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SOLUBLE MODEL OF A BREAK-UP PROCESS

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Abstract. The exact solution of a one-dimensional problem, representing the scattering of a particle by another one which is bound to a fixed centre of force, is given. All the interactions have zero range and are described by boundary conditions. The possible processes are elastic scattering and break-up of the bound system. The elastic scattering and break-up amplitudes are explicitly determined, and the behaviour of the corresponding cross-sections is discussed. At high energies, the incident particle tends to transfer its whole momentum to the bound one, giving rise to a strong peak in the break-up cross-section. The analytic behaviour of the amplitudes is examined. The Riemann surface of the elastic scattering amplitude has three sheets. The break-up threshold gives rise to cubic-root branch points. The remaining singularities are a finite number of poles. The exact amplitudes are compared with those given by the impulse approximation (in first and second order) and by Born's approximation. It is found that these approximations are reliable

only at high incident energies and within the width of the dominant peak of the break-up cross-section.

* * *

1. INTRODUCTION

(a) Relation to previous work

The treatment of collision processes in which one of the partners can break up into two particles (such as ionization, (n, 2n) or (d, np) reactions and many others) encounters considerable difficulties, because at least three-body problems are involved.

The resonance theory of nuclear reactions (Kapur & Peierls 1938, Wigner & Eisenbud 1947) deals only with two-body channels, so that it does not apply to such processes. The dispersion formulas employed in this theory can be derived, at least in the simplest cases, by means of a representation of the scattering amplitudes in terms of their singularities (Humblet 1952, Peierls 1959). For this purpose, one must know the full analytic behaviour of the amplitudes on the associated Riemann surfaces. It is not sufficient to consider only the physical sheet, as is usually done in dispersion-theoretic techniques.

As has been remarked by Peierls (1959), it would be of interest to extend these methods to high-energy physics. A preliminary study of this problem has been made by Hong-Mo (1960). One of the main difficulties which arise is the lack of knowledge concerning the analytic behaviour of the amplitudes in the case of many-body channels.

Break-up processes are usually dealt with by approximation methods, such as Born's approximation or the impulse approximation (Chew 1950). It has not been possible, so far, to ascertain the accuracy of these approximations by comparison with an exact solution.

It would be desirable, therefore, to find an exactly soluble model of a break-up process, in which both the analytic behaviour of the amplitudes and the domain of validity of the approximation methods could be determined.

All attempts to find such a model have been confined to the simplest conceivable case, in which the motion of the particles is restricted to one space dimension and all the interactions have zero range. Almost all the models which have been considered are special cases of the problem characterized by the Hamiltonian

$$H(x,y) = - \frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - A\,\delta(x) - B\,\delta(y) + C\,\delta(x-y), \qquad (1)$$

where A, B and C are non-negative constants.

This Hamiltonian describes a pair of particles of equal mass m, moving in one dimension and interacting with a fixed centre of force at the origin (attractive interaction) as well as with each other (repulsive interaction). The attractive delta-function interaction $-B\,\delta(y)$ gives rise to a single bound state, of binding energy $-\frac{1}{2}\,mB^2/\hbar^2$. It is assumed that particle y (i. e. the particle with coordinate y) is bound to begin with, whereas particle x comes in from $x = \infty$. The incident wave is therefore

$$\psi_{inc}(x,y) = \exp(-ik_o x - \lambda|y|), \tag{2}$$

where $\lambda = mB/\hbar^2$.

Wildermuth (1949) considered the special case in which $A = 0$. The possible processes in this case are elastic scattering and break-up. The break-up threshold is given by $k_o^2 = \lambda^2$. The first and second Born approximations of the elastic scattering and break-up amplitudes were computed (the Born series is an expansion in powers of C). The second Born approximation contains a term proportional to $(k_o^2 - \lambda^2)\log\,[(k_o + \lambda)/(k_o - \lambda)]$, which gives rise to a logarithmic branch point at the threshold. Wildermuth claimed that such a branch point is still present in the exact scattering amplitude. He also conjectured that a break-up threshold always corresponds to a logarithmic branch point, in contrast with the case of a discrete excited state, in which the excitation threshold corresponds to a square-root branch point (Touschek 1949, Peierls 1959). It will be shown in the present paper that this conjecture is incorrect.

Jost (1955) analysed a limiting case of Wildermuth's problem, in which $C \longrightarrow \infty$. This corresponds to the boundary condition: $\psi(x,y) = 0$ for $x = y$, so that the interaction between the parti cles is of the "hard-sphere" type. By an extension of the Wiener-Hopf method, the problem was reduced to a difference equation, which can be solved in principle (Jost 1954). In practice, however, the construction of the solution by this procedure seems to be very hard.

One can also consider a symmetrized version of the problem, in which A = B and $C \longrightarrow \infty$. In this case, there is an additional process which can take place, namely, exchange scattering. It turns out that this problem has a trivial solution: only elastic and exchange scattering occur; there is no break-up. The solution is

$$\psi(x,y) = \varphi(x)\exp(-\lambda|y|) - \varphi(y)\exp(-\lambda|x|) \quad (x \geqslant y), \quad (3)$$

where

$$\begin{aligned} \varphi(x) &= \exp(-ik_o x) + i\lambda(k_o - i\lambda)^{-1}\exp(ik_o x) \quad \text{for} \quad x \geqslant 0, \\ &= k_o(k_o - i\lambda)^{-1}\exp(-ik_o x) \quad \text{for} \quad x \leqslant 0, \end{aligned} \quad (4)$$

is the solution of the scattering problem for the interaction $-B\,\delta(x)$.

To explain this result it is helpful to consider the two-dimensional interpretation of the problem (Morse & Feshbach 1953, p. 1709). The wave function is a solution of the two-dimensional Helmholtz wave equation, except along the coordinate axes and the first bisector. Along the first bisector is a perfectly reflecting mirror, and along the x and y axes are semi-reflecting mirrors, which can support surface waves, such as the incident wave (2). Elastic scattering corresponds to reflection of this wave on the y-axis, and exchange scattering corresponds to reflection on the bisector. Break-up would correspond to the appearance of a circular wave going out radially from the origin, which might be called a "diffracted" wave. That no such wave exists is due to the symmetry of the problem: the solution (3) can be constructed by the method of images. It suffices to add to the incident wave its image (with opposite sign) with respect to the bisector, which gives rise to the last term of (3).

Another problem having a trivial solution was analysed by Danos (cf. Lieb & Koppe 1959). It corresponds to taking $A = B \to -\infty$ in (1), and $C < 0$, so that the interaction between the particles gives rise to a bound state. The incident wave is a surface wave travelling along the bisector, so that the particles are bound together to begin with. There are now perfect mirrors along both axes, giving rise to three images. Again there is no break-up: only elastic scattering takes place.

Lieb and Koppe (1959) considered another problem, which corresponds to taking $A = B = 0$ in (1), and replacing the interaction term by $-2C\,\theta(x+y)\,\delta(x-y)$ $(C > 0)$, where $\theta(x) = 1$ for $x \geqslant 0$, $\theta(x) = 0$ for $x < 0$. The incident wave is again a surface wave travelling along the bisector, coming in from $x = \infty$, $y = \infty$. Break-up occurs if the total energy is positive. The problem can be solved by the Wiener-Hopf method. It is closely related to the problem of diffraction by an imperfectly conducting half-plane, which was solved by Senior (1952). The elastic scattering and break-up cross-sections were explicitly determined, but the corresponding amplitudes and their analytic behaviour were not investigated. It is doubtful, however, whether such an investigation would be worth while, in view of the unphysical character of the interaction, which depends on the sign of the coordinates of the particles.

(b) Formulation of the problem

In the present work, a modified version of Jost's problem will be considered. The interaction between the particles is still of the hard-sphere type, and it is still assumed that the x particle does not interact with the centre of force. However, the interaction between the y particle and the centre of force will be described by the following boundary condition:

$$\frac{\partial \psi}{\partial y}(x,0) = \lambda \psi(x,0). \qquad (5)$$

This boundary condition introduces a considerable simplification as compared with a delta-function interaction, because it restricts the motion of the y particle to a semi-axis (there is no probability current across the x-axis). The interaction (5) still gives rise to a single bound state, in which the y particle is confined to the region $y \geqslant 0$ if $\lambda < 0$, and to the region $y \leqslant 0$ if $\lambda > 0$. In either case, the incident wave in one of these regions is still given by (2).

If $\lambda < 0$, the motion is confined to the octant $x \geqslant y \geqslant 0$. The problem is then a trivial one, and break-up does not occur. To see this, it suffices to continue the wave function symmetrically about the x-axis: it then becomes identical to the solution of Danos's problem (with a rotation of the axes by $\pi/4$).

If $\lambda > 0$, the motion is confined to the region: $y \leqslant 0$, $x \geqslant y$. The problem is no longer trivial, and, as will be shown later, break-up occurs when the total energy is positive. This is the problem which will be dealt with in the present work. It has some features in common with the (three-dimensional) problem of the scattering of slow neutrons by molecularly bound protons (Fermi 1936).

The boundary condition (5) has already been employed by several authors (cf. Eyges 1959 and the references quoted there). It can be considered as a limiting case of an attractive short-range interaction with a hard core. If we consider, for instance, the potential: $V(y) \to \infty$ for $y = 0$; $V(y) = -\frac{1}{2}\hbar^2K^2/m$ for $-a < y < 0$; $V(y) = 0$ for $y < -a$, where $a = \frac{1}{2}\pi/K + \lambda/K^2$, it is readily seen that the boundary condition on the logarithmic derivative of the wave function at $y = -a$ goes over into (5) in the limit as $K \to \infty$.

The same boundary condition is employed in electromagnetic diffraction theory to describe an imperfect conductor of very large conductivity; in this case, λ is a complex parameter (Grünberg 1943, Jones & Pidduck 1950).

Applying the two-dimensional interpretation, the present problem can be described as that of diffraction of a surface wave by a wedge of exterior angle $3\pi/4$. One of the wedge surfaces is perfectly reflecting, and boundary condition (5) is to be satisfied on the other surface.

The problem of diffraction of a plane electromagnetic wave by an imperfectly conducting wedge of arbitrary angle, in the case in which the electric field is parallel to the edge of the wedge, has been solved by Williams (1959) (cf. also Malyughinetz 1960). In this case, λ is a complex parameter with negative real part.

In §2, the exact solution of the present problem will be found by applying Williams' method. The elastic scattering and break-up amplitudes will be explicitly evaluated in terms of elementary functions.

The behaviour of the amplitudes will be discussed in §3. The behaviour of the elastic scattering and break-up cross-sections as a function of the energy and of the momentum distribution between the fragments will be described. The analytic behaviour of the amplitudes will also be discussed. It will be shown that the only branch points in the elastic scattering amplitude are those associated with the break-up threshold, and they are cubic-root branch points, so that the corresponding Riemann surface is three-sheeted. This disproves Wildermuth's conjecture.

In §4, approximate expressions for the amplitudes, corresponding to the first and second order impulse approximations and to Born's approximation, will be derived and compared with the exact ones. In the case of the elastic cross-section, the agreement is not good. Good agreement is found for the break-up cross-section at high energies, within the width of the dominant peak, but not outside of this domain. The approximate amplitudes do not reproduce the analytic behaviour of the exact solution: the break-up threshold appears as a logarithmic branch point, of the same type as that which was found by Wildermuth.

Thus, in the case of the present model, the questions which were raised at the beginning of this § can be completely solved. The extension of these results to the more realistic case of three-dimensional scattering, however, still remains an open problem.

2. SOLUTION OF THE PROBLEM

(a) The boundary-value problem

We shall employ the polar co-ordinate system shown in figure 1. The time factor $\exp(-iEt/\hbar)$, where $E = \frac{1}{2}\hbar^2k^2/m$ is the total energy, will be omitted throughout. The wave function $\psi(\rho, \theta)$ must fulfil the following conditions (cf. § 1b):

$$\text{(i)}\quad \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + k^2\right)\psi(\rho,\theta) = 0 \quad \left(0 < \theta < \tfrac{3}{4}\pi\right); \tag{6}$$

$$\text{(ii)}\quad \frac{1}{\rho}\frac{\partial\psi}{\partial\theta}(\rho,0) = -\lambda\psi(\rho,0) \qquad (\lambda > 0); \tag{7}$$

$$\text{(iii)}\quad \psi\left(\rho,\tfrac{3}{4}\pi\right) = 0; \tag{8}$$

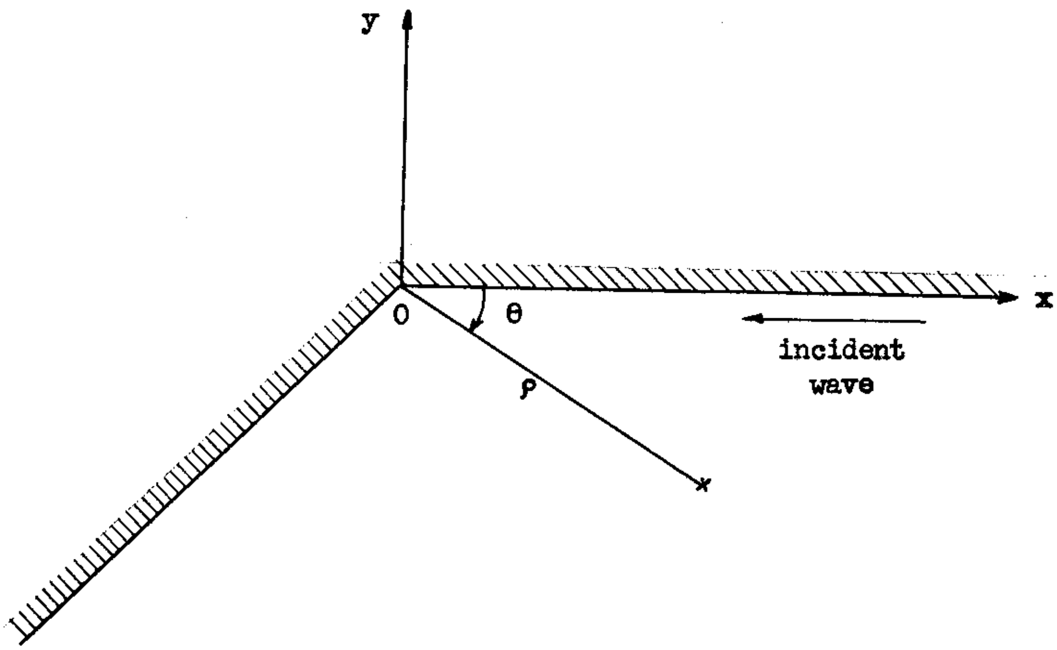

Fig. 1. Co-ordinate system.

(iv) $\psi(\rho,\theta) \approx \left[\exp(-ik_0 \rho \cos\theta) + f(k_0) \exp(ik_0 \rho \cos\theta)\right] .$

$$\cdot \exp(-\lambda\rho \sin\theta) + \left(\frac{2}{\pi k \rho}\right)^{\frac{1}{2}} F(k,\theta) \exp\left[i\left(k\rho - \frac{\pi}{4}\right)\right] \text{ for } \rho \to \infty ; \quad (9)$$

(v) $\psi(\rho,\theta)$ is bounded at the edge of the wedge (i. e. for $\rho \to 0$).

The relation between k_0 and k is

$$k = (k_0^2 - \lambda^2)^{\frac{1}{2}} \qquad (\mathcal{I}\ k \geqslant 0). \qquad (10)$$

Below the break-up threshold, $k = i(\lambda^2 - k_0^2)^{\frac{1}{2}}$, so that the last term of (9) is exponentially damped. Above the threshold, this term represents an outgoing circular wave. The elastic scattering and break-up amplitudes are given by $f(k_0)$ and $F(k,\theta)$, respectively.

Above the threshold, the first two terms in (9) can be omitted for $\theta \neq 0$, since the asymptotic behaviour of $\psi(\rho,\theta)$ for $\rho \to \infty$ is then dominated by the last term. However, the first two terms predominate for $\theta = 0$.

Condition (v) is called the "edge condition" in electromagnetic diffraction theory, where it corresponds to the physical requirement that the electromagnetic energy density in the neighbourhood of the edge be integrable (Meixner 1948). In the case of perfect conductors, this condition is a necessary requirement for the uniqueness of the solution, for it is possible to construct unphysical solutions with singularities at the edge which do not fulfil it (Bouwkamp 1946). Although it is not known *a priori* whether the same is true in the present problem, it can be assumed that the solution satisfies condition (v) [1]. It will be shown in

§2d that $\psi(\rho,\theta) = \underline{O}(\rho^{2/3})$ for $\rho \longrightarrow 0$.

(b) The difference equation

It will be convenient in the derivation of the solution to assume that $E > 0$, so that k is real. The method of solution is similar to that employed in Williams' paper (1959), where additional details can be found. The starting point is the following integral representation of the solution, which is an extension of Sommerfeld's well-known diffraction integral (Frank-von Mises 1935):

$$\psi(\rho,\theta) = \int_C f(w,\theta)\exp(-ik\rho\cos w)\,dw. \tag{11}$$

The domains of the complex w-plane where $\mathcal{R}(-ik\cos w) < 0$ are shaded in figure 2. The path of integration C must begin at infinity in one of these domains and go over to infinity in another one. It will be assumed to lie entirely in the strip $-\pi < \mathcal{R}\, w < 2\pi$.

The function $f(w,\theta)$ must be sufficiently well behaved to justify the manipulations that will follow. The correctness of this assumption can be verified in the solution. It will be assumed that $f(w,\theta)$ is a regular analytic function of w, apart from poles, which will be required to produce the incident and elastically scattered waves. The poles will be assumed to lie all above a straight line parallel to the real axis, and the contour C must lie entirely below this straight line, so that it cannot be crossed by the poles for any value of θ.

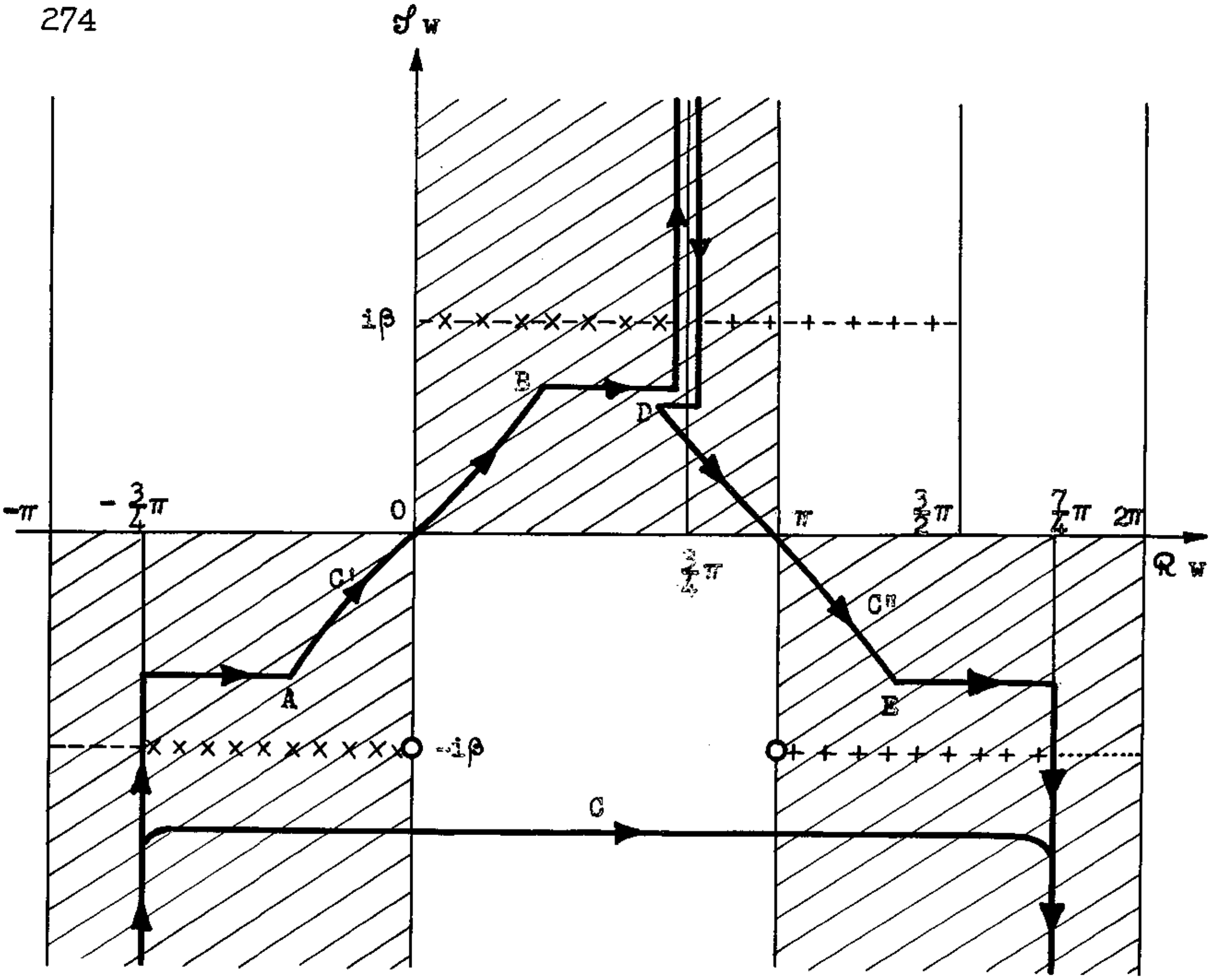

Fig. 2. The paths of integration and the poles of $f(w,\theta)$: ×××× $w = -\theta - i\beta$; ++++ $w = \pi + \theta - i\beta$; -×-×-×- $w = \theta + i\beta$; -+-+- $w = \frac{3}{2}\pi - \theta + i\beta$; - - - - $w = \theta - \frac{3}{2}\pi - i\beta$; ..... $w = \frac{5}{2}\pi - \theta - i\beta$.

In accordance with condition (v), it will be assumed that $f(w,\theta)$ is bounded for all values of θ as w tends to infinity along C [2], and similarly for its first two derivatives with respect to θ.

In order to fulfil condition (i), we must have

$$\rho^2(\Delta + k^2)\psi = \int_C \left(\frac{\partial^2 f}{\partial\theta^2} - \frac{\partial^2 f}{\partial w^2}\right) \exp(-ik\rho\cos w)\, dw = 0. \quad (12)$$

This follows from (6) and (11) by differentiation under the integral sign and partial integration. The integrated part vanishes for $\rho \neq 0$, according to the assumptions.

Equation (12) is satisfied if

$$f(w,\theta) = g_1(w+\theta) + g_2(w-\theta), \quad (13)$$

where g_1 and g_2 are arbitrary functions. Again by partial integration, it follows that

$$\frac{1}{\rho}\frac{\partial\psi}{\partial\theta}(\rho,\theta) = -ik\int_C\left[g_1(w+\theta) - g_2(w-\theta)\right] \exp(-ik\rho\cos w)\sin w\, dw.$$

Thus, condition (ii) will be fulfilled if

$$(\sin w + \cos\alpha)g_1(w) = (\sin w - \cos\alpha)g_2(w), \quad (14)$$

where we have introduced the notation

$$\cos\alpha = i\lambda/k. \quad (15)$$

For $E > 0$, we shall take

$$\alpha = \frac{\pi}{2} - i\beta \quad (\beta > 0). \quad (16)$$

Condition (iii) gives

$$g_2(w) = -g_1\left(w + \frac{3}{2}\pi\right). \tag{17}$$

We conclude that

$$\psi(\rho,\theta) = \int_C \left[G(w+\theta) + \frac{\sin(w-\theta) + \cos\alpha}{\sin(w-\theta) - \cos\alpha}\, G(w-\theta)\right] \exp(-ik\rho\cos w)\,dw \tag{18}$$

satisfies conditions (i), (ii) and (iii), provided that G(w) fulfils the difference equation

$$\frac{G(w)}{G(w+\frac{3}{2}\pi)} = \frac{\cos\alpha - \sin w}{\cos\alpha + \sin w}. \tag{19}$$

This equation can be rewritten as follows:

$$\frac{G(w)}{G(w+\frac{3}{2}\pi)} = \frac{\cos\alpha - \cos(w+\frac{3}{2}\pi)}{\cos\alpha + \sin w} \cdot \frac{\cos\alpha + \sin(w+\frac{3}{2}\pi)}{\cos\alpha - \cos w}, \tag{20}$$

so that its general solution is of the form

$$G(w) = K(w)\, G_o(w), \tag{21}$$

where

$$G_o(w) = \frac{1}{(\cos\alpha + \sin w)(\cos\alpha - \cos w)} \tag{22}$$

and K(w) is the general solution of

$$K(w) = K(w + \frac{3}{2}\pi). \tag{23}$$

Equation (18) now becomes

$$\psi(\rho,\theta) = \int_C \left\{ \frac{K(w+\theta)}{\left[\cos\alpha + \sin(w+\theta)\right]\left[\cos\alpha - \cos(w+\theta)\right]} - \right.$$

$$\left. - \frac{K(w-\theta)}{\left[\cos\alpha - \sin(w-\theta)\right]\left[\cos\alpha - \cos(w-\theta)\right]} \right\} \exp(-ik\rho\cos w)\, dw. \qquad (24)$$

(c) The solution

In order to see the effect of condition (iv), we have to discuss the aymptotic behaviour of (24) for $\rho \to \infty$. For this purpose, we can apply the saddle-point method. The saddle points are $w = 0$ and $w = \pi$, and the corresponding steepest descent curves are represented by the arcs AB and DE in figure 2 (only the neighbourhood of the saddle points is important for $\rho \to \infty$).

In order to apply the saddle-point method, the contour C must be deformed in such a way as to incorporate these arcs. In this process, it has to sweep across poles of the integrand, which are required to produce the incident and elastically scattered waves. As we have seen above, the residues at these poles will contribute to the asymptotic behaviour of ψ only for $\theta = 0$. According to (9), the poles which are required for $\theta = 0$ are given by: $\cos w = k_o/k$, i. e. $w = \alpha - \frac{\pi}{2}$ (incident wave) and $\cos w = -k_o/k$, i. e. $w = \alpha + \frac{\pi}{2}$ (elastically scattered wave). These points are marked by circles in figure 2.

It is readily seen that all the zeros of the denominator of (24) in the lower half of the w-plane lie on the straight line

$\mathscr{I}w = -\beta$, where β is given by (16). The pole at $w = \alpha - \frac{\pi}{2}$ for $\theta = 0$ is supplied by the factor $[\cos\alpha + \sin(w+\theta)]^{-1}$ and the pole at $w = \alpha + \frac{\pi}{2}$ is supplied by the factor $[\cos\alpha - \sin(w-\theta)]^{-1}$. The contour C must lie entirely below the straight line $\mathscr{I}w = -\beta$ (cf. figure 2).

The integrand of (24) must have no poles in the lower half-plane within the strip $0 < \mathscr{R}\, w < \pi$. In fact, the residues at such poles would give rise to exponentially increasing terms for $\rho \to \infty$. The factors $[\cos\alpha - \cos(w \pm \theta)]^{-1}$ in (24), however, give rise to poles within this strip, at $w = \alpha \pm \theta$. We conclude that the numerators of (24) must have zeros at these points:

$$K(\alpha) = 0. \tag{25}$$

Since $K(w \pm \theta)$ must have no poles in the forbidden strip, we can also conclude from (23) that $K(w)$ has no poles in the lower half--plane.

The saddle points at $w = 0$ and $w = \pi$ contribute terms proportional to $\rho^{-\frac{1}{2}}\exp(-ik\rho)$ and $\rho^{-\frac{1}{2}}\exp(ik\rho)$, respectively. The former would correspond to an incoming circular wave, and must therefore be excluded. Thus, the expression within curly brackets in (24) must vanish for $w = 0$, so that

$$K(\theta) = K(-\theta). \tag{26}$$

It follows that $K(w)$ must be an even function of w, so that, according to the above, it can have no poles in the upper half-plane either.

Thus, K(w) must be an entire even periodic function with period $\frac{3}{2}\pi$. On the other hand, in order that condition (v) shall be fulfilled, $f(w,\theta)$ in (11) ought to be bounded for $|\mathcal{I} w| \to \infty$. According to (24), this implies [3]: $K(w) = \underline{O}\left[\exp(2|\mathcal{I} w|)\right]$ for $|\mathcal{I} w| \to \infty$. Therefore, K(w) must be of the form

$$K(w) = A_o + A_1 \cos\left(\frac{4}{3} w\right),$$

where A_o and A_1 are constants. According to (25), $A_o = -A_1 \cos(\frac{4}{3}\alpha)$, so that

$$K(w) = A\left[\cos\left(\frac{4}{3}\alpha\right) - \cos\left(\frac{4}{3} w\right)\right], \tag{27}$$

where A is a normalization factor.

Substituting (27) in (24), we finally get the solution of our problem:

$$\psi(\rho,\theta) = \int_C \left[H(\theta + w) - H(\theta - w)\right] \exp(-ik\rho\cos w)\, dw, \tag{28}$$

where

$$H(w) = \frac{A\left[\cos(\frac{4}{3}\alpha) - \cos(\frac{4}{3} w)\right]}{(\cos\alpha + \sin w)(\cos\alpha - \cos w)}. \tag{29}$$

The location of the poles of the integrand of (28) for all values of θ is shown in figure 2. It is convenient to deform the contour C in such a way as to sweep only across the poles at $w = -\theta - i\beta$ (incident wave) and $w = \pi + \theta - i\beta$ (elastically scattered wave). For this purpose, as shown in figure 2, we deform it into C' + C", where the vertical parts of the paths C' and C" are taken along the straight lines $\mathcal{R} w = -\frac{3}{4}\pi$, $\frac{3}{4}\pi$ and $\frac{7}{4}\pi$. Furthermore, since C' can be taken to be symmetrical about the origin, and the integrand of (28) is an odd function of w, the integral along C' identically vanishes, so that

$$\psi(\rho,\theta) = 2\pi i \sum \text{res} + \int_{C''} ,$$

where $\sum$ res denotes the sum of the residues of the integrand of (28) at the poles $w = -\theta - i\beta$ and $w = \pi + \theta - i\beta$.

The asymptotic behaviour of $\psi(\rho,\theta)$ for $\rho \longrightarrow \infty$ follows from this equation by applying the saddle-point method:

$$\psi(\rho,\theta) = 2\pi i \Big[\text{res } H(w)\big|_{w=-i\beta} \exp(-ik_o\rho\cos\theta) -$$

$$- \text{res } H(w)\big|_{w=-\pi+i\beta} \exp(ik_o\rho\cos\theta)\Big]\exp(-\lambda\rho\sin\theta) +$$

$$+\left(\frac{2\pi}{k\rho}\right)^{\frac{1}{2}}\Big[H(\theta+\pi) - H(\theta-\pi)\Big] \exp\Big[i(k\rho - \frac{\pi}{4})\Big]. \qquad (30)$$

By comparing this with (9), we finally get

$$A = \frac{k_o(k_o - i\lambda)}{\sqrt{3}\,\pi k^2\big[(\zeta\, e^{i\pi})^{2/3} - (\zeta\, e^{i\pi})^{-2/3}\big]} , \qquad (31)$$

$$f(k_o) = -\frac{k_o - i\lambda}{k_o + i\lambda}\left[\frac{\zeta^{2/3} - \zeta^{-2/3}}{(\zeta\, e^{i\pi})^{2/3} - (\zeta\, e^{i\pi})^{-2/3}}\right], \qquad (32)$$

$$F(k,\theta) = \frac{k_o(k_o - i\lambda)\sin(\frac{4}{3}\theta)}{\big[(\zeta\, e^{i\pi})^{2/3} - (\zeta\, e^{i\pi})^{-2/3}\big](k\sin\theta - i\lambda)(k\cos\theta + i\lambda)} , \qquad (33)$$

where

$$\zeta = \frac{k_o - \lambda}{k_o + \lambda} \quad . \tag{34}$$

The exact elastic scattering and break-up amplitudes are given by (32) and (33), respectively.

(d) Behaviour near the edge

The behaviour of the solution near the edge of the wedge can be determined by means of an expansion in a series of Bessel functions. This expansion can be derived in the same way as the corresponding expansion of Sommerfeld's branched wave functions (Frank-von Mises 1935).

Let B be a constant such that the contour C lies entirely below the line $\mathscr{I}\, w = -B$ $(B > \beta)$. The constant B can be chosen arbitrarily large. For sufficiently large B, the integrand of (28) can be expanded as follows:

$$H(\theta + w) - H(\theta - w) = 4iA\Big[\cos(\tfrac{2}{3}\,\theta)\exp(-\tfrac{2}{3}iw) + 2^{3/2}\cos\alpha\cos(\tfrac{5}{3}\,\theta + \tfrac{\pi}{4}).$$

$$.\exp(-\tfrac{5}{3}iw) - 2\cos(\tfrac{4}{3}\,\alpha)\cos(2\theta)\exp(-2iw) - 4\cos^2\alpha\sin(\tfrac{8}{3}\theta)\exp(-\tfrac{8}{3}iw) + \ldots\Big] \tag{35}$$

Substituting this in (28), and employing Sommerfeld's integral representation of the Bessel functions

$$J_p(z) = \frac{1}{2\pi}\int_C \exp\left[-iz\cos w - ip(w - \tfrac{\pi}{2})\right]dw, \tag{36}$$

we get the <u>convergent</u> expansion

$$\psi(\rho,\theta) = 8\pi i A\Big[\exp(-i\pi/3)\cos(\tfrac{2}{3}\,\theta)J_{\frac{2}{3}}(k\rho) - 2^{3/2}\exp(i\pi/6)\cos\alpha.$$

$$.\cos(\tfrac{5}{3}\,\theta + \tfrac{\pi}{4})\,J_{\frac{5}{3}}(k\rho) + 2\cos(\tfrac{4}{3}\,\alpha)\cos(2\theta)\,J_2(k\rho) + 4\exp(-i\pi/3).$$

$$.\cos^2\alpha\sin(\tfrac{8}{3}\,\theta)\,J_{\frac{8}{3}}(k\rho) + \ldots\Big]. \qquad (37)$$

In particular, for $k\rho \ll 1$, (37) becomes

$$\psi(\rho,\theta) = \frac{8\pi i A}{\Gamma(\frac{5}{3})}\Big[\exp(-i\pi/3)\cos(\tfrac{2}{3}\,\theta)(\tfrac{1}{2}\,k\rho)^{2/3} - \tfrac{3}{5}\,2^{3/2}\exp(i\pi/6)\cos\alpha.$$

$$.\cos(\tfrac{5}{3}\,\theta + \tfrac{\pi}{4})(\tfrac{1}{2}\,k\rho)^{5/3} + \Gamma(\tfrac{5}{3})\cos(\tfrac{4}{3}\,\alpha)\cos(2\theta)(\tfrac{1}{2}k\rho)^2 + \ldots\Big]. \qquad (38)$$

Thus, $\psi(\rho,\theta) = \underline{O}(\rho^{2/3})$ at the edge, whereas grad ψ has a singularity in $\rho^{-1/3}$. A singularity of the same order appears in the case of electromagnetic diffraction by a totally reflecting rectangular wedge (Nussenzveig 1959, p. 12). It is clear by comparing (35) with (38) that the behaviour of $f(w,\theta)$ at infinity in (11) is determined by the behaviour of $\psi(\rho,\theta)$ near the edge. All the assumptions which have been made in the derivation can be verified in the above results.

3. BEHAVIOUR OF THE AMPLITUDES

(a) The elastic scattering and break-up cross-sections

It follows from (9) that the elastic scattering cross-section [4] is given by

$$\sigma_e(k_o) = |f(k_o)|^2. \tag{39}$$

Below the break-up threshold, we take, according to (10), $k_o - \lambda = (\lambda - k_o) \exp(i\pi)$. It can readily be verified that, with the corresponding replacement: $\zeta \to |\zeta| \exp(i\pi)$, (32) remains valid below the threshold, and gives, as ought to be expected,

$$\sigma_e(k_o) = 1 \qquad (k_o < \lambda). \tag{40}$$

Above the threshold, according to (32) and (39),

$$\sigma_e(k_o) = 1 - \frac{3\,\zeta^{4/3}}{1 + \zeta^{4/3} + \zeta^{8/3}} \qquad (k_o \geqslant \lambda). \tag{41}$$

The behaviour of σ_e as a function of the dimensionless parameter $K = k_o/\lambda$, which is the square root of the ratio of the incident energy to the binding energy, is shown in figure 3. The elastic cross-section decreases quite rapidly as the incident energy increases. In particular, for $K \gg 1$, (41) becomes

$$\sigma_e \approx \frac{64}{27} K^{-2} \; (K \gg 1). \tag{42}$$

Thus, at high energies, the elastic cross-section is inversely proportional to the incident energy.

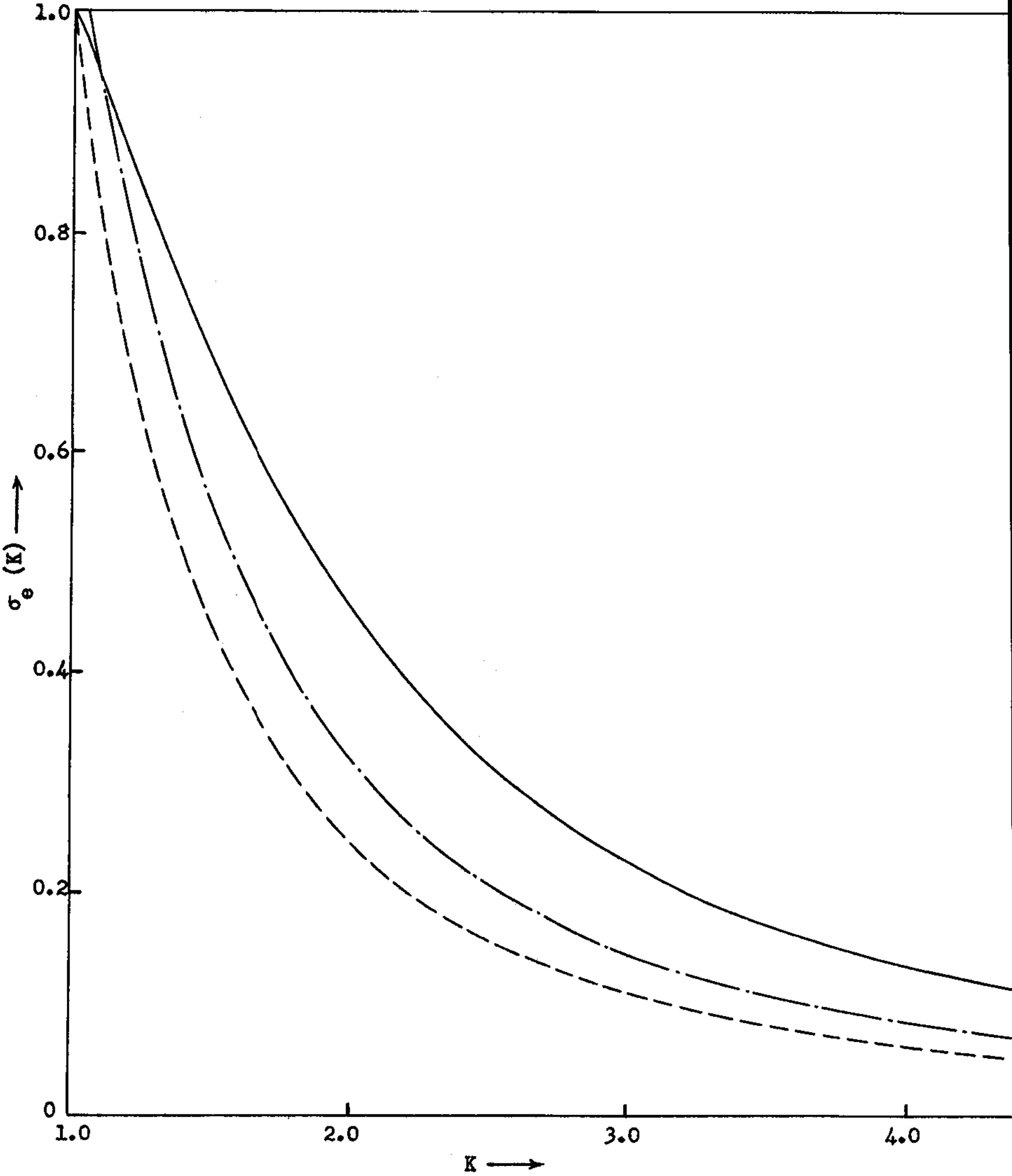

Fig. 3. The elastic scattering cross-section: —— exact solution; - - - impulse approximation; -.-.-.- iterated impulse approximation.

The amplitude $F(k,\theta)$ corresponds to a break-up process in which particles x and y come out with momenta $\hbar k_x = \hbar k \cos\theta$ and $\hbar k_y = -\hbar k \sin\theta$, respectively. It follows from (9) that the probability for such an event to take place in the interval $(\theta, \theta + d\theta)$ is $\sigma(k,\theta)d\theta$, where

$$\sigma(k,\theta) = \frac{4\lambda}{\pi k_o} |F(k,\theta)|^2 \tag{43}$$

is the break-up cross-section expressed as a function of k and θ. It is also possible to express it as a function of other pairs of variables: for instance, the probability for break-up to take place in the interval $(k_x, k_x + dk_x)$ is $\sigma(k,k_x)dk_x$, where

$$\sigma(k,k_x) = |k_y|^{-1} \sigma(k,\theta) \quad (k_x = k \cos\theta). \tag{44}$$

It follows from (43), (32) and (33) that

$$\sigma(k,\theta) = \frac{4}{3\pi} [1 - \sigma_e(k_o)] \frac{K(K^2+1)\sin^2(\frac{4}{3}\theta)}{[(K^2-1)^2 \sin^2\theta \cos^2\theta + K^2]}. \tag{45}$$

The unitarity condition above the break-up threshold takes the form

$$\sigma_e(k_o) + \int_0^{3\pi/4} \sigma(k,\theta)\, d\theta = 1. \tag{46}$$

To verify that (45) satisfies this condition, it suffices to employ the following result, which can be derived by the method of residues:

$$\int_0^{3\pi/4} \frac{\sin^2(\frac{4}{3}\theta)\, d\theta}{[(K^2-1)^2\sin^2\theta \cos^2\theta + K^2]} = \frac{3}{2}\int_0^{2\pi} \frac{\sin^2(2\theta)\, d\theta}{[(K^2-1)^2\sin^2(3\theta) + 4K^2]} =$$

$$= \frac{3\pi}{4K(K^2+1)} \quad . \tag{47}$$

The behaviour of $\sigma(K,\theta)$ as a function of θ, for several values of K, is shown in figure 4. It vanishes both for $\theta = 0$ and for $\theta = 3\pi/4$. For incident energies slightly above threshold, the total break-up cross-section is small and $\sigma(K,\theta)$ is a slowly-varying function of θ. The curve has a rather flat peak at $\theta \approx 3\pi/8$. The total break-up cross-section increases with the incident energy, and the peak is shifted towards $\theta = \frac{\pi}{2}$. This peak gets narrower and more pronounced as the energy increases. For $K \gg 1$, (45) becomes

$$\sigma(K,\theta) = \frac{4K}{3\pi} \quad \frac{\sin^2(\frac{4}{3}\,\theta)}{\left[1 + \frac{1}{4}\,K^2\sin^2(2\theta)\right]} \quad (K \gg 1), \tag{48}$$

which gives rise to a peak of height K/π and width $\Delta\theta \sim 2/K$ at $\theta = \frac{\pi}{2}$. In the limit $K \to \infty$, we find

$$\lim_{K \to \infty} \sigma(K,\theta) = \delta(\tfrac{\pi}{2} - \theta). \tag{49}$$

These results have a simple physical interpretation: if the incident energy is much larger than the binding energy, there tends to be a complete momentum transfer from the incident particle to the bound one. Under such circumstances, one would expect that the behaviour of the bound particle would approach that of a free one, and it is well known that a complete momentum exchange takes place in a collision between two free particles with a "hard-sphere" interaction (head-on collision between two billiard balls).

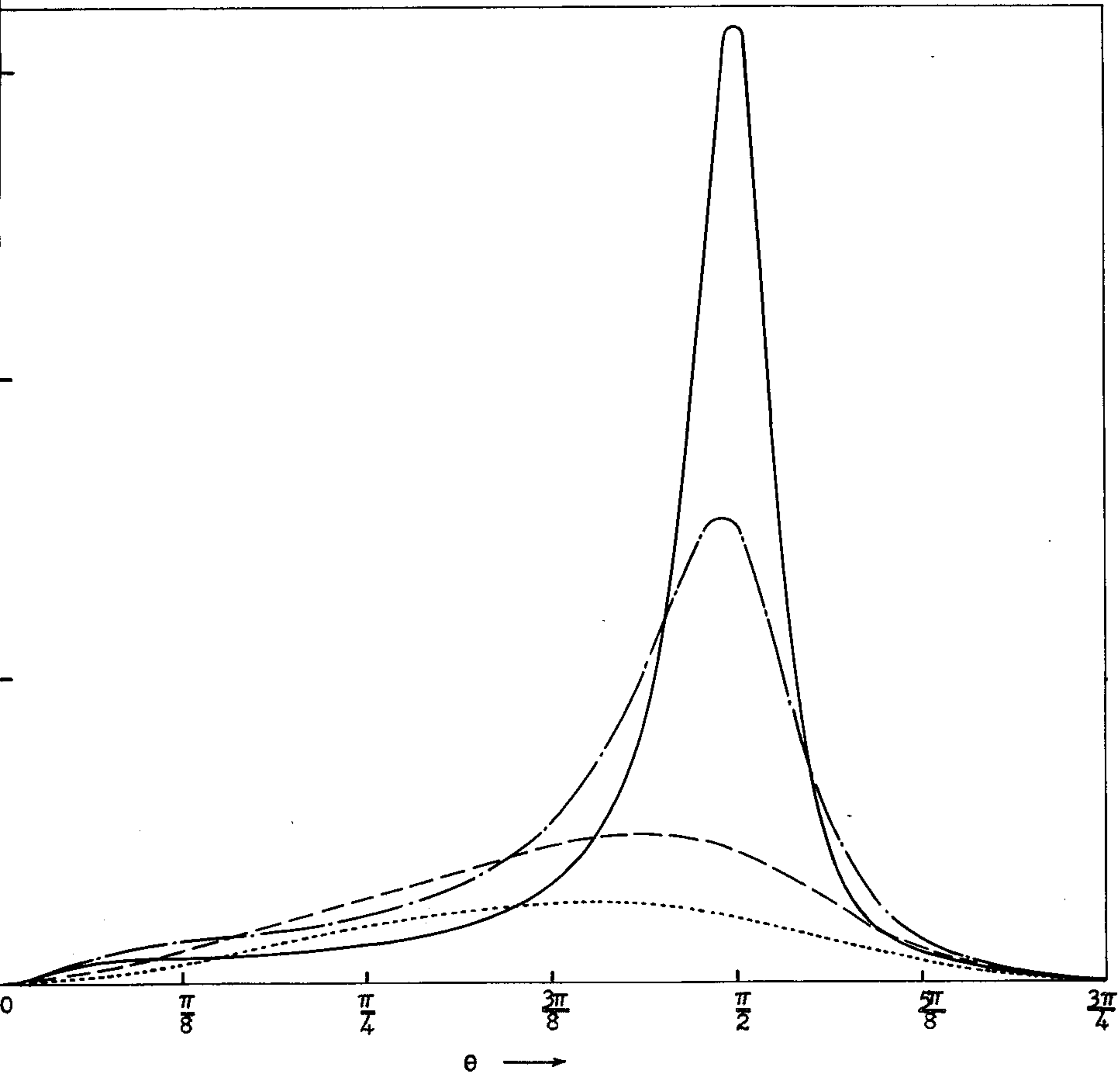

Fig. 4. The break-up cross-section as a function of θ for several values of K:
.... K = 1.5; - - - - K = 2; -.-.-.- K = 5; ——— K = 10.

(b) Analytic behaviour

The variables in terms of which one expresses the scattering amplitudes in order to study their analytic behaviour can be chosen in various ways. For the elastic scattering amplitude, a convenient variable is k_o, the incident wave number.

According to (32) and (34), $f(k_o)$ is an algebraic function of k_o having second-order [5] branch points at $k_o = \pm\lambda$ (break-up threshold). This disproves Wildermuth's conjecture, according to which these points would be logarithmic branch points (cf. § 1a).

The Riemann surface of $f(k_o)$ has three sheets, which are joined together at the two branch points. We shall take the branch cut along the real axis, from $k_o = -\lambda$ to $k_o = \lambda$. The sheets will be numbered according to the range of values taken by arg ζ: sheet I: $-\pi < \arg\zeta \leqslant \pi$; sheet II: $\pi < \arg\zeta \leqslant 3\pi$; sheet III: $3\pi < \arg\zeta \leqslant 5\pi$. Sheet I is the physical sheet, where $f(k_o)$ still represents the elastic scattering amplitude below the break-up threshold (on the upper edge of the cut).

The only singularities of $f(k_o)$, besides the two branch points, are the following poles (the roman index denotes the sheet on which the pole is located) : $k_o = (-i\lambda)_I$: simple pole; $k_o = (\infty)_{II}$: simple pole; $k_o = (0)_{III}$: simple pole; $k_o = (-i\lambda)_{III}$: double pole.

It is noteworthy that the only branch points are those to be expected on kinematic grounds. It is not immediately clear, however, whether the nature of these branch points can be predicted on kinematic grounds alone. As will become apparent in §4a, there exists a close relationship between the three-sheeted character of $f(k_o)$ and the three-sheeted character of the wave function in configuration space. The latter, according to (37), is connected with the behaviour of the solution near the edge of the wedge ($\rho \longrightarrow 0$). Thus, the nature of the branch points is determined essentially by three-body collisions, and it may depend to some extent on the nature of the interactions.

An interesting question has been raised by Peierls (1959) in connection with the well-known redundant singularities of non-relativistic scattering amplitudes, which are found in the case of potentials having exponential or Yukawa tails. The question is whether such singularities still occur in local field theory, where the basic interactions have zero range, but the colliding objects may have appreciable size because of the fields surrounding them.

In the present example, all the interactions have zero range, but the exponential tail of the bound state corresponds to a "size" of the compound system. The situation is of course quite different from that of field theory, but it may be worth while to note that the redundant singularities characteristic of an exponential potential do not appear in the present case. The argument is not quite conclusive, however, because the wave function is confined to the region $y \leqslant 0$, and a "one-sided" exponential potential in one dimension does not give rise to redundant singularities.

The analytic behaviour of the break-up amplitude can be discussed, for instance, in terms of the variables (k_o, θ).

According to (33), $F(k_o,\theta)$ is, for fixed θ, an algebraic function of k_o having fifth-order branch points at $k_o = \pm\lambda$. The associated Riemann surface has six sheets, which can be obtained by combining the three sheets of the Riemann surface of $f(k_o)$ with the two sheets of the Riemann surface of the factor $k = (k_o^2 - \lambda^2)^{\frac{1}{2}}$ in the denominator of (33). Besides the two branch points, there are a small number of poles, which can easily be obtained from (33). Some of them are independent of θ.

For a fixed value of k_o, $F(k_o,\theta)$ is a periodic function of θ with period 6π, whose only singularities are poles. The location of these poles follows at once from (33).

4. APPROXIMATION METHODS

In this §, we shall investigate the accuracy of some approximation methods usually employed in the solution of problems of this kind. These methods are based upon the replacement of the exact wave function by an approximate expression in an otherwise exact representation of the scattering amplitudes. As a preliminary step, we shall now derive this representation for the present problem.

(a) Integral representation of the amplitudes

The scattered wave $\psi_s(x,y) = \psi(x,y) - \exp(-ik_o x + \lambda y)$ fulfils the following conditions, which are equivalent to conditions (i) to (v), §2: (i) $(\Delta + k^2)\,\psi_s = 0$ $(x > y;\ y < 0)$; (ii) $\frac{\partial\psi_s}{\partial y}(x,0) =$

$= \lambda \psi_s(x,0) (x > 0)$; (iii) $\psi_s(x,x) = -\exp(-ik_o x + \lambda x)$ $(x < 0)$; (iv) ψ_s contains only outgoing waves at infinity; (v) ψ_s is bounded at the edge of the wedge.

Let $G(xy|x'y')$ be the Green's function satisfying the following conditions: (i') $(\Delta_{x'y'} + k^2)\, G = 0$ $(x' > y';\ y' < 0)$; (ii') $\frac{\partial G}{\partial y'}(xy|x'0) = \lambda G(xy|x'0)$; (iii') G is purely outgoing for $(x^2 + y^2)^{\frac{1}{2}} \rightarrow \infty$; (iv') G is bounded at the edge of the wedge.

The scattered wave can be expressed as follows in terms of G:

$$\psi_s(x,y) = \int_{-\infty}^{0} \exp(-ik_o x' + \lambda x') \left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right) G(xy|x'y')\right]_{x'=y'} dx' + \int_{-\infty}^{0} G(xy|x'x') \left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right) \psi_s(x', y')\right]_{x'=y'} dx' \ . \tag{50}$$

Note that $\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)_{x'=y'} dx' = -\frac{\partial}{\partial n'} ds'$, where n' is the direction of the normal to the first bisector $(x' = y')$, pointing out of the region $x' \geqslant y'$, and ds' is the line element along the bisector.

The Green's function can be constructed by expansion in eigenfunctions. The complete orthonormal set of eigenfunctions in the y variable is given by

$$\chi_o(y) = (2\lambda)^{\frac{1}{2}} \exp(\lambda y), \tag{51}$$

$$\chi(k_y,\ y) = (2/\pi)^{\frac{1}{2}} \sin(k_y y + \eta) \qquad (k_y \geqslant 0), \tag{52}$$

where $y \leqslant 0$ and [6]

$$\exp(2i\eta) = -\frac{k_y - i\lambda}{k_y + i\lambda} \ . \tag{53}$$

It will be convenient from now on to assume that k^2 has an infinitesimal positive imaginary part; thus, according to (iv), ψ_s must be bounded at infinity. The Green's function is then given by

$$G(xy|x'y') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_x \frac{e^{ik_x(x-x')}}{k_o^2 - k_x^2} \chi_o(y)\,\chi_o(y') +$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_x \int_0^{\infty} dk_y \frac{e^{ik_x(x-x')}\chi(k_y,y)\,\chi(k_y,y')}{k^2 - k_x^2 - k_y^2}. \qquad (54)$$

Performing the integrations with respect to k_x, and substituting the result in (50), we get [7]

$$(2\lambda)^{\frac{1}{2}}\,\psi_s(x,y) = \varphi_o(x)\,\chi_o(y) + \int_0^{\infty} \varphi(k_y, x)\,\chi(k_y, y)\,dk_y, \qquad (55)$$

where

$$\varphi_o(x) = \frac{i\lambda}{k_o}\int_{-\infty}^{0} e^{ik_o|x-x'|+\lambda x'}\Big\{\left[ik_o\epsilon(x-x')+\lambda\right] e^{-ik_o x'+\lambda x'} -$$

$$- \left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)\psi_s(x',y')\right]_{x'=y'}\Big\}\,dx' \qquad (56)$$

and

$$\varphi(k_y, x) = i(\lambda/\pi)^{\frac{1}{2}}\,k_x^{-1}\int_{-\infty}^{0} e^{ik_x|x-x'|}\Big\{e^{-ik_o x'+\lambda x'}\left[k_y\cos(k_y x'+\eta) +\right.$$

$$\left. + ik_x\epsilon(x-x')\sin(k_y x'+\eta)\right] - \sin(k_y x'+\eta)\left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)\psi_s(x',y')\right]_{x'=y'}\Big\}dx'. \qquad (57)$$

In these equations, $\epsilon(x) = 1$ for $x > 0$, $\epsilon(x) = -1$ for $x < 0$, and

$$k_x = (k^2 - k_y^2)^{\frac{1}{2}} \quad (\mathcal{R}\,k_x \geqslant 0;\ \mathcal{I}\,k_x > 0). \qquad (58)$$

According to (55), the probability amplitudes associated with elastic scattering and with break-up leading to the eigenstate $\chi(k_y,y)$ of particle y are $\varphi_o(x)$ and $\varphi(k_y, x)$, respectively. Thus,

$$\varphi_o(x) \rightarrow f(k_o)\exp(ik_o x) \quad \text{for } x \rightarrow \infty , \tag{59}$$

where $f(k_o)$ is the elastic scattering amplitude. Similarly, let

$$\varphi(k_y,x) \rightarrow f_{\pm}(k,k_y)\exp(\pm ik_x x) \quad \text{for } x \rightarrow \pm\infty , \tag{60}$$

where the upper (lower) signs go together. Then, the probability for break-up in the interval (k_y,k_y+dk_y), with the x-particle coming out in the positive (negative) x-direction, is $\sigma_{\pm}(k,k_y)dk_y$, where

$$\sigma_{\pm}(k,k_y) = \frac{k_x}{k_o}|f_{\pm}(k,k_y)|^2. \tag{61}$$

The asymptotic behaviour of ψ_s for $(x^2+y^2)^{\frac{1}{2}} \rightarrow \infty$ can be obtained by inserting (59) and (60) in (55) and by applying the method of stationary phase (Erdélyi 1956). By comparing the result with (9), one finds

$$F(k,\theta) = \tfrac{1}{2}(\pi/\lambda)^{\frac{1}{2}}\exp\left[-i\eta(\overline{k_y})\right]\overline{k_x}\, f_{\pm}(k,\overline{k_y}), \tag{62}$$

where $\overline{k_x} = k|\cos\theta|$, $\overline{k_y} = k\sin\theta$ is the stationary phase point, and f_+ or f_- is to be employed according as to whether $\theta < \frac{\pi}{2}$ or $\theta > \frac{\pi}{2}$, respectively. It is readily seen that (61) and (62) agree with (43) and (44).

By letting $|x| \longrightarrow \infty$ in (56) and (57), and by comparing the results with (59) and (60), one finally gets the following exact representation of the scattering amplitudes:

$$f(k_o) = -\frac{i\lambda}{k_o}\int_{-\infty}^{0} e^{-ik_o x'+\lambda x'}\left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)\psi(x', y')\right]_{x'=y'} dx', \quad (63)$$

$$f_{\pm}(k, k_y) = -\frac{i}{k_x}(\lambda/\pi)^{\frac{1}{2}}\int_{-\infty}^{0} e^{\mp ik_x x'}\sin(k_y x'+\eta)\cdot$$

$$\cdot\left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)\psi(x', y')\right]_{x'=y'} dx'. \quad (64)$$

These expressions correspond to the well-known representation of transition amplitudes which was introduced by Lippmann and Schwinger (1950). To show this, let $\phi_b = \exp(ik_o x)\,\chi_o(y)$, $\phi_b = \exp(\pm ik_x x)\,\chi(k_y, y)$ denote the final-state wave functions in the absence of interaction corresponding to elastic scattering and to break-up, respectively. Let $k_{xb} = k_o$ in the former case, and $k_{xb} = k_x$ in the latter one. Let $\Psi_a^{(+)}(x,y) = (2\lambda)^{\frac{1}{2}}\,\psi(x,y)$ denote the total wave function corresponding to the incident wave $\exp(-ik_o x)\cdot\chi_o(y)$. Then, (63) and (64) can be rewritten in the form

$$f_{ba} = (2ik_{xb})^{-1}(\phi_b, U\Psi_a^{(+)}) = (2ik_{xb})^{-1}\int \phi_b^*(\xi,\eta)U(\xi)\Psi_a^{(+)}(\xi,\eta)d\xi\, d\eta, \quad (65)$$

where $\xi = x - y$, $\eta = x + y$, and the integration is extended over the whole range of values of ξ and η. The effective interaction $U(\xi)$ between the particles x and y is given by

$$U(\xi) = \delta(\xi)\frac{\partial}{\partial\xi} = -\delta'(\xi). \quad (66)$$

Note that we consider only functions defined in the domain $\xi \geqslant 0$, so that $\int_0^\infty \delta(\xi)\, d\xi = 1$.

The interaction (66) is a sort of "pseudopotential" associated with the one-dimensional "hard-sphere" boundary condition (Fermi 1936). The factor $(2ik_{xb})^{-1}$ in (65) is a characteristic factor which appears in one-dimensional scattering amplitudes (cf. Morse & Feshbach 1953, p. 1071).

The representations (63) and (64) establish a connection between the analytic behaviour of the amplitudes and the analytic continuation of the wave function for complex energies. In particular, as was mentioned in §3b, it is clear that the threshold branch points are closely related to the branched character of the wave function (37) in configuration space.

(b) The impulse approximation

The impulse approximation was introduced by Chew (1950). It amounts, in the present problem, to the replacement of the exact wave function in (63) and (64) by an approximate wave function $\psi^{(0)}(x,y)$, which represents the scattering of an x-particle of momentum $\hbar k_0$ by a wave packet of free y-particles having the same momentum distribution as the bound-state wave function $\theta(-y)\exp(\lambda y)$ (θ denotes the Heaviside step function). This corresponds to the assumption that the interaction between the y-particle and the centre of force can be neglected during the collision, its only effect being the generation of the momentum distribution of the y-particle.

The usual criteria for the validity of this approximation (Chew 1950, Chew & Wick 1951), as adapted to the present problem, would be the following: (I) the range of the interactions must be much smaller than the "radius" of the bound state; (II) the incident energy must be much larger than the binding energy. In the present example, condition I is obviously fulfilled (in a particularly favourable way) and condition II is equivalent to the requirement: $k_o \gg \lambda$. Thus, according to these criteria, the impulse approximation should be a very good approximation for $K = k_o/\lambda \gg 1$.

Since the interaction between particles x and y is of the "hard sphere" type, the wave function $\psi^{(0)}(x,y)$ is given by

$$\psi^{(0)}(x,y) = \theta(-y)\exp(-ik_o x + \lambda y) - \theta(-x)\exp(\lambda x - ik_o y). \qquad (67)$$

Substituting this result in (63) and (64) and taking into account (62), we find the expressions for the amplitudes in the impulse approximation:

$$f^{(0)}(k_o) = -\frac{i\lambda}{k_o}\left(\frac{k_o - i\lambda}{k_o + i\lambda}\right), \qquad (68)$$

$$F^{(0)}(k,\theta) = -\frac{i}{2}\;\frac{k(k_o - i\lambda)\sin\theta}{(k\sin\theta - i\lambda)(k\cos\theta + i\lambda)}. \qquad (69)$$

The corresponding cross-sections follow from (39) and (43):

$$\sigma_e^{(0)}(k_o) = K^{-2}, \qquad (70)$$

$$\sigma^{(0)}(k,\theta) = \frac{1}{\pi}\left[1 - \sigma_e^{(0)}(k_o)\right]\frac{K(K^2+1)\sin^2\theta}{\left[(K^2-1)^2\sin^2\theta\cos^2\theta + K^2\right]}. \qquad (71)$$

Notice the similarity between (71) and (43).

The behaviour of $\sigma_e^{(0)}$ as a function of K is shown by the curve in dashed line in figure 3. Although its qualitative behaviour agrees with that of the exact solution, the quantitative agreement is far from good. Even at high incident energies, according to (42), there is a discrepancy by more than a factor of two.

The behaviour of $\sigma^{(0)}(k,\theta)$ as a function of θ for K = 1.5 and for K = 5 is shown by the curves in dashed line in figures 5 and 6, respectively. The curves in full line represent the exact solution.

For K = 1.5, which is still close to the threshold, the agreement is good for $\theta \lesssim 3\pi/8$, but becomes very bad beyond this value. For K = 5, the agreement is reasonably good within the width of the peak centred at $\theta \approx \pi/2$, but becomes quite poor outside of it. The situation is similar for larger values of K. It is readily seen that the result (49) is also valid for $\sigma^{(0)}$ in the limit as $K \rightarrow \infty$.

Thus, in the present example, the impulse approximation for the elastic cross-section is not a good approximation, even at high incident energies. It gives good results for the break-up cross-section at high energies within the width of the dominant peak of the cross-section, but not outside of it.

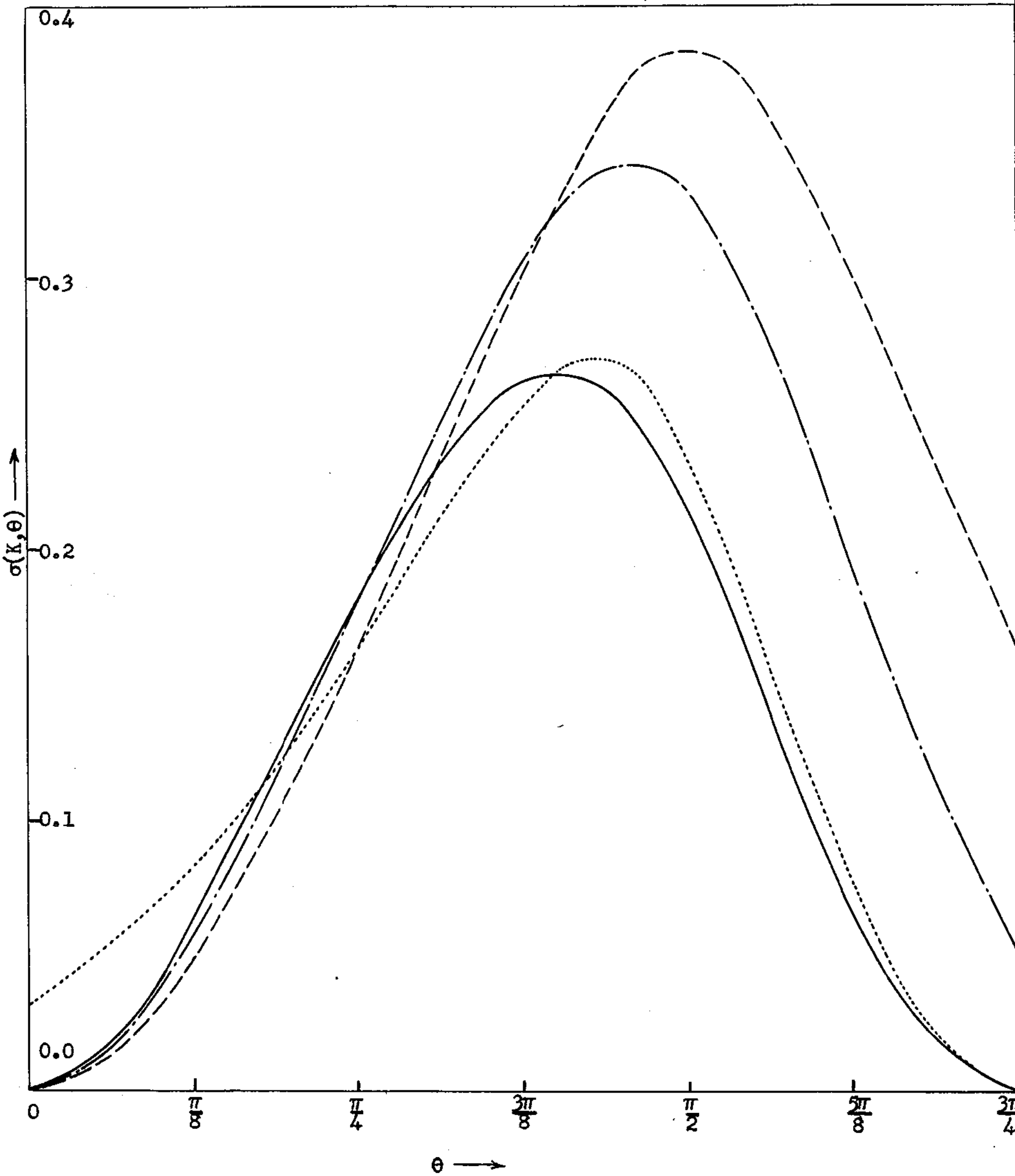

Fig. 5. The break-up cross-section for K = 1.5: —— exact solution; - - - -impulse approximation; .-.-.-. iterated impulse approximation; .... Born approximation

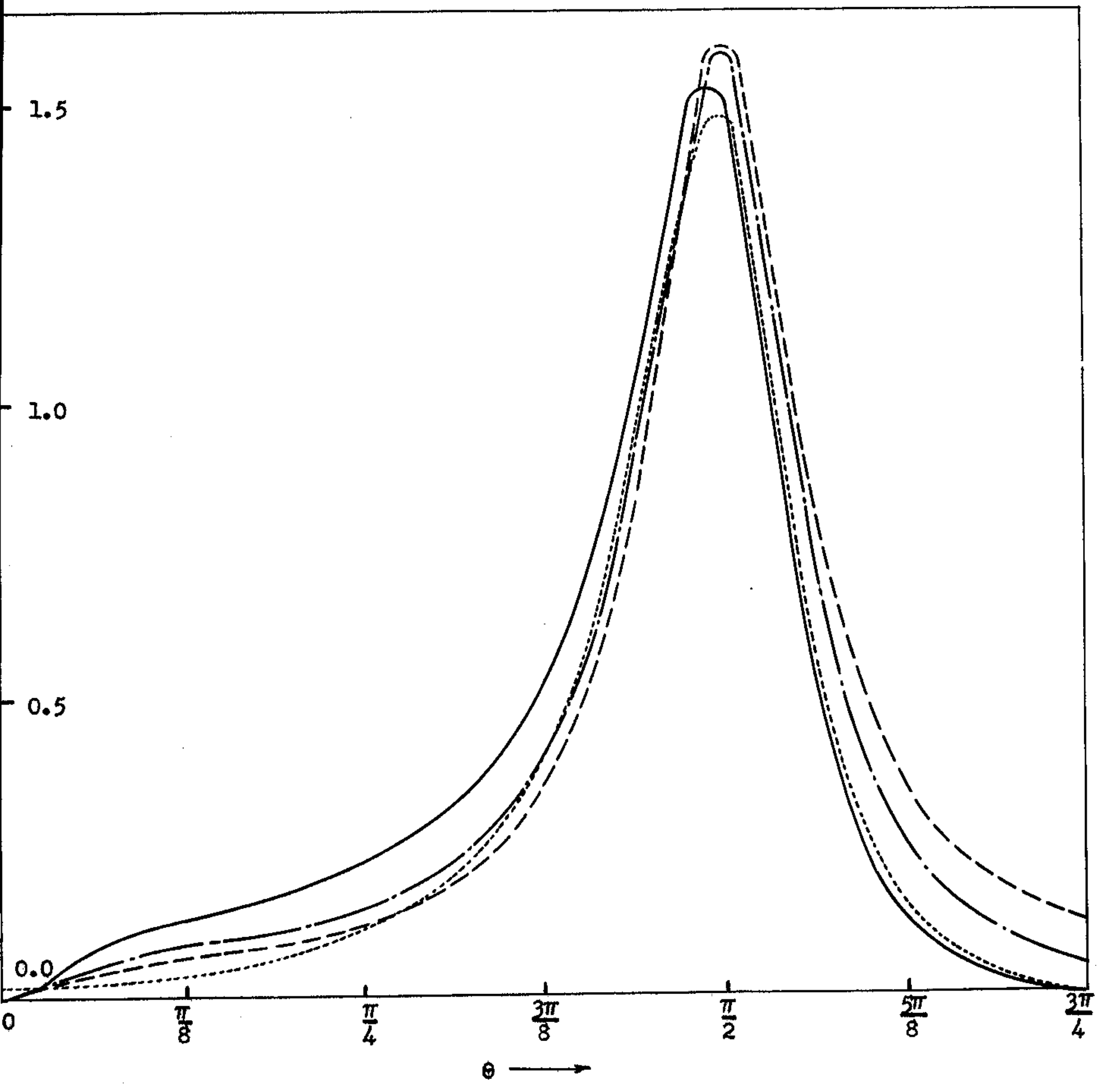

Fig. 6. The break-up cross-section for K = 5: ——— exact solution; - - - - impulse approximation; -.-.-. iterated impulse approximation; .... Born approximation.

(c) The iterated impulse approximation

According to (67), the scattered wave in the impulse approximation is given by: $\psi_s^{(0)} = -\,\theta(-x)\exp(\lambda x - ik_o y)$. If we replace ψ_s by $\psi_s^{(0)}$ in the r. h. s. of (50), we obtain a new approximation $\psi_s^{(1)}$. Substituting ψ by $\psi^{(1)} = \theta(-y)\exp(-ik_o x + \lambda y) + \psi_s^{(1)}$ in (63) and (64), we get the iterated impulse approximation.

In order to compute $\psi_s^{(1)}$, it is convenient to express the Green's function (54) as an integral over k_x, by performing first the integration with respect to k_y. This leads to

$$G(xy|x'y') = \frac{1}{4\pi i}\int_{-\infty}^{+\infty} \exp\left[ik_x'(x-x')\right]\left\{\exp(ik_y'|y-y'|) + \right.$$

$$\left. + \left(\frac{k_y' + i\lambda}{k_y' - i\lambda}\right)\exp\left[-ik_y'(y+y')\right]\right\}\frac{dk_x'}{k_y'}, \tag{72}$$

where

$$k_y' = (k^2 - k_x'^2)^{\frac{1}{2}} \qquad (\mathcal{R}\,k_y' \geqslant 0;\ \mathcal{I}\,k_y' > 0). \tag{73}$$

Substituting ψ_s by $\psi_s^{(0)}$ and G by (72) in (50), and carrying out the integration over x', we obtain

$$\psi_s^{(1)}(x,y) = \frac{(k_o - i\lambda)}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp\left[i(k_x'x - k_y'y)\right]}{(k_x' + i\lambda)(k_y' - i\lambda)}\,dk_x'. \tag{74}$$

Replacing ψ by $\psi^{(1)}$ in (63) and (64), and performing the integration with respect to x', we get

$$f^{(1)}(k_o) = \frac{1}{2}f^{(0)}(k_o) - \frac{\lambda(k_o - i\lambda)}{2\pi k_o}\ I, \tag{75}$$

$$f_+^{(1)}(k,k_y) = \frac{1}{2}f_+^{(0)}(k,k_y) + i(\lambda/\pi)^{\frac{1}{2}}\frac{(k_o - i\lambda)}{4\pi k_x}\exp\left[-i\eta(k_y)\right]J, \tag{76}$$

(and similarly for $f_-^{(1)}$), where

$$I = \int_{-\infty}^{+\infty} \frac{(k_x'+k_y')\, dk_x'}{(k_x'+i\lambda)(k_y'-i\lambda)(k_x'-k_y'-k_o-i\lambda)} , \qquad (77)$$

$$J = \int_{-\infty}^{+\infty} \frac{(k_x'+k_y')}{(k_x'+i\lambda)(k_y'-i\lambda)} \left[\frac{1}{k_x'-k_y'-k_x-k_y} + \left(\frac{k_y - i\lambda}{k_y + i\lambda}\right) \frac{1}{k_x'-k_y'-k_x+k_y}\right] dk_x' . \qquad (78)$$

The integrals I and J will be computed in the Appendix. Substituting the results, given by (A4) and (A7), in (75) and (76), and taking into account (62), (68) and (69), we finally get the iterated impulse approximation:

$$f^{(1)}(k_o) = \left[1 + \frac{(k_o^2 - \lambda^2)}{4\pi k_o} \log\left(\frac{k_o + \lambda}{k_o - \lambda}\right) + \frac{i\lambda}{4k_o}\right] f^{(0)}(k_o), \qquad (79)$$

$$F^{(1)}(k,\theta) = \left[1 - \frac{\lambda}{4\pi k_o} \log\left(\frac{k_o + \lambda}{k_o - \lambda}\right) + \frac{\theta \cot \theta}{2\pi} + \frac{i\lambda}{4k_o}\right] F^{(0)}(k,\theta), \qquad (80)$$

$$\sigma_e^{(1)}(k_o) = \left\{ \left[1 + \frac{(K^2 - 1)}{4\pi K} \log\left(\frac{K + 1}{K - 1}\right)\right]^2 + \frac{1}{16\, K^2} \right\} \sigma_e^{(0)}(k_o), \qquad (81)$$

$$\sigma^{(1)}(k,\theta) = \left\{ \left[1 - \frac{1}{4\pi K} \log\left(\frac{K + 1}{K - 1}\right) + \frac{\theta \cot \theta}{2\pi}\right]^2 + \frac{1}{16\, K^2} \right\} \sigma^{(0)}(k,\theta). \qquad (82)$$

The behaviour of $\sigma_e^{(1)}$ as a function of K is shown in figure 3 by the dash-and-dotted curve. This curve begins a little to the right of K = 1, because (81) cannot be applied in a small neighbourhood of this point, where it violates the condition: $\sigma_e \leqslant 1$. Beyond this neighbourhood, $\sigma_e^{(1)}$ is a better approximation than $\sigma_e^{(0)}$, but the deviations from the exact solution are still quite large.

Similar remarks apply to the behaviour of $\sigma^{(1)}(k,\theta)$ as a function of θ, which is shown, for K = 1.5 and K = 5, by the dash-and-dotted curves in figures 5 and 6. It gives better results than the impulse approximation, but it is still a poor approximation in the same regions as the former one.

Let us now consider the analytic behaviour of the iterated impulse approximation. The most interesting feature is the appearance of branch points at the break-up threshold, on account of the logarithmic terms in (79) and (80). These terms have precisely the same form as those found by Wildermuth (1949) in the second Born approximation of his problem (cf. §1a). The substitution of an algebraic branch point by a logarithmic one in an approximate solution obtained by iteration is a not unfamiliar phenomenon (cf. Nussenzveig 1959, pp. 26 and 40), which shows that it may be misleading to draw inferences about the analytic behaviour from such an approximation.

(d) The Born approximation

The Born approximation in the usual sense would be obtained by replacing $\Psi(x', y')$ by the incident wave in (63) and (64). It is readily seen that the result would differ from the impulse approximation only by a factor of 1/2. The same relation exists between the second Born approximation, defined in this way, and the iterated impulse approximation. Thus, Born's first and second order approximations, in the usual sense, are very bad approximations, as ought to be expected for a "hard-sphere" interaction.

However, there is a different definition of the Born approximation which leads to more accurate results in the present problem. Instead of expressing the scattered wave in terms of its normal derivative on the bisector, as in (50), one can express it in terms of its value on the x-axis. It suffices to employ the Green's function $G_0(xy|x'y')$ which fulfils the condition: $G_0(xy|x'x') = 0$, instead of condition (ii'), §4a. This function is given by (Morse & Feshbach 1953, p. 813)

$$G_0(xy|x'y') = \frac{1}{4i}\left[H_0^{(1)}\left(k\sqrt{(x-x')^2+(y-y')^2}\right) - H_0^{(1)}\left(k\sqrt{(x-y')^2+(y-x')^2}\right)\right], \tag{83}$$

where $H_0^{(1)}(z)$ is Hankel's function of the first kind of order zero.

In the place of (50), we get

$$\psi_s(x,y) = \int_{-\infty}^{0} \exp(-ik_0x' + \lambda x')\left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right) G_0(xy|x'y')\right]_{x'=y'} dx' +$$

$$+ \int_0^{\infty} \left[\frac{\partial G_0}{\partial y'}(xy|x'0) - \lambda G_0(xy|x'0)\right]\psi_s(x',0)\, dx'. \tag{84}$$

For $K \gg 1$, it is to be expected that the amplitude of the scattered wave on the x-axis will be very small, as compared with that of the incident wave [8]. We shall therefore define the Born approximation as that which is obtained by taking $\psi_s(x',0) = 0$ in (84).

If we then let $\rho \to \infty$ along the direction θ, we find, replacing Hankel's functions by their asymptotic expansions in (84), the following expression for the asymptotic behaviour of the scattered wave in the Born approximation:

$$\psi_s^{(B)}(x,y) \approx -\left(\frac{2}{\pi k\rho}\right)^{\frac{1}{2}} \exp\left[i(k\rho - \tfrac{\pi}{4})\right] \frac{k}{2} (\sin\theta + \cos\theta) \cdot$$

$$\cdot \int_{-\infty}^{0} \exp\left[-i(k_o + i\lambda)x' + ik(\sin\theta - \cos\theta)x'\right] dx' \quad (\text{for } \rho \to \infty).$$

There is no elastically scattered wave in this approximation. Comparison with (9) shows that the break-up amplitude in the Born approximation is given by

$$F^{(B)}(k,\theta) = \frac{ik(\sin\theta + \cos\theta)}{2\left[k(\sin\theta - \cos\theta) - k_o - i\lambda\right]}, \tag{85}$$

and the corresponding cross-section is

$$\sigma^{(B)}(k,\theta) = \frac{(K^2-1)\left[1+\sin(2\theta)\right]}{2\pi K\left[K^2 - K(K^2-1)^{\frac{1}{2}}(\sin\theta - \cos\theta) - (K^2-1)\sin\theta\cos\theta\right]}. \tag{86}$$

The behaviour of $\sigma^{(B)}(k,\theta)$ as a function of θ for $K = 1.5$ and for $K = 5$ is shown by the dotted curves in figures 5 and 6. For small values of θ, the deviation from the exact solution is larger than that of the previously considered approximations (note that $\sigma^{(B)} \neq 0$ for $\theta = 0$). Then, up to $\theta \sim 3\pi/8$ or $\pi/2$, the accuracy is about the same as that of these approximations. Finally, for $\theta > \pi/2$, in contrast with the former approximations, there is good agreement with the exact solution. In particular, $\sigma^{(B)} = 0$ for $\theta = 3\pi/4$. For $K \to \infty$, $\sigma^{(B)}$ also tends to the correct limit (49). Thus, on the whole, the Born approximation gives better results than the previous ones.

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APPENDIX - COMPUTATION OF THE INTEGRALS I AND J

The integral I defined in (77) can be rewritten as follows:

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk_x'}{k_x'^2 + \lambda^2} - \frac{i\lambda}{2}\int_{-\infty}^{+\infty} \frac{(2k_x' + k_o + i\lambda)dk_x'}{(k_x'^2 + \lambda^2)(k_x'^2 - k_o^2)} \; -$$

$$- \frac{1}{2(k_o + i\lambda)}\left[k_o^2\int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 + \lambda^2)k_y'} + \lambda^2\int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 - k_o^2)k_y'}\right]. \qquad \text{(A1)}$$

The first two integrals can be computed by residues; note that $\mathcal{J} k_o > 0$. On the other hand, according to (73),

$$\int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 + \lambda^2)k_y'} = \frac{1}{\lambda k_o} \left[\tan^{-1} \left(\frac{k_o k_x'}{\lambda k_y'} \right) \right]_{-\infty}^{+\infty} = \frac{\pi}{\lambda k_o} - \frac{i}{\lambda k_o} \log \left(\frac{k_o + \lambda}{k_o - \lambda} \right), \tag{A2}$$

$$\int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 - k_o^2)k_y'} = - \frac{1}{\lambda k_o} \left[\tan^{-1} \left(\frac{\lambda k_x'}{k_o k_y'} \right) \right]_{-\infty}^{+\infty} = \frac{i}{\lambda k_o} \log \left(\frac{k_o + \lambda}{k_o - \lambda} \right). \tag{A3}$$

Substituting these results in (A1), we finally get

$$I = \frac{\pi i}{(k_o + i\lambda)} \left[1 + \frac{1}{2\pi} \left(\frac{k_o}{\lambda} - \frac{\lambda}{k_o} \right) \log \left(\frac{k_o + \lambda}{k_o - \lambda} \right) + \frac{i\lambda}{2k_o} \right]. \tag{A4}$$

The integral J defined in (78) can be rewritten as follows:

$$J = \frac{k_y}{(k_y + i\lambda)} \left\{ \int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x' - k_x)(k_x' + i\lambda)} + \mathcal{P} \int_{-\infty}^{+\infty} \frac{(k_x'^2 + i\lambda k_x - k^2)}{(k_x'^2 - k_o^2)(k_x'^2 - k_y^2)} dk_x' - \right.$$

$$\left. - \frac{1}{k_x + i\lambda} \left[\lambda^2 \int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 - k_o^2)k_y'} + k_x^2 \mathcal{P} \int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 - k_y^2)k_y'} \right] \right\}. \tag{A5}$$

Note that $\mathcal{J} k_x > 0$ according to (58), but k_y is real (cf. (55) and (60)), so that the second and fourth integrals in (A5) are Cauchy principal values. We have:

$$\mathcal{P} \int_{-\infty}^{+\infty} \frac{dk_x'}{(k_x'^2 - k_y^2)k_y'} = \frac{1}{2k_x k_y} \mathcal{P} \left[\log \left(\frac{k_y k_y' - k_x k_x'}{k_y k_y' + k_x k_x'} \right) \right]_{-\infty}^{+\infty}$$

$$= - \frac{2i}{k_x k_y} \tan^{-1} \left(\frac{k_y}{k_x} \right). \tag{A6}$$

Substituting (A3) and (A6) in (A5), and computing the first two integrals by the method of residues, we get

$$J = \frac{2\pi i k_y}{(k_x+i\lambda)(k_y-i\lambda)}\left[1 - \frac{\lambda}{2\pi k_o}\log\left(\frac{k_o+\lambda}{k_o-\lambda}\right) + \frac{1}{\pi}\frac{k_x}{k_y}\tan^{-1}\left(\frac{k_y}{k_x}\right) + \frac{i\lambda}{2k_o}\right]. \tag{A7}$$

* * *

FOOTNOTES

1. In the case of the Schrödinger equation, the physical requirement is that the flux of the probability current through a circle of radius ρ centred at the origin shall vanish as $\rho \to 0$ (Pauli 1933). This would still allow a singularity $\psi(\rho, \theta) = \underline{O}(\rho^{-\gamma})$ at the edge, with $\gamma < \frac{1}{2}$.

2. According to the previous footnote, one might have: $f(w,\theta) = \underline{O}[\exp(i\gamma w)]$ $(\gamma < \frac{1}{2})$ for $|w| \to \infty$ along C.

3. According to the preceding footnotes, one might have $K(w) = \underline{O}\left\{\exp\left[(2+\gamma)|\mathcal{I}\ w|\right]\right\}$ $(\gamma < \frac{1}{2})$, but this would not affect the conclusion (27).

4. The "cross-section" of an event in one-dimensional scattering is the probability for this event to take place.

5. According to the usual terminology, a branch point such as that of $z^{1/p}$ is of order $p - 1$.

6. Since $y \leq 0$, the phase shift in the usual sense is $-\eta(k_y)$.

7. The factor $(2\lambda)^{\frac{1}{2}}$ in (55) is due to the normalization of the incident wave.

8. Notice, however, that this is not true on the bisector, on account of the boundary condition $\psi(x,x) = 0$. The natural approximation in this case is not the Born approximation, but the impulse approximation, given by (67).

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