CBPF-NF-014/95

Exact Solutions of the Dirac Equations in an Anisotropic Cosmological Background

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Abstract

Exact solutions of the Dirac equations are given for an anisotropic Bianchi VI_0 background. The direction of propagation of the spinor obeys the geometrical symmetries of the cosmological model.

Key-words: Dirac's Equation; Cosmology; Anisotropic Background.

PACS numbers: 11.10.Qr, 04.20

I. Introduction

Recently, there is an increasing interest to study the propagation of neutrinos and particles of spin 1/2 in curved spacetime. The most studied cosmological background is the Robertson-Walker one. This is natural since they are the simplest models that explain a large amount of astronomical data[1]. We can refer to Parker[2], Isham & Nelson[3], Ford[4], Audretsch & Schäfer[5], Kovalyov & Légare[6], Barut & Duru[7] and Villalba & Percoco[8].

The interest to study the Klein-Gordon and Dirac equations in anisotropic models has increased since Hu & Parker[10] have shown that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the Universe expands.

Although the Universe seems homogeneous and isotropic at present, there are no observational data guaranteeing the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that sustain the existence of an anisotropic phase that approaches an isotropic one[9].

Chimento and Mollerach[11] have studied the Dirac equations in anisotropic Bianchi I models (see also Srivastava[12]). They posed the problem that, for each value of momentum, there are only two independent solutions of the Dirac equations. Castagnino et al[13] solved this problem by showing that it is a consequence of the Ansatz the authors have chosen for the Dirac spinor.

On an other hand, Castagnino et al[13] claimed to have proved that there are no solutions of the Dirac equations with one null component for all times in anisotropic metrics. However, their proof is also based in a particular Ansatz. In the present paper, we give a counter example of this statment by taking a particular choice of the direction of the momentum in a metric similar to the one used by these authors. The same procedure can be applied to their metric.

The background used in this paper is a Bianchi type VI_0 model, which has been widely studied from the cosmological point of view. We integrate the Dirac equations for a spinor propagating in a special direction of the spacetime[14], which allows a simple separation of variables due to the symmetries of the Bianchi type VI_0 model.

In section 2, we establish the Dirac equations with the spinor obeying the Ansatz (11). In section 3, we present the solutions when $m = 0$ and, in section 4, we present the solutions when $m \neq 0$. We use the Dirac representation for the γ matrices

II. The Dirac Equations

We will study the Dirac equations in a spacetime with the line element given by

$$ds^2 = dt^2 - \frac{t^2}{(m-n)^2}\, dx^2 - \frac{dy^2}{t^{2(m+n)}e^{2x}} - \frac{e^{2x}}{t^{2(m+n)}}\, dz^2 \ . \tag{1}$$

There is a wide class of solutions of the Einstein equations using this metric. Ellis & MacCallum[15] have found dust solutions and Collins[16] has found perfect fluid solutions with the equation of state $p = (\gamma - 1)\mu$. Dunn & Tupper[17] have found exact solutions of the Einstein-Maxwell equations for fluid matter and electromagnetic field with sources, admitting a spacelike 4-current with positive electrical conductivity.

We choose a tetrad basis of 1-forms $\sigma^a (a = 0, 1, 2, 3)$, given by

$$\begin{aligned}
\sigma^0 &= dt \\
\sigma^1 &= \frac{t\ dx}{m-n} \\
\sigma^2 &= \frac{dy}{t^{m+n} e^x} \\
\sigma^3 &= \frac{e^x dz}{t^{m+n}}\ .
\end{aligned} \tag{2}$$

In this basis, the line element is given by

$$ds^2 = \eta_{ab} \sigma^a \sigma^b\ , \tag{3}$$

where $\eta_{ab} = diag(+, -, -, -)$.

The Dirac equations in curved spacetime[18,19] are

$$(\gamma^a h^\mu{}_a \partial_\mu - \gamma^a \Gamma_a + iM)\Psi = 0\ , \tag{4}$$

where $h^\mu{}_a$ establish the connection between tetrad indices (Latin indices) and world indices (Greek indices). The components $h^\mu{}_a$ are obtained from

$$dx^\mu = h^\mu{}_a \sigma^a\ . \tag{5}$$

The matrices γ^a are the Dirac matrices of Minkowski spacetime. Γ_a are given by the relation:

$$\Gamma_a = -\frac{1}{4}\ \Gamma_{bca} \gamma^b \gamma^c + h^\mu{}_a a_\mu, \tag{6}$$

where a_μ is a vector which we assume equal zero (the spinor has no charge). Γ_{bca} are the Ricci rotation coefficients defined by

$$d\sigma^a = \Gamma^a{}_{bc}\ \sigma^b \wedge \sigma^c\ , \tag{7}$$

and antisymmetric in the first 2 indices: $\Gamma_{abc} = -\Gamma_{bac}$. For the line element (1), the Ricci coefficients are given by

$$\begin{aligned}
\Gamma_{011} &= \frac{1}{t} \\
\Gamma_{022} &= \Gamma_{033} = -\frac{m+n}{t} \\
\Gamma_{122} &= = -\Gamma_{133} = -\frac{m-n}{t}\ .
\end{aligned} \tag{8}$$

From eqs. (6) and (8), we obtain

$$\gamma^a \Gamma_a = \frac{m+n-1/2}{t}\ \gamma^0\ . \tag{9}$$

The Dirac equations (4) turn out to be:

$$\left(\partial_t + \frac{m-n}{t}\ \alpha^1 \partial_x + t^{m+n} e^x \alpha^2 \partial_y + \frac{t^{m+n}}{e^x}\ \alpha^3 \partial_z + \frac{1/2-m-n}{t} + iM\gamma^0\right)\Psi = 0\ , \tag{10}$$

and it can be solved by separation of variables if we impose the following form for the solutions

$$\Psi_k(\vec{x},t) = \sqrt{m-n}\ \frac{e^{ik_1x}t^{m+n-1/2}}{(2\pi)^{3/2}} \begin{pmatrix} f_I \\ f_{II} \end{pmatrix} \ , \tag{11}$$

where f_I and f_{II} are two-component spinors that depend on t only. In this case, eq. (10) reduces to

$$\left(\partial_t + iM\gamma^0 + \frac{ik_1(m-n)}{t}\ \alpha^1\right) \begin{pmatrix} f_I \\ f_{II} \end{pmatrix} = 0 \ , \tag{12}$$

which has no dependence on space variables. This equation is equivalent to

$$\begin{pmatrix} \partial_t + iM & 0 & 0 & \frac{ik_1(m-n)}{t} \\ 0 & \partial_t + iM & \frac{ik_1(m-n)}{t} & 0 \\ 0 & \frac{ik_1(m-n)}{t} & \partial_t - iM & 0 \\ \frac{ik_1(m-n)}{t} & 0 & 0 & \partial_t - iM \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0 \ . \tag{13}$$

The components f_1 and f_2 obey the differential equation

$$\left(\partial_t^2 + \frac{1}{t}\ \partial_t + M^2 + \frac{iM}{t} + \frac{k_1^2(m-n)^2}{t^2}\right)(f_{1,2}) = 0 \tag{14}$$

and the components f_3 and f_4 are given by

$$f_{4,3} = \frac{it}{k_1(m-n)}\ (\partial_t + iM) f_{1,2} \tag{15}$$

III. The Neutrino Solutions

For $M = 0$, eq. (14) reduces to

$$\left(\partial_t^2 + \frac{1}{t}\ \partial_t + \frac{k_1^2(m-n)^2}{t^2}\right) f = 0 \ . \tag{16}$$

The solutions are given

$$f = a\ e^{ik_1(m-n)\ln t} + b\ e^{-ik_1(m-n)\ln t} \ , \tag{17}$$

where a and b are constants.

The four independent normalized solutions of the form (11) are

$$\Psi_1^{\pm} = \frac{\sqrt{m-n}\ e^{ik_1x}t^{m-n-1/2}}{(2\pi)^{3/2}} \begin{pmatrix} e^{\pm ik_1(m-n)\ln t} \\ 0 \\ 0 \\ -e^{\pm ik_1(m-n)\ln t} \end{pmatrix} \tag{18}$$

$$\Psi_2^{\pm} = \frac{\sqrt{m-n}\ e^{ik_1x}t^{m+n-1/2}}{(2\pi)^{3/2}} \begin{pmatrix} 0 \\ e^{\pm ik_1(m-n)\ln t} \\ e^{\pm ik_1(m-n)\ln t} \\ 0 \end{pmatrix} \ . \tag{19}$$

IV. The case $M \neq 0$

For $M \neq 0$, eq. (14) can be written as

$$\left(\partial_t^2 + M^2 + \frac{iM}{t} + \frac{\frac{1}{4} + k_1^2(m-n)^2}{t^2}\right)(\sqrt{t}\ f_{1,2}) = 0\ . \tag{20}$$

With the change of variable $\eta = \pm 2iMt$, eq. (2) is equivalent to the Whittaker differential equation[20]

$$\left(\partial_\eta^2 - \frac{1}{4} \pm \frac{1}{2\eta} + \frac{\frac{1}{4} + k_1^2(m-n)^2}{\eta^2}\right)(\sqrt{t}\ f_{1,2}) = 0\ . \tag{21}$$

Therefore, the two independent solutions are

$$f_{1,2}^{\pm} = \frac{1}{\sqrt{t}}\ W_{[\pm\frac{1}{2}, ik_1(m-n)]}(\pm 2iMt)\ . \tag{22}$$

Using (2) and the identities of the first derivative of the Whittaker function[20], we obtain

$$f_{4,3}^{\pm} = \frac{(ik_1(m-n))^{\pm 1}}{\sqrt{t}}\ W_{[\mp\frac{1}{2}, ik_1(m-n)]}(\pm 2iMt)\ . \tag{23}$$

The four independent normalized solutions of the form (11) are

$$\Psi_1^{\pm} = \frac{A^{\pm}\sqrt{m-n}\ t^{m+n-1} e^{ik_1 x}}{(2\pi)^{3/2}} \begin{pmatrix} W_{[\pm\frac{1}{2}, ik_1(m-n)]}(\pm 2iMt) \\ 0 \\ 0 \\ (ik_1(m-n))^{\pm 1} W_{[\mp\frac{1}{2}, ik_1(m-n)]}(\pm 2iMt) \end{pmatrix} \tag{24}$$

$$\Psi_2^{\pm} = \frac{A^{\pm}\sqrt{m-n}\ t^{m+n-1} e^{ik_1 x}}{(2\pi)^{3/2}} \begin{pmatrix} 0 \\ W_{[\pm\frac{1}{2}, ik_1(m-n)]}(\pm 2iMt) \\ (ik_1(m-n))^{\pm 1} W_{[\mp\frac{1}{2}, ik_1(m-n)]}(\pm 2iMt) \\ 0 \end{pmatrix}\ . \tag{25}$$

where $A^+ = 1/\sqrt{2iM}$ and $A^- = ik_1(m-n)/\sqrt{-2iM}$. The asymptotic form of (24) and (25) for large times is

$$\Psi_1^{+} = \frac{\sqrt{m-n}\ t^{m+n-\frac{1}{2}} e^{ik_1 x}}{(2\pi)^{3/2}} \begin{pmatrix} e^{-iMt} \\ 0 \\ 0 \\ \frac{ik_1(m-n)}{2iMt}\ e^{-iMt} \end{pmatrix} \tag{26}$$

$$\Psi_1^{-} = \frac{\sqrt{m-n}\ t^{m+n-\frac{1}{2}} e^{ik_1 x}}{(2\pi)^{3/2}} \begin{pmatrix} -\frac{ik_1(m-n)}{2iMt}\ e^{iMt} \\ 0 \\ 0 \\ e^{iMt} \end{pmatrix}\ . \tag{27}$$

$$\Psi_2^+ = \frac{\sqrt{m-n}\; t^{m+n-\frac{1}{2}}\; e^{ik_1 x}}{(2\pi)^{3/2}} \begin{pmatrix} 0 \\ e^{-iMt} \\ \frac{ik_1(m-n)}{2iMt}\; e^{-iMt} \\ 0 \end{pmatrix} \tag{28}$$

$$\Psi_2^- = \frac{\sqrt{m-n}\; t^{m+n-\frac{1}{2}}\; e^{ik_1 x}}{(2\pi)^{3/2}} \begin{pmatrix} 0 \\ -\frac{ik_1(m-n)}{2iMt}\; e^{iMt} \\ e^{iMt} \\ 0 \end{pmatrix} \tag{29}$$

Acknowledgments

The author would like to thank F.R. Carvalho and J.A. Heläyel Neto for helpfull discussions during this work and M.O. Calvão for revising the final version.

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