MOTIONS IN THE RELATIVISTIC FIELDS OF A CHARGED DUST

A.F. da F. Teixeira

Centro Brasileiro de Pesquisas Físicas
22290 Rio de Janeiro, Brazil

ABSTRACT

The general relativistic motion of arbitrarily charged test particles is investigated, in the spherically symmetric fields of a charged, static, incoherent matter with T^0_0 = const. The condition for existence of stable circular orbits is established, inside and outside the diffused source. The null geodesics are also investigated, as a limiting case.

1. INTRODUCTION

As is widely known[1-4], the Einstein-Maxwell's equations predict that an electrically charged dust distribution can only be mantained in static equilibrium when the densities of charge and mass bear a uniform ratio (±1, in relativistic units), provided there is no singularity in the distribution. However, the shape and the concentration of the distribution are not determined from the field equations alone[5]. Systems with spherical symmetry have been studied by Bonnor[6], who expressed the concentration as a function of the first and second radial derivatives of the gravitational potentials. His expression then permits to obtain physically plausible exact solutions, by appropriate choices of the gravitational field[7]. However, it appears more natural to start from physically acceptable sources of field, and obtain the solution for the fields by integration of the field equations[8].

In this paper we investigate the charged dust counterpart of Schwarzschild's incompressible fluid sphere (T_0^0=const), and describe the motion of arbitrarily charged particles under the relativistic fields. In Sec. 2 we describe some properties of the exact solution of the field equations. In Sec. 3 we consider the motion of particles and light rays under the combined effects of the fields, and compare these motions with less general results obtained by previous authors. In Sec. 4, the stability of the circular orbits is studied, and is also compared with previous results. Finally, remarks concerning the mass of the system are made in Sec. 5, together with some comments concerning the radial coordinate used throughout the paper.

2. ELECTRIC AND GRAVITATIONAL FIELDS

The energy momentum tensor corresponding to an electrically charged dust distribution is

$$T^{\mu}_{\nu} = \rho\, u^{\mu}u_{\nu} + (F^{\mu}_{\alpha}\, F^{\alpha}_{\nu} - \tfrac{1}{4}\, F^{\alpha}_{\beta}\, F^{\beta}_{\alpha}\, \delta^{\mu}_{\nu})/(4\pi) \quad , \tag{1}$$

where ρ is matter density with four-velocity u^{μ}, and $F_{\mu\nu}$ is electromagnetic field. For static, spherically symmetric systems of the form (1) we can write[5]

$$ds^2 = e^{2\eta}\, dt^2 - e^{-2\eta}\, (dr^2 + r^2\, d\Omega^2) \quad , \tag{2}$$

$$u^{\mu}u_{\nu} = \delta^{\mu}_{0}\delta^{0}_{\nu} \quad , \quad F_{\mu\nu} = (\delta^{1}_{\mu}\delta^{0}_{\nu} - \delta^{0}_{\mu}\delta^{1}_{\nu})\; d\Phi/dr \quad . \tag{3}$$

The three quantities ρ, η and Φ depend only on r. The independent field equations then reduce to

$$(2\eta'' + 4\eta'/r - \eta'^2)\; e^{2\eta} = 8\pi\rho + \Phi'^2 \quad , \tag{4}$$

$$\eta'^2 e^{2\eta} = \Phi'^2 \quad , \tag{5}$$

where a prime means d/dr.

The solution of (4) and (5) when $\rho = 0$ is already known[5,7]:

$$\eta = - \ell n(1 + M/r) \quad , \quad \Phi = \pm\, (1 + r/M)^{-1} \quad . \tag{6}$$

The parameter M is the mass of the system, and the electric charge is $\pm M$.

Inside the sphere of radius R the three functions η, Φ, ρ

have to satisfy only two equations, so one constraint is necessary to obtain explicit solutions. A most reasonable physical assumption is

$$T_0^0 \equiv \rho + \Phi'^2/(8\pi) = \text{const} \quad . \tag{7}$$

We then obtain the exact internal solution

$$e^\eta = (1 + 2\mu)^{-2} \left[1 + \mu + \mu r^2/R^2\right] \quad , \tag{8}$$

$$\Phi = \pm\ (1 - e^\eta) \quad , \tag{9}$$

$$\rho = (M/V)(1 + 2\mu)^{-4} \left[1 + \mu - \tfrac{1}{3}\mu r^2/R^2\right] \quad , \tag{10}$$

where $V = 4\pi R^3/3$ and $2\mu = M/R$. The constants of integration were adjusted to make the potentials η, Φ, and their first radial derivatives continuous through the boundary of the sphere.

We observe in (10) that the density $\rho(r)$ is always positive and finite, and decreases monotonically outwards. When $M \ll R$ the sphere is nearly uniform, with density M/V. From (6), (8) and (9) we also remark that both potentials η and Φ vary monotonically from a finite value at the center of the sphere to a zero value at infinity.

3. CIRCULAR ORBITS OF CHARGED PARTICLES AND LIGHT RAYS

We now consider the motion of a particle with mass $m \ll M$ and charge q (with $|q| \ll M$), under the combined effects of gravitation and electrostatic field. The Lagrangian and corresponding equations of motion are

$$L = -\frac{1}{2} m\, g_{\mu\nu}u^{\mu}u^{\nu} - q\, u^{\mu}\Phi_{\mu} \quad , \tag{11}$$

$$(m\, u_{\lambda;\nu} - q\, F_{\lambda\nu})\, u^{\nu} = 0 \quad , \tag{12}$$

where a semicolon denotes covariant derivative.

All motions are planar, and are taken on the equatorial plane for simplicity. Two constants of motion are readily found, corresponding to the conservation of energy and angular momentum:

$$\alpha = u_0 - \sigma e^{\eta} \quad , \quad \beta = u_3 \quad , \tag{13}$$

where $\sigma = (q/m)$ sign Φ. From its definition, σ is positive (negative) when q and the charge of the sphere have same (opposite) sign. The component u_1 satisfies the equation (12) for $\lambda = 1$,

$$du^1/ds = \eta' \left[(u^1)^2 - (u_0)^2 - r^2(u^3)^2 + \sigma e^{\eta}u_0\right] + r(u^3)^2 \quad , \tag{14}$$

and can be calculated directly from the first integral $u^{\mu}u_{\mu} = 1$:

$$(u^1)^2 = (u_0)^2 - r^2(u^3)^2 - e^{2\eta} \quad . \tag{15}$$

The differential equation of the orbit is then obtained from (13) and (15):

$$(d\kappa/d\phi)^2 = \beta^{-2}e^{-2\eta}\left[(\sigma + \alpha e^{-\eta})^2 - 1\right] - \kappa^2 \quad , \quad \kappa \equiv 1/r \; . \tag{16}$$

We are particularly interested in the circular orbits. Setting $u^1 = 0$ and $du^1/ds = 0$ in (13) to (15), we find that the radius ξ of a circular orbit is given implicitly by

$$\xi\ \eta'(\xi) = v^2 \left[1 + v^2 - \sigma(1 - v^2)^{1/2}\right]^{-1} , \qquad (17)$$

where the constant $v^2 = 1 - (u^0 u_0)^{-1}$ gives the velocity of the particle[9-12].

In the external region we obtain, from (17) and (6),

$$\xi_{ext} = (M/v^2)\left[1 - \sigma(1 - v^2)^{1/2}\right] , \qquad (18)$$

provided this value is greater than R. The expression (18) tends to its nonrelativistic analogue, $M(1 - \sigma)/v^2$, when $v^2 << 1$. For uncharged test particle we set $\sigma = 0$ in (18) and obtain $\xi_{ext} = M/v^2$, which exactly coincides with the nonrelativistic result. Since $v^2 < 1$, the circular orbits of uncharged particles satisfy $\xi_{ext} > M$, as already pointed out by Armenti[13].

In the internal region we obtain, from (17) and (8),

$$\xi^2_{int} = R^2 v^2 (1/2 + R/M) \left[1 + v^2/2 - \sigma(1 - v^2)^{1/2}\right]^{-1} , \qquad (19)$$

provided this value is less than R^2. This expression tends to its nonrelativistic analogue, $R^3 v^2 M^{-1}(1 - \sigma)^{-1}$, when $v^2 << 1$ and $M << R$.

The circular orbits for a photon are obtained simply by setting $v = 1$ in the preceding equations. From (18) we find an external, unstable orbit of radius M, as is already known[13]. As a new result, a circular photon orbit inside the incoherent matter is predicted by (19), with radius $R\left[\frac{1}{3}(1+2R/M)\right]^{1/2}$. The interesting feature of this photon orbit is its stability, as is proved in the next Section. It should be remarked that both external and internal circular photon orbits only exist when

$M \geq R$. In the case where $M = R$, the two orbits coalesce on the boundary of the sphere, $\xi = R$.

4. STABILITY OF THE CIRCULAR ORBITS

To investigate the stability of the circular orbits we follow the standard method described by Armenti and Havas[14], using the Lagrangian of the motion (11). We write the Routhian

$$R = L - u^0 \partial L/\partial u^0 - u^3 \partial L/\partial u^3 = T - U \quad , \tag{20}$$

and find that

$$T = \frac{1}{2} m\, e^{2\eta}(u_1)^2 \quad , \qquad 2U/m = -(\sigma + \alpha e^{-\eta})^2 + \beta^2 r^{-2} e^{2\eta} \quad . \tag{21}$$

We next evaluate $\delta U = U(\xi + \varepsilon) - U(\xi)$, where ξ is the radius corresponding to a circular orbit, and where $\varepsilon^2 \ll \xi^2$. The circular orbit is stable when δU is a positive quadratic form in ε. Calculations involving (21), (17), (15), (13) and $u^1 = 0$ then give the following condition for stability:

$$v^2 < 3 + \xi(\eta' + \eta''/\eta')(1 - \xi\eta')^{-2} \quad , \tag{22}$$

with the derivatives η' and η'' calculated on the radius ξ.

In the internal region we find, from (8), that both η' and η'' are positive definite. Then the right hand side of (22) is always greater than 1, showing that all circular orbits of charged test particles and light rays inside the incoherent matter are stable.

In the external region we use (22) and (6), and find

$$\xi_{ext} > 2M/(1 - v^2) \qquad (23)$$

as condition for stability. Taken together, the relations (18) and (23) generalize the result $\xi_{ext} > 3M$, obtained for uncharged test particles by Armenti[13] and by Dadhich and Kale[15]. These relations also give an upper bound for the specific charge of a particle in a stable, circular orbit with velocity v:

$$\sigma < (1 - 3v^2)(1 - v^2)^{-3/2} \quad . \qquad (24)$$

This expression corrects the constraint $\sigma < 1$ obtained in nonrelativistic physics.

5. COMMENTS

The internal solution (8) - (10) contains only two independent parameters: the mass M and the radius R of the sphere. Differently from the internal Schwarzschild solution, these parameters can assume arbitrary finite values without producing metrical singularity.

The mass parameter M is related to the matter density ρ according to[6]

$$M = \int \rho \, (-g_3)^{1/2} \, d^3x \quad , \qquad (25)$$

where $g_3 = \det g_{ij}$ $(i,j = 1,2,3)$ and $d^3x = dr\, d\theta\, d\phi$. As shown by Tolman[16], it is also related to the total source of gravitation, which includes the electrostatic energy:

$$M = \int (2T_0^0 - T)(-g_4)^{1/2} \, d^3x \quad , \qquad (26)$$

where $g_4 = \det g_{\mu\nu}$ $(\mu,\nu = 0\text{-}3)$.

In the literature of static, spherically symmetric systems, we generally find the line element written as

$$ds^2 = e^{2\eta}dt^2 - e^{2\alpha}d\lambda^2 - \lambda^2 d\Omega^2 \quad , \qquad (27)$$

with η and α functions of the radial coordinate λ . A discontinuity in the derivative of the radial gravitational potential, $d\alpha/d\lambda$, is usually encountered on the boundary of systems containing diffuse distribution of matter[17,18]. The absence of such discontinuity in the line element (2) is a possibility afforded by electrically charged, incoherent matter[19,20], together with our choice of the radial coordinate. Our coordinate r is related to λ according to $r = \lambda \exp \eta$, which takes the simple form $r = \lambda - M$ in the region external to the incoherent matter.

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