Notas de Física Nº 14

NON RELATIVISTIC EQUATION FOR PARTICLES WITH SPIN 1*

J. J. Giambiagi and J. Tiomno
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, D.F.

March 22, 1954

The problem of obtaining the non relativistic equation for particles with spin 1/2, correct to any desired order of approximation, has been properly solved only recently by Foldy and Wouthuysen.[1]

In this paper an appropriate extension of the Foldy-Wouthuysen (F.W.) method is developed in order to obtain the non relativistic limit of the Proca [2] equations for spin 1 particles in interaction with the electromagnetic potentials. The method is used to obtain the equation up to terms of the order $\frac{1}{m^2}$ in the expansion in powers of $\frac{1}{m}$. The Hamiltonian obtained contains, besides the usual terms of the case of spinless particles, only a magnetic moment interaction term $\frac{e}{2m}\vec{H}\cdot\vec{M}$ where $\vec{H}$ is the magnetic field and $\vec{M}$ the spin operator for spin 1 particles (so the gyromagnetic factor is 1)[3]. No terms of the order $\frac{1}{m^2}$ such as spin orbit coupling or quadrupole moment interaction exists. We were not able to find a physical justification for the lack of these terms, in particular for that of the

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* To be published in "An. Acad. Bras. Ciências".

spin orbit coupling.[4]

Relativistic equations for spin 1 particles.

The Proca equations for spin 1 particles in vacuum can be written as.

$$-\frac{\partial \vec{A}}{\partial t} = m\vec{E} + \operatorname{grad} A_0$$
$$\frac{\partial \vec{E}}{\partial t} = m\vec{A} + \operatorname{rot} \vec{H} \qquad \text{(1a)}$$

$$m\vec{H} = \operatorname{rot} \vec{A}$$
$$m A_0 = -\operatorname{div} \vec{E} \qquad \text{(1b)}$$

$$\frac{\partial A_0}{\partial t} + \operatorname{div} \vec{A} = 0$$
$$\frac{\partial \vec{H}}{\partial t} + \operatorname{rot} \vec{E} = 0 \qquad \text{(1c)}$$
$$\operatorname{div} \vec{H} = 0$$

where A_0 and $\vec{A}$ form a quadrivector, $\vec{E}$ and $\vec{H}$ an antisymmetric tensor of second rank. We use natural units: $\hbar = c = 1$.

Equations (1a) and (1b) are really the fundamental ones, as (1c) are immediate consequences of them.

If the particle interactswith the electromagnetic potentials φ , $\vec{A}$ we have to substitute, in equations (1a) and (1b) (but not in (1c)), the ordinary derivatives $\frac{\partial}{\partial t}$ and $\vec{\nabla}$ by

$$\partial_t = \frac{\partial}{\partial t} + ie\varphi \qquad \vec{\partial} = \vec{\nabla} - ie\vec{A} \qquad (2)$$

The new equations (1c) must then be deduced from those. They are not obtained from (1c) by the substitution (2).

The expressions for the density of energy W and the density of charge ρ are given by[5]

$$W = m(\vec{A}^*\vec{A} + \vec{E}^*\vec{E} + A_0^*A_0 + \vec{H}^*\vec{H}) + \rho\varphi \tag{3a}$$

$$\rho = ie(\vec{E}^*\vec{A} - \vec{A}^*\vec{E}) \tag{3b}$$

The density of energy in the absence of an electric field is everywhere positive, but the density of charge can be positive or negative.

<u>Hamiltonian Operator</u>

If we consider equations (1b), which do not involve time derivatives, as definition equations and take them into (1a) we obtain the second order equations

$$\begin{aligned} \partial_t \vec{E} &= m\vec{A} + \frac{1}{m}\vec{\partial}\wedge\vec{\partial}\wedge\vec{A} \\ -\partial_t \vec{A} &= m\vec{E} - \frac{1}{m}\vec{\partial}(\vec{\partial}\cdot\vec{E}) \end{aligned} \tag{4}$$

These equations can be written in matrix form:

$$iB\frac{\partial\psi}{\partial t} = H\psi \tag{5}$$

$$H = m + eB\varphi - \frac{\vec{\partial}\cdot\vec{\partial}}{2m} - \frac{t^{ij} - t^{ji}}{2m}\partial_i\partial_j + \frac{\theta}{2m}Q^{ij}\partial_i\partial_j \tag{6}$$

where ψ is a six-component wave-function:

$$\psi = \begin{pmatrix} \vec{A} \\ \vec{E} \end{pmatrix} \qquad \vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \qquad \vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$B = i \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array} \right) \qquad \theta = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array} \right) \tag{7}$$

1 being the 3 x 3 unit matrix. T^{ij} are the matrices

$$T^{ij} = \left(\begin{array}{c|c} t^{ij} & 0 \\ \hline 0 & t^{ji} \end{array} \right) \tag{8}$$

t^{ij} being the 3 x 3 matrices whose r,s elements are [6]

$$(t_{ij})_{rs} = \delta_{ir}\delta_{js} \tag{8a}$$

Finally the matrices Q^{ij} are defined by

$$Q^{ij} = T^{ij} + T^{ji} - \delta^{ij} \times 1 = Q^{ji} \tag{9}$$

The matrices T_{ij} and Q_{ij} commute with B and θ.

We note also the following properties:

$$B^2 = \theta^2 = 1 \qquad B\theta + \theta B = 0 \tag{10}$$

$$T^{ij}\,T^{rs} = \delta^{jr}\,T^{ij} \tag{11}$$

$$(Q^{ij}a^i a^j)^2 = (a^i a_i)^2 \quad \text{if} \quad [a_i, a_j] = 0 \tag{12}$$

The last three operators in (6) take the form:

$$\left(\begin{array}{ccc|ccc}
-\partial_y^2-\partial_z^2 & \partial_y\partial_x & \partial_z\partial_x & & & \\
\partial_x\partial_y & -\partial_x^2-\partial_z^2 & \partial_z\partial_y & & 0 & \\
\partial_x\partial_z & \partial_y\partial_z & -\partial_x^2-\partial_y^2 & & & \\
\hline
 & & & -\partial_x^2 & -\partial_x\partial_y & -\partial_x\partial_z \\
 & 0 & & -\partial_y\partial_x & -\partial_y^2 & -\partial_y\partial_z \\
 & & & -\partial_z\partial_x & -\partial_z\partial_y & -\partial_z^2
\end{array}\right)$$

We shall cal the hermitian operator H, the Hamiltonian operator for the spin 1 particle because its integral over all space is equal to the energy

$$\int \psi^* H \psi \, dv = \int W \, dv = \mathcal{E}nergy \tag{13}$$

In the proof of (13) the expression (3a) and (6) for W and

H were used and integration by parts was performed assuming the usual conditions at infinity.

Also we consider eB as the operator for the charge because we have

$$\psi^* eB \psi = ie(\vec{E}^* \vec{A} - \vec{A}^* \vec{E}) = \rho \qquad \text{(charge density)};$$

This is indeed the operator for the charge which appears, multiplied by the electric potential φ, in the expression (6) of the Hamiltonian. The equation for conservation of charge:

$$\frac{\partial}{\partial t}\int \psi^* B \psi \, dv = 0$$

can be obtained in the usual way from equation (5) and its adjoint.

Before proceeding further it is convenient to make a unitary transformation in order to bring B into a diagonal form. This is accomplished by a transformation

$$\psi \longrightarrow U\psi$$

where

$$U = \frac{1}{\sqrt{2}}(1 + \theta B)$$

In this new representation the Hamiltonian retains the form (6) with the new definitions for the operators B and θ:

$$B = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \qquad \theta = \left(\begin{array}{c|c} 0 & i \\ \hline -i & 0 \end{array}\right) \qquad (14)$$

If the new ψ is written

$$\psi = \begin{pmatrix} \vec{\psi}_1 \\ \vec{\psi}_2 \end{pmatrix}$$

we have the following expression for the density of charge:

$$\rho = e\psi^{*}B\psi = e(\psi_1^{*}\psi_1 - \psi_2^{*}\psi_2) \tag{15}$$

This representation is somewhat similar to the usual representation of the Dirac equation. In that case, the charge density being positive, the upper and lower components of the wave function represent respectively, states of positive and negative energy for particles at rest. Here, the energy density being positive in vacuum, the upper and lower components of the wave function represent respectively, states of positive and negative charge for particles at rest.

Generalization of the Foldy-Wouthuysen Method

In the F. W. method for spin 1/2 particles a two component equation is obtained from the 4 component Dirac equation by a convenient unitary transformation which eliminates the odd operators so that one can impose the supplementary condition that two of the components of the wave function vanish. The reason for the use of unitary transformation is that we want to keep invariant the form of the total probability, or of the total charge.

In the case of spin 1 we wish to obtain from our 6 component equation, a 3 component one by a similar procedure.*

However, as now we wish to keep invariant the form of the expression for the total charge

$$Q = e\int \psi^{*}B\psi\, dv$$

the transformation T, $\psi \rightarrow T\psi \qquad T = e^{iS}$

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* Here we call odd the operators $\mathcal{O}$ which anticommute with B, and even, those $\mathcal{E}$ which commute with B.

will not necessarily be unitary. For instance, if S is an <u>odd</u> operator, then, as we need

$$B \rightarrow (T^{-1})^{+} B T^{-1} \equiv B$$

it is necessary that

$$SB = BS^{+} = -BS$$

Then S should be <u>anti-hermitian</u>:

$$S^{+} = -S \qquad\qquad T^{+} = T \tag{16}$$

After we are able to eliminate the <u>odd</u> terms from the Hamiltonian we can impose the supplementary condition (for positively charged particles):

$$B\psi = \psi \tag{17}$$

i.e., that the three lower components of ψ vanish (in the representation given by (14)).

We consider separately the case of the free particle and that of a particle in presence of an electromagnetic field, as in the first case an exact transformation of the required type can be found while in the second case only approximate transformations can be found.

<u>Free Particle</u>

In this case equations (5), (6) reduce to

$$iB\frac{\partial\psi}{\partial t} = m\psi + \frac{p^{2}}{2m} - \frac{1}{2m}\,\theta Q^{ij} p_i p_j \tag{18}$$

with

$$\vec{p} = \frac{1}{i}\vec{\nabla}$$

If we call $\mathcal{O}$ the odd operator in (18) we find, from (12):

$$\mathcal{O}^2 = \frac{p^4}{4m^2} \tag{19}$$

Now we try a transformation

$$\phi = e^{iS}\psi \qquad\qquad S = -\frac{i}{2m}\mathcal{O}\omega \tag{20}$$

where ω is a function of p/m; S, being an odd operator, has to be anti-hermitian, as we have shown, so ω is a real function. We have

$$i e^{-\frac{\mathcal{O}}{2m}\omega} B \frac{\partial}{\partial t} e^{-\frac{\mathcal{O}}{2m}\omega}\phi = e^{-\frac{\mathcal{O}}{2m}\omega} H e^{-\frac{\mathcal{O}}{2m}\omega}\phi$$

Using the commutation properties of $\mathcal{O}$ we find

$$iB\frac{\partial\phi}{\partial t} = e^{-\frac{\mathcal{O}}{m}\omega} H\phi = H'\phi \tag{21}$$

From (19) we have

$$e^{-\frac{\mathcal{O}}{2m}\omega} = ch\,\frac{\omega p^2}{2m} - \frac{2m\mathcal{O}}{p^2}\, sh\,\frac{\omega p^2}{2m^2}$$

so that

$$H = \left(m + \frac{p^2}{2m}\right) ch\,\frac{\omega p^2}{2m^2} - \frac{p^2}{2m}\, sh\,\frac{\omega p^2}{2m} +$$
$$+ \mathcal{O}\left[ch\,\frac{\omega p^2}{2m^2} - \left(\frac{2m^2}{p^2} + 1\right) sh\,\frac{\omega p^2}{2m}\right]$$

The odd terms of H' vanish if we take:

$$\omega = \frac{2m^2}{p^2} \operatorname{arcth} \frac{p^2}{p^2+2m^2} \tag{22}$$

Then, as

$$\operatorname{sh} \frac{\omega p^2}{2m^2} = \frac{p^2}{2m\sqrt{p^2+m^2}}$$

$$\operatorname{ch} \frac{\omega p^2}{2m^2} = \frac{p^2+2m^2}{2m\sqrt{p^2+m^2}}$$

we find

$$H' = \sqrt{p^2+m^2} \tag{23}$$

So we have obtained the square root equation:

$$iB\frac{\partial \phi}{\partial t} = \sqrt{p^2+m^2}\,\phi \tag{24}$$

Equation (24) has solutions such that the 3 upper components of ϕ represent positively charged particles (if B is diagonal), and the lower components represent negatively charged particles.

<u>Positive particles in the presence of an electromagnetic field</u>.

In the general case equations (5) and (6) can be written:

$$iB\frac{\partial \psi}{\partial t} = m\psi + e\varphi B\psi + \frac{p^2}{2m}\psi + \frac{e}{2m}\vec{\mathcal{H}}\cdot\vec{m}\Psi - \frac{\theta}{2m}Q^{ij}\delta_i\delta_j\Psi \tag{25}$$

where $$\vec{p} = \frac{1}{i}\vec{\nabla} - e\vec{A}$$

$\vec{H}$ is the magnetic field and $\vec{m}$ is the spin operator for spin 1 particles:

$$\partial_i \partial_j - \partial_j \partial_i = ie\mathcal{H}_k \qquad m_k = \frac{1}{i}(t_{ij} - t_{ji}). \tag{26}$$

for i,j,k a cyclic permutation of 1,2,3.

Equation (25) can be put in the form

$$iB\frac{\partial \Psi}{\partial t} = H\Psi = (m + \mathcal{E} + \mathcal{O})\Psi \tag{27}$$

where $\mathcal{O}$ is the odd part of H.

Consider, now, the canonical transformation generated by the antihermitian operator

$$S = -\frac{i}{2m}\mathcal{O} \tag{28}$$

We find for the Hamiltonian in the new representation (similar to the free particle case):

$$H' = e^{-iS} H e^{-iS} - iBe^{iS}\left(\frac{\partial e^{-iS}}{\partial t}\right) \tag{29}$$

If $\mathcal{O}$ is of no lower order in 1/m than (1/m^{0}) (as is the case for equation (25)), and if S can be considered small, then we can make an expansion in powers of 1/m of the Hamiltonian in the new representation:

$$H' = H - i\{S,H\} + \frac{i^2}{2}\{S,\{S,H\}\} + \cdots$$
$$- B\frac{\partial S}{\partial t} + \frac{i}{2}B\left[S,\frac{\partial S}{\partial t}\right] + \frac{i^2}{3!}B\left[S,\left[S,\frac{\partial S}{\partial t}\right]\right] + \cdots \tag{30}$$

All terms in this development are hermitian. If we take instead of (28)

$$S = -\frac{i}{2m}\cdot\text{(odd terms in H of lowest order in 1/m)} \quad (31)$$

we find, as in the F.W. method, that we may successively remove odd terms from the Hamiltonian to any desired order.

Here we want to obtain the non relativistic equation up to terms in $1/m^2$.

Starting from equation (25), we first make the transformation generated by

$$S = -\frac{i}{2m}\left(-\frac{\theta}{2m} Q^{ij} p_i p_j\right) \quad (32)$$

obtaining the new Hamiltonian

$$H_1 = m + e\varphi + \frac{p^2}{2m} + \frac{e}{2m}\vec{\mathcal{H}}\cdot\vec{m} - \frac{i}{4m^2} B\theta Q^{ij}(\mathcal{E}_i p_j + p_i\mathcal{E}_j) \quad (33)$$

to terms of order $(1/m)^2$. $\vec{\mathcal{E}}$ is the electric field.

The last term of the order $1/m^2$ is odd and is eliminated in the next transformation generated by

$$S_2 = -\frac{i}{2m}\left[\frac{-ieB\theta}{4m^2} Q^{ij}(\mathcal{E}_i p_j + p_i\mathcal{E}_j)\right]$$

the resulting Hamiltonian correct to terms in $1/m^2$ being

$$H_2 = m + eB\varphi + \frac{p^2}{2m} + \frac{e}{2m}\vec{\mathcal{H}}\cdot\vec{m} \quad (34)$$

Now, as there are no odd operators in the equation

$$i B \frac{\partial \psi}{\partial t} = H_2 \psi$$

we can impose the supplementary condition (17), for positive particles and we obtain after the transformation

$$\psi = e^{imt} \phi .$$

the non relativistic 3 component equation

$$i \frac{\partial \phi}{\partial t} = \left(e\varphi + \frac{p^2}{2m} + \frac{e}{2m} \vec{m} \cdot \vec{\mathcal{H}} \right) \phi \qquad (35)$$

with the usual normalization condition

$$\int \phi^* \phi \, dv = 1$$

Equation (35) which is correct up to terms in $1/m^2$ is very similar to the N.R. approximation of Dirac equation but has <u>no terms in $1/m^2$</u> such as a spin-orbit interaction

$$\frac{e}{m^2} \frac{\vec{r} \wedge \vec{p} \cdot \vec{m}}{r^3}$$

which seems difficult to understand; also there is no quadrupole moment interaction term

$$\frac{e}{m^2} \left(Q^{ij} + \frac{1}{3} \delta^{ij} \right) \frac{\partial \mathcal{E}_i}{\partial x_j}$$

or even the divergence term $\frac{e}{m^2}$ div$\mathcal{E}$ as one might expect.

[1] Foldy, L. and Wouthuysen, S. A., Phys. Rev. 78 , 29, 1950.

[2] Proca, A., J. de Physique 7, 347, 1936.

[3] Proca, A. J. de Physique, 9, 61, 1938. (In this paper the same results as in our paper are obtained by the method of elimination of "small" components, which is known to be unsatisfactory for higher orders than $1/m^2$.

[4] When most of this work was accomplished, our attention was called to an abstract in the Bull. Am. Phys. Soc. 29, Nº 149 (1954), in whic Case, K.M., reports the solution of the same problem with similar results.

[5] Wentzel, G., Quantum Theory of Fields, Interscience Pub., 1949, New York.

[6] Tiomno, J., Notas de Física Nº 9, 1952