(x_{\mathbf{b}}) + i x_{\mathbf{b}} \phi u(x_{\mathbf{b}}) k_{ij} \xi^i \xi^j) \\ &= \widehat{\phi u}(k_{\mathbf{b}}) + (\widehat{\phi u})'(k_{\mathbf{b}}) k_{ij} \xi^i \xi^j . \end{aligned}$$

The proof follows one making use of repeated integrations-by-parts generalizing the fact $-i k_{\mathbf{b}}^{-1} \left(\frac{d}{dx_{\mathbf{b}}} e^{ik_{\mathbf{b}}x_{\mathbf{b}}} \right) = e^{ik_{\mathbf{b}}x_{\mathbf{b}}}$

$$\widehat{\phi u}(k) = \frac{(i)^{|\beta|}}{k_{\mathbf{b}}^{\beta}} \left\{ \int dx_{\mathbf{b}} e^{-ik_{\mathbf{b}}x_{\mathbf{b}}} (D_{x_{\mathbf{b}}}^{\beta} (\phi u(x_{\mathbf{b}})) + D_{x_{\mathbf{b}}}^{\beta} (x_{\mathbf{b}} \phi u(x_{\mathbf{b}})) k_{ij} \xi^i \xi^j) \right\} .$$

Taking the absolute value of both sides and using the Banach algebra property of \mathcal{G}_L , we get the estimate:

$$\begin{aligned} \left| \widehat{\phi u}(k) \right| &\leq \left| \widehat{\phi u}(k_{\mathbf{b}}) \right| + \left| (\widehat{\phi u})'(k_{\mathbf{b}}) \right| |k_{ij}| \\ &\leq (1 + |k_{\mathbf{b}}|)^{-|\beta|} \left(\sup_{\substack{|\beta| \leq m \\ x_{\mathbf{b}} \in K'}} |D_{x_{\mathbf{b}}}^{\beta} (\phi u(x_{\mathbf{b}}))| + \sup_{\substack{|\beta| \leq m \\ x_{\mathbf{b}} \in K'}} |D_{x_{\mathbf{b}}}^{\beta} (x_{\mathbf{b}} \phi u(x_{\mathbf{b}}))| |k_{ij}| \right) . \end{aligned} \tag{6.3}$$

This inequality clearly implies our assertion. Hence, in order that (6.3) be smooth, we only need that $\widehat{\phi u}(k)$ be rapidly decreasing as $|k_{\mathbf{b}}| \rightarrow \infty$. The proof may be

generalized to include the case in which $s(x)$ is a multi-valued function of the body and L is finite arbitrarily. We finish the proof by observing that as expected the soul part of k has a polynomial behaviour. \square

LEMMA 6.5. *By replacing $\mathcal{G}_L^{m,0}$ by $\mathcal{G}_L^{m,n}$ in the Lemma 6.4, then in this case the following estimate holds:*

$$\left| \widehat{\phi u}(k, \theta, \bar{\theta}) \right| \leq (1 + |k_{\mathbf{b}}|)^{-N} C(N, \phi_{(\gamma)}) \|\theta_1\| \|\bar{\theta}_1\| \cdots \|\theta_n\| \|\bar{\theta}_n\| .$$

Proof. First, we note that both u and ϕ are G^∞ superfunctions which can be expanded as a polynomial in the odd coordinates whose coefficients are functions defined over the even coordinates.

$$u(x, \theta, \bar{\theta}) = \sum_{(\gamma)=0}^{\Gamma} z(u_{(\gamma)})(x)(\theta)^{(\gamma)} \quad \text{and} \quad \phi(x, \theta, \bar{\theta}) = \sum_{(\gamma)=0}^{\Gamma} z(\phi_{(\gamma)})(x)(\theta)^{(\gamma)} .$$

Then, the proof follows essentially by similar arguments to the proof of the previous lemma, taking into account the polynomial behaviour of odd variables, θ and $\bar{\theta}$. In fact, $\phi u(x, \theta, \bar{\theta})$ is *linear* function in each odd coordinates separately, because each odd coordinate is nilpotent, and no higher power of a odd coordinate can appear, i.e., $\phi u(x, \theta, \bar{\theta})$ is analytic in the odd coordinates. This suggests that to take the Fourier transform of $\phi u(x, \theta, \bar{\theta})$ on the even variables must be sufficient to infer on the smoothness properties of $\phi u(x, \theta, \bar{\theta})$:

$$\begin{aligned} \widehat{\phi u}(k, \theta, \bar{\theta}) &= \sum_{(\gamma)=0}^{\Gamma} \sum_{(\mu)=0}^L (\widehat{\phi u})_{(\gamma),(\mu)}(k_{\mathbf{b}})(\xi)^{(\mu)}(\theta)^{(\gamma)} \\ &= \sum_{(\gamma)=0}^{\Gamma} \left[\int dx_{\mathbf{b}} e^{ik_{\mathbf{b}}x_{\mathbf{b}}} ((\phi u)_{(\gamma)}(x_{\mathbf{b}}) + i x_{\mathbf{b}}(\phi u)_{(\gamma)}(x_{\mathbf{b}}) k_{ij} \xi^i \xi^j + \cdots) \right] (\theta)^{(\gamma)} . \end{aligned} \tag{6.4}$$

Then, taking the absolute value of both sides of (6.4), we obtain from the Banach algebra property of \mathcal{G}_L and for each integer N the estimate:

$$\begin{aligned} \left| \widehat{\phi u}(k, \theta, \bar{\theta}) \right| &= \left| \sum_{(\gamma)=0}^{\Gamma} \sum_{(\mu)=0}^L (\widehat{\phi u})_{(\gamma),(\mu)}(k_{\mathbf{b}})(\xi)^{(\mu)}(\theta)^{(\gamma)} \right| \\ &\leq \sum_{(\gamma)=0}^{\Gamma} \sum_{(\mu)=0}^L \left| (\widehat{\phi u})_{(\gamma),(\mu)}(k_{\mathbf{b}}) \right| \|(\theta)^{(\gamma)}\| \\ &\leq (1 + |k_{\mathbf{b}}|)^{-N} C(N, \phi_{(\gamma)}) \|\theta_1\| \|\bar{\theta}_1\| \cdots \|\theta_n\| \|\bar{\theta}_n\| . \end{aligned} \tag{6.5}$$

This proves the lemma. \square

So, the odd sector of superspace does not produce any effect on the singular structure of u . Combining the results above, we have proved:

THEOREM 6.6. *The singularities of a superdistribution u are located at specific values of the body of x , the coordinates of the **physical spacetime**, independently of the odd coordinates.* \square

Comment. That the body of the superspace is responsible for carrying all its singular structure is not too surprising. Apparently, there exists no reason to have superspaces whose topological properties are substantially different from its body, which is responsible for carrying all observables, reflecting some measurable properties of the model. \blacktriangle

We sum up the preceding discussion as follows:

DEFINITION 6.7 (Wavefront Set of a Superdistribution). *The wavefront set $WF(u)$ of a superdistribution u in a superspace \mathcal{M} is the complement of the set of all regular directed points in the cotangent bundle $T^*\mathcal{M}_0$, where $\mathcal{M}_0 = \epsilon(\mathcal{M})$ is the body of superspace, excluding the trivial point $k_{\mathbf{b}} = 0$.*

There is a more precise version of Definition 6.7. As we have seen in Section 3 all of the foregoing definitions and statements about supermanifolds may be converted into corresponding definitions and statements about ordinary manifolds, since associated with a supermanifold \mathcal{M} of dimension (m, n) is a family of ordinary manifolds, of dimensions $N = 2^{L-1}(m+n)$, ($L = 1, 2, \dots$). The resulting manifold is called the L th skeleton of \mathcal{M} and denoted by $\mathcal{S}_L(\mathcal{M})$ [26]. With the aid of the family of skeletons we can define the pushforward (or direct image) of a superdistribution. Let $X \subset \mathcal{S}_L(\mathcal{M})$ and $Y \subset \mathcal{M}_0$ be open sets and let ϵ be the natural projection from $\mathcal{S}_L(\mathcal{M})$ (or \mathcal{M}) to \mathcal{M}_0 , the body map. If we introduce local coordinates $x = (x_1, \dots, x_N)$ in X , then Y is defined by $x_{\mathbf{b}} = (x_1, \dots, x_m)$. There is a local relationship between the body and the skeletons given by

$$\mathcal{S}_L(X) \stackrel{\text{diff.}}{=} Y \times \mathbb{R}^{2^{L-1}(m+n)-m} .$$

Now, let u be a superdistribution on X , then the pushforward ϵ_*u defined by $\epsilon_*u(\varphi) = u(\epsilon^*\varphi)$, $\varphi \in C_0^\infty(Y)$, it is a superdistribution on Y . Using these concepts, we can establish the following

COROLLARY 6.8. *Let $\epsilon : X \subset \mathcal{S}_L(\mathcal{M}) \rightarrow Y \subset \mathcal{M}_0$ be the body projection, and let $u \in \mathcal{D}'(X)$. Then*

$$WF(\epsilon_*u) \subset \{(x_{\mathbf{b}}, k_{\mathbf{b}}) \in T^*\mathcal{M}_0 \setminus 0 \mid \exists x' = (x_{m+1}, \dots, x_{N'}), (x_{\mathbf{b}}, x', k_{\mathbf{b}}, 0) \in WF(u)\} ,$$

where $N' = 2^{L-1}(m+n) - m$.

Proof. If $x = (x_{\mathbf{b}}, x')$, where $x_{\mathbf{b}} \in Y$, $x' \in \mathbb{R}^{N'}$ and $\epsilon : X \rightarrow Y$ is the body map, then the Jacobian matrix is of the form $\epsilon'_x = (1, 0)$ and the statement follows by Theorem 6.3. Thus, with any superspace \mathcal{M} and body of superspace \mathcal{M}_0 the singularities of a superdistribution $\epsilon_* u$ are located in a natural way in the set of projections of those points of the wavefront set of the superdistribution u where singular directions are parallel to the $x_{\mathbf{b}}$ -axis. \square

Example. For the model of Wess-Zumino, which consist of a chiral superfield Φ in self-interaction, the Feynman superpropagators, in flat superspace, are [54]:

$$\begin{aligned}\Delta_{\Phi\Phi}^{\text{F}}(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') &= -i m \delta^2(\theta - \theta') e^{i(\theta\sigma^\mu\bar{\theta} - \theta'\sigma^\mu\bar{\theta}')\partial_\mu} \Delta_{\text{F}}(x - x'), \\ \Delta_{\Phi\bar{\Phi}}^{\text{F}}(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') &= e^{i(\theta\sigma^\mu\bar{\theta} + \theta'\sigma^\mu\bar{\theta}' - 2\theta\sigma^\mu\bar{\theta}')\partial_\mu} \Delta_{\text{F}}(x - x'), \\ \Delta_{\bar{\Phi}\bar{\Phi}}^{\text{F}}(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') &= i m \delta^2(\bar{\theta} - \bar{\theta}') e^{-i(\theta\sigma^\mu\bar{\theta} - \theta'\sigma^\mu\bar{\theta}')\partial_\mu} \Delta_{\text{F}}(x - x'),\end{aligned}\tag{6.6}$$

where $\delta^2(\theta - \theta') = (\theta - \theta')^2$, with $x, \theta, \bar{\theta}$ having the form (2.2) and (2.3), respectively. According to our analysis, the wavefront set of Feynman superpropagators have the form,

$$WF(\Delta_{\text{susy}}^{\text{F}}) = \{(x_{\mathbf{b}}, k_{\mathbf{b}}; x'_{\mathbf{b}}, -k'_{\mathbf{b}}; x, 0; x', 0) \mid (x_{\mathbf{b}}, k_{\mathbf{b}}; x'_{\mathbf{b}}, -k'_{\mathbf{b}}) \in WF(\Delta_{\text{susy}}^{\text{F}}|_{\mathcal{M}_0})\},$$

where $\text{susy} = (\Phi\Phi; \bar{\Phi}\Phi; \bar{\Phi}\bar{\Phi})$, $x = (x_{m+1}, \dots, x_{N'})$, $x' = (x'_{m+1}, \dots, x'_{N'})$, $\Delta_{\text{susy}}^{\text{F}}|_{\mathcal{M}_0} \equiv \epsilon_* \Delta_{\text{susy}}^{\text{F}}$ is the direct image of Feynman superpropagators on the body of superspace, and $WF(\Delta_{\text{susy}}^{\text{F}}|_{\mathcal{M}_0}) \subset O \cup D$ [8], with the off-diagonal piece given by

$$\begin{aligned}O = \{(x_{\mathbf{b}}, k_{\mathbf{b}}; x'_{\mathbf{b}}, -k'_{\mathbf{b}}) \in T^*\mathcal{M}_0^2 \mid (x_{\mathbf{b}}, k_{\mathbf{b}}) \sim (x'_{\mathbf{b}}, k'_{\mathbf{b}}), x_{\mathbf{b}} \neq x'_{\mathbf{b}}, \\ k_{\mathbf{b}} \in \bar{V}_{\pm} \text{ if } x_{\mathbf{b}} \in J_{\pm}(x'_{\mathbf{b}})\},\end{aligned}$$

where the equivalence relation $(x_{\mathbf{b}}, k_{\mathbf{b}}) \sim (x'_{\mathbf{b}}, k'_{\mathbf{b}})$ means that there is a lightlike geodesic γ connecting $x_{\mathbf{b}}$ and $x'_{\mathbf{b}}$, such that at the point $x_{\mathbf{b}}$ the covector $k_{\mathbf{b}}$ is tangent to γ and $k'_{\mathbf{b}}$ is the vector parallel transported along the curve γ at $x'_{\mathbf{b}}$ which is again tangent to γ .

The diagonal piece is given by

$$D = \{(x_{\mathbf{b}}, k_{\mathbf{b}}; x_{\mathbf{b}}, -k_{\mathbf{b}}) \in T^*\mathcal{M}_0^2 \setminus 0 \mid x_{\mathbf{b}} \in \mathcal{M}_0, k_{\mathbf{b}} \in T^*\mathcal{M}_0^2 \setminus 0\}.$$

For this reason, the Feynman superpropagators are singular only for pairs of points on the body of superspace that can be connected by a lightlike geodesic. \blacktriangle

We end this section quoting the main lesson on the microlocal analysis that we can use, i.e., the one about how the wavefront set may be lifted from superdistributions on open sets of $\mathcal{G}_L^{m,n}$ to superdistributions on a smooth supermanifold \mathcal{M} . Such

an extension can be achieved in analogy with the ordinary case. Let \mathcal{O} be an open neighbourhood of $z \in \mathcal{M}$, which is assumed without loss generality to be covered by a single coordinate patch, and $u \in \mathfrak{D}'(\mathcal{O})$ be a superdistribution. Then, there exists a diffeomorphism $\chi : \mathcal{O} \rightarrow U \subset \mathcal{G}_L^{m,n}$, so that $\chi^*u \in \mathfrak{D}'(U)$ is the superdistribution pulled back by χ . Therefore $WF(\chi^*u) = \chi^*WF(u)$. Now, let ϕ be a supersmooth function with compact support contained within \mathcal{O} with $\phi(z) \neq 0$ – one should keep always in mind that each component $\phi_{(\gamma)}(\epsilon(x))$ of $\phi(z)$ is a smooth function and with support contained within $\mathcal{O}_{\mathbf{b}}$, where $\mathcal{O}_{\mathbf{b}}$ denotes an open neighbourhood of $x_{\mathbf{b}} \in \mathcal{M}_0$. Hence, the superdistribution $u\phi$ can be seen as a superdistribution on $\mathcal{G}_L^{m,n}$ which is of compact support, and given that there are no points belonging to the $WF(u)$, the Fourier transform, $\widehat{u\phi}$, of $u\phi$ is well defined as a superdistribution on $\mathcal{G}_L^{m,n}$ and satisfies the Lemma 6.5.

7 A Type of Microlocal Spectral Condition

We come back to the question of the Hadamard superstates. As repeatedly stated in this paper, Hadamard states have acquired a prominent status in connection with the spectral condition, and are recognized as defining the class of physical states for quantum field theories on a globally hyperbolic spacetime. Important progress in understanding the significance of Hadamard states was achieved by Radzikowski (with some gaps filled by Köhler [10]) who succeeded in characterizing the class of these states in terms of the wavefront set of their two-point function ω_2 satisfying a certain condition. He called this condition the wavefront set spectral condition (WFSSC). He proposed that a quasifree state ω of the Klein-Gordon field over a globally hyperbolic manifold is a Hadamard state if and only if its two-point distribution ω_2 has wavefront set

$$WF(\omega_2) = \{(x_1, k_1); (x_2, k_2) \in T^*\mathcal{M}_0^2 \setminus \{0\} \mid (x_1, k_1) \sim (x_2, -k_2) \text{ and } k_1^0 \geq 0\} , \quad (7.1)$$

so that x_1 and x_2 lie on a single null geodesic γ , $(k_1)^\mu = g^{\mu\nu}(k_1)_\nu$ is tangent to γ and future pointing, and when k_1 is parallel transported along γ from x_1 to x_2 yields $-k_2$. If $x_1 = x_2$, we have $k_1^2 = 0$ and $k_1 = k_2$. Radzikowski in fact showed that this condition is similar to the spectral condition of axiomatic quantum field theory [48].

Note that equation (7.1) restricts the singular support of $\omega_2(x_1, x_2)$ to points x_1 and x_2 which are null related. Hence, ω_2 must be smooth for all other points. This is known to be true for theory of quantized fields on Minkowski space for space-like related points. The key is the Bargman-Hall-Wightman theorem which shows that this is obtainable by applying complex Lorentz transformations to the primitive domain of analyticity determined by the spectral condition. However, a similar prediction on

the smoothness does not exist for time-like related points. Radzikowski suggested to extend the right-hand side of equation (7.1) to all causally related points, in order to include possible singularities at time-like related points.

The microlocal characterization of Hadamard states may be applied equally well to a n -point function, with $n > 2$. This generalization was achieved by Brunetti *et al.* [11]. They suggested a prescription which we recall now. Let \mathcal{G}_m denote the set of all finite graphs,^j into some Lorentz manifold \mathcal{M}_0 , whose vertices represent points in the set $V = \{x_1, \dots, x_m\} \in \mathcal{M}_0$, and whose edges e represent connections between pairs x_i, x_j by smooth curves (geodesics) $\gamma(e)$ from x_i to x_j . To each edge e one assigns a covariantly constant causal covector field k_e which is future directed if $i < j$, but not related to the tangent vector of the curve. If e^{-1} denotes the edge with opposite direction as e , then the corresponding curve $\gamma(e^{-1})$ is the inverse of $\gamma(e)$, which carries the momentum $k_{e^{-1}} = -k_e$.

DEFINITION 7.1 (μ SC [11]). *A state ω with m -point distribution ω_m is said to satisfy the Microlocal Spectral Condition if, and only if, for any m*

$$WF(\omega_m) \subseteq \Gamma_m ,$$

where Γ_m is the set $\{(x_1, k_1), \dots, (x_m, k_m)\}$ for which there exists a graph G_m as described above with $k_i = \sum k_e(x_i)$ where the sum runs over all edges which have the point x_i as their sources. The trivial momentum configuration $k_1 = \dots = k_m = 0$ is excluded.

Passing from a smooth manifold to a smooth supermanifold, it seems reasonable to require that a superstate satisfies a certain type of microlocal spectrum condition. A completely analogous statement to the Definition 7.1 can be achieved, once more with the aid of the family of skeletons, $\mathcal{S}_L(\mathcal{M})$, and the graph theory. Let \mathcal{G}_r be a set of finite “*supergraphs*,” into some $\mathcal{S}_L(\mathcal{M})$, whose vertices represent points in the set $V = \{x_1, \dots, x_r\} \in \mathcal{S}_L(\mathcal{M})$. Locally the traditional notion of a supergraph drawing is that its vertices are represented by points in the hyperplane $\mathbb{R}^{2^{L-1}(m+n)}$, its edges are represented by curves – that are piecewise linear – between these points, and different curves meet only in common endpoints. If $\epsilon_0 : \mathbb{R}^{2^{L-1}(m+n)} \rightarrow \mathbb{R}^m$ is the canonical projection, then $\tilde{G} = \epsilon_0 G$ is a graphy composed by the projection of those points of a supergraph whose edges e represent connections between pairs $x_{\mathbf{b}_i}, x_{\mathbf{b}_j} \in \mathbb{R}^m$ by curves from $x_{\mathbf{b}_i}$ to $x_{\mathbf{b}_j}$. Then, according to Brunetti *et al* [11], an immersion of a graph \tilde{G} into the body manifold \mathcal{M}_0 is an assignment of vertices of \tilde{G} to points in \mathcal{M}_0 , and of the edges of \tilde{G} to piecewise smooth curves in \mathcal{M}_0 , $e \rightarrow \gamma(e)$ with

^jA graph is a pair $G = (V, E)$, where the elements of V are the *vertices* (or *nodes*, or *points*) of the graph G , and the elements of E are its *edges* (or *lines*). The numbers of vertices of a graph is its order, and graphs are finite or infinite according to their order.

source $s(\gamma(e)) = x_{\mathbf{b}}(s(e))$ and target $t(\gamma(e)) = x_{\mathbf{b}}(t(e))$, respectively, together with a covariantly constant causal covector field $k_{\mathbf{b}_e}$ on γ such that: (i) if e^{-1} denotes the edge with opposite direction as e , then the corresponding curve $\gamma(e^{-1})$ is the inverse of $\gamma(e)$; (ii) for every edge e the covector $k_{\mathbf{b}_e}$ is directed toward future if $x_{\mathbf{b}}(s(e)) < x_{\mathbf{b}}(t(e))$; (iii) $k_{\mathbf{b}_{e^{-1}}} = -k_{\mathbf{b}_e}$. Using this construction, we establish:

DEFINITION 7.2. A superstate ω^{susy} with r -point superdistribution ω_r^{susy} is said to satisfy a type of Microlocal Spectral Condition if, and only if, for any r

$$WF(\omega_r^{\text{susy}}) = \left\{ (x_{\mathbf{b}_1}, x'_1, k_{\mathbf{b}_1}, 0); \dots; (x_{\mathbf{b}_r}, x'_r, k_{\mathbf{b}_r}, 0) \mid WF(\epsilon_* \omega_r^{\text{susy}}) \subseteq \tilde{\Gamma}_r \right\},$$

where $\tilde{\Gamma}_r$ is the set $\{(x_{\mathbf{b}_1}, k_{\mathbf{b}_1}); \dots; (x_{\mathbf{b}_r}, k_{\mathbf{b}_r})\}$ for which there exists a graph \tilde{G} as described above with $k_{\mathbf{b}_i} = \sum k_{\mathbf{b}_e}(x_{\mathbf{b}_i})$ where the sum runs over all edges which have the point $x_{\mathbf{b}_i}$ as their sources. The trivial momentum configuration $k_{\mathbf{b}_1} = \dots = k_{\mathbf{b}_r} = 0$ is excluded.

Remarks. We would like to call attention to two important points:

- The Definition 7.2 is in agreement with the DeWitt's remark which asserts that, in physical applications of supersymmetric quantum field theories, the spectral condition of the GNS-Hilbert superspace is restricted to the ordinary GNS-Hilbert space that sits inside the GNS-Hilbert superspace.
- The Definition 7.2 provides us with a “global” microlocal spectral condition. In our setting the word “global” means that the singular support of all component fields is embodied in $WF(\epsilon_* \omega_m^{\text{susy}})$. This is typical feature of supersymmetric theories in superspace language. For instance, for the chiral superfield of Wess-Zumino [54], in analogy to the scalar component field, the Hadamard condition for a spinorial component field is formulated in terms of its two-point distribution ω_2 . The latter are obtainable by applying the adjoint of the spinorial operator to a suitable auxiliary Hadamard state of the squared spinorial equation. For fixed spinor indices the wavefront set of the latter is contained in r.h.s. of equation (7.1) and derivatives do not enlarge the wavefront set. \blacktriangle

Next we give an example of an application of our definition. We restrict ourselves to the simplest case of massive chiral/antichiral fields of the Wess-Zumino model in flat superspace, leaving other cases as the Wess-Zumino model, or supersymmetric gauge theories in curved superspace for future works.

★ The Free Wess-Zumino Model in Flat Superspace

The simplest $N = 1$ supersymmetric model in four dimension is the free model of Wess-Zumino [54], which consists of a chiral superfield $\Phi(x, \theta, \bar{\theta})$, resp. antichiral superfield $\bar{\Phi}(x, \theta, \bar{\theta})$, obeying the differential constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$, resp. $D_{\alpha}\bar{\Phi} = 0$. As usual,

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} - i\sigma_{\alpha\dot{\alpha}}^{\mu}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^{\alpha}\sigma_{\alpha\dot{\alpha}}^{\mu}\partial_{\mu}, \quad (7.2)$$

is a supersymmetric covariant derivatives. Our notations and conventions are those of [56]. The elements of the $N = 1$ superspace are parametrized by even and odd coordinates $z^M = (x^{\mu}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$, with $\mu = (0, \dots, 3)$, $\alpha = (1, 2)$, $\dot{\alpha} = (\dot{1}, \dot{2})$, where θ and its complex conjugate $\bar{\theta}$, are odd coordinates and by construction they anticommute with each other. In this case the body manifold is \mathbb{R}^m and the body map is the augmentation map $\epsilon : \mathcal{G}_L^{m,n} \rightarrow \mathbb{R}^m$.

The superfield $\Phi(z)$ is a function mapping superspace into the even part of a Grassmann algebra [25]. With the help of the commutation rule $\bar{D}_{\dot{\alpha}}(e^{-i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}}\phi) = e^{-i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}}(-\partial/\partial\bar{\theta}^{\dot{\alpha}})\phi$, the chiral superfield can be expanded in powers of the odd coordinates as

$$\Phi(z) = e^{-i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}}(\varphi(x) + \theta\psi(x) + \theta^2 F(x)), \quad (7.3)$$

with $\varphi \stackrel{\text{def}}{=} 2^{-1/2}(A + iB)$ and $F \stackrel{\text{def}}{=} 2^{-1/2}(D - iE)$. A , B and ψ are respectively the scalar, pseudoscalar and spin-1/2 physical component fields of Φ , whereas D and E are their scalar and pseudoscalar auxiliary components. The latter are necessary for a classical off-shell closure of the supersymmetry algebra (they do not corresponding to propagating degrees of freedom in that appear through non-derivative terms).

As above, the antichiral superfield $\bar{\Phi}(z)$, with the help of the commutation rule $D_{\alpha}(e^{i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}}\phi) = e^{i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}}(\partial/\partial\theta^{\alpha})\phi$, can be expanded in component fields:

$$\bar{\Phi}(z) = e^{i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}}(\varphi^*(x) + \bar{\theta}\bar{\psi}(x) + \bar{\theta}^2 F^*(x)). \quad (7.4)$$

The quantum version of the Wess-Zumino model is based on the classical field equations

$$\frac{1}{16}\bar{D}^2\bar{\Phi} + \frac{m}{4}\bar{\Phi} = 0, \quad \frac{1}{16}D^2\Phi + \frac{m}{4}\Phi = 0. \quad (7.5)$$

Applying the operator D^2 to the first equation (resp. \bar{D}^2 to the second equation), multiplying the second equation by $4m$ (resp. the first equation), and using the commutation relation $[D^2, \bar{D}^2] = 8iD\sigma^{\mu}\bar{D}\partial_{\mu} + 16\Box$; one may combine them in order to find

$$(\Box_x + m^2)\Phi = 0, \quad (\Box_x + m^2)\bar{\Phi} = 0. \quad (7.6)$$

To our classical superfields Φ and $\bar{\Phi}$, we associate quantum superfields, an

operator-valued “superdistributions,” smeared with “supertest” functions,

$$\begin{aligned} F(z) &= e^{-i\theta\sigma^\mu\bar{\theta}\partial_\mu}(f(x) + \theta\chi(x) + \theta^2h(x)) , \\ \bar{F}(z) &= e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}(f^*(x) + \bar{\theta}\bar{\chi}(x) + \bar{\theta}^2h^*(x)) , \end{aligned} \quad (7.7)$$

with $F(z), \bar{F}(z) \in G_0^\infty(U, \mathcal{G}_L)$, the \mathcal{G}_L -valued superfunctions on an open set $U \subset \mathcal{G}_L^{m,n}$ which have compact support.

For all $F(z), G(z) \in G_0^\infty(U, \mathcal{G}_L)$, we define the commutation relations

$$\begin{aligned} [\Phi(\bar{F}), \Phi(\bar{G})] &= \int d\mu(z)d\mu(z') \Delta_{\Phi\Phi}^{\text{PJ}}(z, z')\bar{F}(z)\bar{G}(z') , \\ [\bar{\Phi}(F), \Phi(\bar{G})] &= \int d\mu(z)d\mu(z') \Delta_{\bar{\Phi}\Phi}^{\text{PJ}}(z, z')F(z)\bar{G}(z') , \\ [\bar{\Phi}(F), \bar{\Phi}(G)] &= \int d\mu(z)d\mu(z') \Delta_{\bar{\Phi}\bar{\Phi}}^{\text{PJ}}(z, z')F(z)G(z') . \end{aligned} \quad (7.8)$$

where $d\mu(z) \stackrel{\text{def}}{=} d^8z = d^4xd^2\theta d^2\bar{\theta}$. We call $\Delta_{\Phi\Phi}^{\text{PJ}}$, $\Delta_{\bar{\Phi}\Phi}^{\text{PJ}}$ and $\Delta_{\bar{\Phi}\bar{\Phi}}^{\text{PJ}}$ the Pauli-Jordan superdistributions, fundamental solutions of the homogeneous equations (7.6). In fact they are two-point distributions, elements of $\mathfrak{D}'(U)$.

The vacuum expectation value of the product $\Phi(F)\Phi(G)$ satisfies the relation

$$(\Omega, \Phi(F)\Phi(G)\Omega) = (w_2^{\text{susy}}(z, z'), F(z)G(z')) . \quad (7.9)$$

The distribution $w_2^{\text{susy}}(z, z')$ extends the Wightman formalism. For this reason, we call $w_2^{\text{susy}}(z, z')$ Wightman superdistribution of two-points.

The Wightman superdistribution of n -points will be symbolically written under the form [49]:

$$w_n^{\text{susy}}(z_1, \dots, z_n) = (\Omega, \Phi(x_1; \theta_1, \bar{\theta}_1) \dots \Phi(x_n; \theta_n, \bar{\theta}_n) \Omega) , \quad (7.10)$$

and

$$w_n^{\text{susy}}(F_n) = \int \prod_{i=1}^n d\mu_i w_n^{\text{susy}}(z_1, \dots, z_n) F_n(z_1, \dots, z_n) . \quad (7.11)$$

In this definition, we have fixed the order in which we take the superdistribution and the supertest function.

PROPOSITION 7.3. – *The two-point Hadamard, Pauli-Jordan and Wightman superdistributions have the following dependence in $x, \theta, \bar{\theta}$:*

$$\begin{aligned} \Delta_{\Phi\Phi}^X(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') &= -im\delta^2(\theta - \theta')e^{i(\theta\sigma^\mu\bar{\theta} - \theta'\sigma^\mu\bar{\theta}')\partial_\mu} \Delta_X(x - x') , \\ \Delta_{\bar{\Phi}\Phi}^X(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') &= e^{i(\theta\sigma^\mu\bar{\theta} + \theta'\sigma^\mu\bar{\theta}' - 2\theta\sigma^\mu\bar{\theta}')\partial_\mu} \Delta_X(x - x') , \\ \Delta_{\bar{\Phi}\bar{\Phi}}^X(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') &= im\delta^2(\bar{\theta} - \bar{\theta}')e^{-i(\theta\sigma^\mu\bar{\theta} - \theta'\sigma^\mu\bar{\theta}')\partial_\mu} \Delta_X(x - x') , \end{aligned} \quad (7.12)$$

where $X = (\text{Had}, \text{PJ}, \text{W})$.

Idea of Proof. We start from (6.6) and use the fact that in terms of even and odd solutions of the homogeneous wave equation, the function $\Delta_{\mathbb{F}}(x - x')$ can be write as

$$\Delta_{\mathbb{F}}(x - x') = \frac{1}{2} [i \Delta_{\text{Had}}(x - x') + \varepsilon(x^0 - x^{0'}) \Delta_{\text{PJ}}(x - x')] \quad (7.13)$$

Then, by replacing (7.13) in (6.6), we immediately get the Hadamard and Pauli-Jordan superdistribution as stated. The Wightman superdistribution is obtained directly from the fact the $\Delta_{\text{PJ}}(x - x') = \Delta_{\mathbb{W}}(x - x') - \Delta_{\mathbb{W}}(x - x')$ and $\Delta_{\text{Had}}(x - x') = -i(\Delta_{\mathbb{W}}(x - x') + \Delta_{\mathbb{W}}(x - x'))$.^k \square

PROPOSITION 7.4. *Let ω^{susy} be a state for the quantum Wess-Zumino model on flat superspace, whose r -point superdistributions ω_r^{susy} satisfy the Wightman axioms.^l Then ω^{susy} satisfies the Definiton 7.2.*

Proof. This is an immediate consequence of Corollary 6.8 above and Theorem 4.6 of [11]. \square

8 Final Considerations

Having proposed an extension of some structural aspects that have successfully been applied in the development of the theory of quantum fields propagating on a general spacetime manifold so as to include superfield models on a supermanifold, it would be interesting to consider the perturbative treatment of interacting quantum superfield models, in particular the formulation of renormalization theory on supermanifolds. The main problem which still remains in this rather restrictive framework is the mathematically consistent definition of all powers of Wick “superpolynomials” and their time-ordered products for the noninteracting theory, which serve as building blocks for a perturbative definition of interacting superfields. Another work devoted to its solution is in progress [62], such that covariance with respect to supersymmetry is manifestly preserved. The renormalization scheme underlying our construction is the one of Epstein-Glaser. It is formulated, unlike the other renormalization schemes, in configuration space. Therefore, it becomes appropriate to define carefully perturbative renormalization on a generic spacetime manifold. Recently, Brunetti and Fredenhagen [16] (with some gaps filled by Hollands and Wald [61]) have shown that the Wick polynomials and their time-ordered products can be defined in globally hyperbolic spacetimes. By the methods of this paper we can define powers of Wick “superpolynomials” and their time-ordered products for the noninteracting theory.

^kWe remark that our convention is opposite to that one of Fulling [5], where the Wightman function is defined without the factor i .

^lIn [49, 50] is explained how to generalize the standard Wightman axioms so as to include supersymmetry.

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