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# Two-dimensional models as testing ground for principles and concepts of local quantum physics

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**Abstract**

In the past two-dimensional models of QFT have served as theoretical laboratories for testing new concepts under mathematically controllable condition. In more recent times low-dimensional models (e.g. chiral models, factorizing models) often have been treated by special recipes in a way which sometimes led to a loss of unity of QFT. In the present work I try to counteract this apartheid tendency by reviewing past results within the setting of the general principles of QFT. To this I add two new ideas: (1) a modular interpretation of the chiral model Diff(S)-covariance with a close connection to the recently formulated local covariance principle for QFT in curved spacetime and (2) a derivation of the chiral model temperature duality from a suitable operator formulation of the angular Wick rotation (in analogy to the Nelson-Symanzik duality in the Ostertwalder-Schrader setting) for rational chiral theories. The SL(2,Z) modular Verlinde relation is a special case of this thermal duality and (within the family of rational models) the matrix S appearing in the thermal duality relation becomes identified with the statistics character matrix S. The relevant angular Euclideanization" is done in the setting of the Tomita-Takesaki modular formalism of operator algebras.

I find it appropriate to dedicate this work to the memory of J. A. Swieca with whom I shared the interest in two-dimensional models as a testing ground for QFT for more than one decade.

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# 1 Early history of two-dimensional solved models and the problem of learning the right lessons

Local quantum physics of systems with infinitely many interacting degrees of freedom leads to situations whose understanding often requires new physical intuition and mathematical concepts beyond that acquired in quantum mechanics and perturbative constructions in quantum field theory. In this situation two-dimensional soluble models turned out to play an important role. On the one hand they illustrate new concepts and sometimes remove misconceptions in an area where new physical intuition is still in process of being formed. On the other hand rigorously soluble models confirm that the underlying physical postulates are mathematically consistent, a task which for interacting systems with infinite degrees of freedom is mostly beyond the capability of pedestrian methods or brute force application of hard analysis on models whose natural invariances has been mutilated by a cut-off.

In order to underline these points and motivate the interest in two-dimensional QFT, let us briefly look at the history, in particular at the physical significance of the three oldest two-dimensional models of relevance for statistical mechanics and relativistic particle physics which are in chronological order: the Lenz-Ising model, Jordan's model of bosonization/fermionization and the Schwinger model ($QED_2$).

The Lenz-Ising (L-I) model was proposed in 1920 by Wihelm Lenz [1] as the simplest discrete statistical mechanics model with a chance to go beyond the P. Weiss phenomenological Ansatz involving long range forces and instead explain ferromagnetism in terms of non-magnetic short range interactions. Its one-dimensional version was solved 4 years later by his student Ernst Ising. In his 1925 university of Hamburg thesis, Ising [2] not only showed that his chain solution could not account for ferromagnetism, but he also proposed some (as it turned out much later) not entirely correct intuitive arguments to the extend that this situation prevails to the higher dimensional lattice version. His advisor Lenz as well as Pauli (at that time Lenz's assistant) accepted these reasonings and as a result there was considerable disappointment among the three which resulted in Ising's decision (despite Lenz's high praise of Ising's thesis) to look for a career outside of research.

As a contribution to the many historical reminiscences on the occasion of the 2005 Einstein year, there is one episode which indirectly connects Einstein to the beginnings of theoretical physics in Hamburg [1]. The university of Hamburg was founded in 1919, but the town fathers deemed it unnessary to create a separate chair for theoretical physics. The newly appointed mathematicians (Artin, Blaschke, Hecke) had the splendid idea to invite Einstein (who at that time already enjoyed general fame) for a public talk. The event was an overwhelming success, the talk on the concepts underlying general relativity ended with a round of lively discussions, especially questions coming from Hecke. The subsequent public pressure on the city government led to a change of their decision. Einstein's indirect role as a catalyzer

[1] I learned about this episode through an email correspondence with K. Reich [3].

was rapidly vindicated since the theoretical physics activities at Hamburg university shortly thereafter led to the important discovery of the exclusion principle and the construction of one of the most fruitful two-dimensional models.

For many years a reference by Heisenberg [4] (to promote his own proposal as an improved description of ferromagnetism) to Ising's negative result was the only citation; the situation begun to change when Peierls [5] drew attention to "Ising's solution" and the results of Kramers and Wannier [6] cast doubts on Ising's intuitive arguments beyond the chain solution. The rest of this fascinating episode i.e. Lars Onsager's rigorous two-dimensional solution exhibiting ferromagnetic phase transition, Brucia Kaufman's simplification which led to conceptual and mathematical enrichments (as well as later contributions by many other illustrious personalities) hopefully remains a well-known part of mathematical physics history even beyond my own generation.

This work marks the beginning of applying rigorous mathematical physics methods to solvable two-dimensional models as the ultimate control of intuitive arguments in statistical mechanics and quantum field theory. The L-I model continued to play an important role in the shaping of ideas about universality classes of critical behavior; in the hands of Leo Kadanoff it became the key for the development of the concepts about order/disorder variables (The microscopic version of the famous Kramers-Wannier duality) and also of the operator product expansions which he proposed as a concrete counterpart to the more general field theoretic setting of Ken Wilson. Its massless version (and the related so-called Coulomb gas representation) became a role model in the Belavin-Polyakov-Zamolodchikov [7] setting of minimal chiral models and it remained up to date the only model with non-abelian braid group commutation relations for which the n-point correlators can and have been written down explicitly in terms of elementary functions. Chiral theories confirmed the pivotal role of "exotic" statistics in low dimensional QFT by exposing the appearance of *plektonic* statistics subject to the laws of Artin's braid group as a novel manifestation of Einstein causality. As free field theories in higher dimensions are fixed by their mass, spin and internal symmetries, the structure of chiral theories is almost entirely determined by their braid group statistics data i.e. they are in some sense as free (of genuine interactions, or as "kinematical") as possible under the condition that they must realize nontrivial braid group statistics. The latter is incompatible with a linear equation of motion and the ensuing absence of correlations beyond two-point functions, a fact which also has been established in the physically important case of braid group statistics in d=1+2 where the statistics carrying fields are necessarily semiinfinite string-like [8].

Another conceptually rich model which lay dormant for almost two decades as the result of a misleading speculative higher dimensional generalization by its protagonist is the bosonization/fermionization model first proposed by Pascual Jordan [9]. This model establishes a certain equivalence between massless two-dimensional Fermions and Bosons; it is related to Thirring's massless 4-fermion coupling model[2] and also to Luttinger's one-dimensional model of an electron gas [11][12]. One reason why even nowadays hardly anybody knows Jordan's contribution (besides lack of comprehension of its content) is certainly the ambitious but unfortunate title "The neutrino theory of light" under which he published a series of

[2] In its original massless and conformally invariant version it reached its mathematical perfection in the work of Klaiber's Boulder lecture [10]. Klaiber also enriched the model by an additional parameter which allows a realization of the Thirring model in a anyonic statistics mode with the expected spin-statistics relation between the anomalous Lorentz spin and the anyonic commutation relation.

papers; besides some not entirely justified criticism of content, the reaction of his contemporaries consisted in a good-humored carnivalesque Spottlied (mockery song) about its title [13]. Its mathematical content, namely the realization that the current of a free zero mass Dirac fermion in d=1+1 is linear in canonical Boson creation/annihilation operators ("Bosonization") and that such zero mass Fermions (Jordan's "neutrinos") permit a formal representation in terms of ordered exponential expressions in a free Boson field ("Fermionization") turned out to play an important illustrative role in the context of two-dimensional conformal theories and their chiral decomposition. The massive version of the Thirring model became the role model of integrable relativistic QFT and shed additional light on two-dimensional Bosonization [14].

Both discoveries demonstrate the usefulness of having controllable low-dimensional models; at the same time their complicated history also illustrates the danger of rushing to premature "intuitive" conclusions about extensions to higher dimensions. The search for the appropriate higher dimensional analogon of a two-dimensional observation is an extremely subtle but very worthwhile endeavour because if done correctly it often leads to considerable conceptual progress. In the aforementioned two historical examples the true physical message of those models only became clear through hard mathematical work and profound conceptual analysis by other authors many years after the discovery of the original model.

A review of the early conceptual progress through the study of solvable two-dimensional models would be incomplete without mentioning Schwinger's proposed solution of two-dimensional quantum electrodynamics, afterwards referred to as the Schwinger model. Schwinger used this model in order to argue that gauge theories are not necessarily tied to zero mass vector particles; in opposition to the majority in the physics community (including Pauli[3]) he thought that it is conceivable that there may exist a strong coupling regime of a QED-like gauge coupling which converts the massless photons into a massive vectormeson and he used massless $QED_2$ to illustrate his point that the gauge theory setting does not exclude massive vectormesons. His solution in terms of indefinite metric correlation functions [15] was however quite far removed from the interpretation of its physical aspects. Some mathematical physics work and conceptual clarification was necessary [16] to unravel its physical content with the result that the would-be charge of that $QED_2$ model was "charged-screened" and hence its apparent chiral symmetry "broken", in other words the model exists only in the so-called Schwinger-Higgs phase with massive free scalar particles accounting for its physical content[4]). Another closely related aspect of this model which also arose in the Lagrangian setting of 4-dimensional gauge theories was that of the $\theta$-angle. Extended multi-component versions of this model were used in for the study of problems of charge-screening versus confinement [17]. Although one believes that the basic features of this difference between the Schwinger-Higgs screening mechanism versus (fractional) fundamental flavor confinement continue to apply in the 4-dimensional standard model, the lack of an intrinsic meaning of notions of spin as well as statistics in massive d=1+1 models prevent simple-minded analogies.

Thanks to its property of being superrenormalizable, the Schwinger model also served as a useful

[3]For a long time it was thought that the use of abelian or nonabelian gauge theories (as proposed by Yang and Mills and before (in 1938) by Oscar Klein) in particle physics of massive vectormesons was not possible. For this reason Sakurai presented his ideas about vector mesons without using a gauge theoretical setting.

[4]As an illustration of historical prejudices against Schwinger's ideas it is interesting to note that Swieca apparently was not able to convince Peierls (on a visit to Brazil) about the possibility of having massive vectormesons in a gauge theory (private communication by J. A. Swieca around 1975).

testing ground for the Euclidean integral formulation in the presence of Atiyah-Singer zero modes and their role in the Schwinger-Higgs chiral symmetry breaking [17]. These classical topological aspects of the functional integral formulation attracted a lot of attention beginning in the late 70s and through the 1980s but, as most geometrical aspects of the Euclidean functional integral, their intrinsic physical significance remained controversial. Even in those superrenormalizable 2-dim models, where the measure theory underlying Euclidean functional integration can be mathematically controlled [19], there is no good reason why within such a setting topological properties derived from continuity requirements should assert themselves outside of quasiclassical approximations [20]. This is no problem in the operator algebra approach where no topological or differential geometrical property is imposed but certain geometric structures (spacetime- and internal- symmetry properties) are encoded in the causality and spectral principles of observable algebras.

In passing it is worthwhile to mention that Schwinger's idea on charge screening found a rigorous formulation in a structural theorem which links the issue of charge (carried by physical particles) versus charge screening to the spectral property near zero of the mass operator [21]. Mass generation via charge screening in 4-dimensional *perturbation* theory is not possible without additional physical (Higgs) degrees of freedom.

The most coherent and systematic attempt at a mathematical control of two-dimensional models came in the wake of Wightman´s first rigorous programmatic formulation of QFT [22]. This formulation stayed close to the ideas underlying the impressive success of renormalized QED perturbation theory, although it avoided the direct use of ideas of Lagrangian quantization. The early attempts towards a "constructive QFT" found their successful realization in two-dimensional QFT (the $P\varphi_2$ models [19]). Only in low dimensional theories the presence of Hilbert space positivity and energy positivity can be reconciled with the kind of mild short distance singularity behavior (super-renormalizability) which the methods of constructive QFT requires. Despite interesting later additions after the appearance of the cited book, this barrier has essentially persisted. For this reason the main attention will be focussed on alternative constructive methods which are free of this restrictions; they have the additional advantage to reveal more details about the conceptual structure of QFT beyond the assertion of their existence. The best illustration of the constructive power of these new methods comes from massless d=1+1 conformal and chiral QFT as well as from massive factorizing models. Their presentation and that of the contained messages for general QFT will form the backbone of this article.

There are several books and review articles [23][24][25] on d=1+1 conformal as well as on massive factorizing models [17][18]. To the extend that concepts and mathematical structures are used which permit no known generalization to higher dimensions (e.g. Kac-Moody algebras, loop groups, integrability, presence of an infinite number of conservation laws), their line of approach will not be followed in this report since our primary interest will be the use of two-dimensional models of QFT as "theoretical laboratories" of general QFT as stated in the abstract.

Our aim is two-fold; on the one hand we intend to illustrate known principles of general QFT in a mathematically controllable context and on the other hand we want to identify new concepts whose adaptation to QFT in d=1+1 lead to their solvability. In emphasizing the historical side of the problem by using the oldest solved two-dimensional models as a vehicle for the introduction of relevant concepts, I also hope to uphold the awareness of the unity and historical continuity in QFT in times of rapidly changing fashions in the age of electronic communications.

## 2 General concepts and their two-dimensional adaptation

The general framework of QFT, to which the rich world of controllable two-dimensional models contributes as a testing ground, exists in two quite different but nevertheless closely related formulations: the setting in terms of pointlike covariant fields due to Wightman [22], and the more algebraic setting initiated by Haag and Kastler based on spacetime-indexed operator algebras [26] and concepts which developed over a long period of time with contributions of many other authors. Whereas the Wightman approach aims directly at the (not necessarily observable) quantum fields, the operator algebraic setting is more ambitious. It starts from physically motivated assumptions about the algebraic structure of local (spacelike commuting) observables[5] and structurally reconstructs (i.e. demonstrates rigorously that all concepts and implementing objects of particle physics are in place) the full field theory (including the operators carrying the superselected charges) in the spirit of a local representation theory of the assumed structure of the local observables. This has the advantage that the somewhat mysterious concept of an inner symmetry (as opposed to "outer" (spacetime) symmetry) can be traced back to its physical roots which is the representation theoretical structure of the local observable algebra. In the Lagrangian quantization approach the inner symmetry is part of the input (the multiplicity index of field components on which subgroups of SU(n) or SO(n) act linearly) and hence it is not possible to even formulate this fundamental question. Whenever the sharp separation (the Coleman-Mandula theorems [28]) of inner versus outer symmetry becomes blurred as a result of the appearance of braid group statistics in low spacetime dimensions, the Lagrangian quantization setting is inappropriate and even the Wightman framework has to be extended. In that case the algebraic approach is the most appropriate.

The two most important physical properties which are shared between the Wightman approach (WA) [22] and the operator algebra (AQFT) setting [26], are the spacelike locality (often referred to as Einstein causality) and the restriction to the stability ensuring positive energy representations of the Poincaré group which implement the covariance of the Wightman fields respectively the local observable algebras.

- Spacelike commutativity of quantum fields or of local observable algebras:

$$\begin{aligned} &[\psi(x),\varphi(y)]_{\mp}=0,\ (WA) \\ &\mathcal{A}(\mathcal{O}')\subseteq\mathcal{A}(\mathcal{O})',\ \mathcal{O}\ open\ nbhd.\ (AQFT) \end{aligned} \tag{1}$$

- Positive energy reps. of the Poincaré group $\mathcal{P}$ :

$$\begin{aligned} &U(a,\Lambda)\psi(x)U(a,\Lambda)^{*}=D^{-1}(\Lambda)\psi(\Lambda x+a),\ (WA) \\ &U(a,\Lambda)\mathcal{A}(\mathcal{O})U(a,\Lambda)^{*}=\mathcal{A}(\mathcal{O}_{(a,\Lambda)}),\ (AQFT) \\ &U(a)=e^{iPa},\ specP\subseteq V_{+},\ P\Omega=0 \end{aligned} \tag{2}$$

Here $\psi(x),\varphi(x)$...are (singular) field operators (operator-valued distributions) in a Wightman QFT which are assumed to either commute or anticommute for spacelike distances and a structural theorem

[5] The minimal requirement on observable fields or localized algebras is that they are "bosonic" i.e. commute for spacelike distances whereas a maximal definition would require the absence of any internal symmetry.

[22] ties the commutator relation to finite dimensional representations of the Lorentz group $\mathcal{L}$ whereas the anticommutator has to be used for projective representations (which turn out to be usual vector-representations of the two-fold covering $\widetilde{\mathcal{L}}$). The observable algebra consists of a family of (weakly closed) operator algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{K}}$ indexed by a family of convex causally closed spacetime regions $\mathcal{O}$ (with $\mathcal{O}'$ denoting the spacelike complement and $\mathcal{A}'$ the von Neumann commutant) which act in one common Hilbert space; it is suffient to know these local algebras in the vacuum representation i.e. without loss of generality one can identify $\mathcal{A}(\mathcal{O})$ with $\pi_0(\mathcal{A}(\mathcal{O}))$. Certain properties cannot be naturally formulated in the pointlike field setting (e.g. Haag duality for simply connected causally complete regions $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$), but the connection between the two formulations of local quantum physics is nevertheless quite close; in particular in case of two-dimensional theories there are convincing arguments that one can pass between the two without having to impose additional technical requirements. There exists a recent generalization of this algebraic framework which incorporates the Einstein local covariance principle in which the above setting re-emerges as a special case [29]. In section 5 we will present a chiral illustration of these new concepts.

The above two requirements are often (depending on the kind of structural properties one wants to derive) complemented by additional impositions which, although not carrying the universal weight of principles, nevertheless represent natural assumptions whose violation (even though not prohibited by the principles) would cause paradigmatic attention and warrants special explanations. Examples are "weak additivity", "Haag duality" and "the split property". Weak additivity i.e. the requirement $\vee\mathcal{A}(\mathcal{O}_i) = \mathcal{A}(\mathcal{O})$ if $\mathcal{O} = \cup\mathcal{O}_i$ expresses "the global results from amalgamating the local" aspect which is inherent in the pointlike field formulation, but needs to be added in the algebraic approach.

Haag duality is the statement that the commutant not only contains the algebra of the causal complement (Einstein causality), but is even equal to it, i.e. $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$. Its obvious connection with the measurement process (it assigns a localization to measurements which are commensurable with the totality of all measurements which are performable in a prescribed spacetime region $\mathcal{O}$) suggests to look for a profound physical explanation whenever it is violated e.g. its violation for observables localized in a simply connected causally complete region signals spontaneous symmetry breaking in the associated charge-carrying field algebra [26]. The possibilities of spontaneous symmetry breaking in d=1+1 are severely restricted. The Bisognano-Wichmann property for wedge localized algebras (3) assures that Haag duality can be enforced by a symmetry-reducing (to the unbroken subgroup) extension via dualization. In conformal theories it is always satisfied independent of spacetime dimension [30]. Its violation for multi-local region reveals the charge content of the model by enforcing charge-anticharge splittings in the neutral observable algebra [31].

The split property [32], namely the algebraic isomorphism to a quantum mechanical type tensor factorization $\mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) \simeq \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$ for regions $\mathcal{O}_i$ with a (arbitrarily small) spacelike separation between them, is a result of the adaptation of the "finite number of degrees of freedom per unit cell in phase space" requirement of QM to QFT which leads to the so-called "nuclearity property" [26]. Looming behind all these properties is the inexorable vacuum polarization; in order to prevent its infinity-creating reponse to sharp localization which destroys the strong property of quantum mechanical tensor-factorization for nonoverlapping regions, one needs the finite spacelike distance required by the split property.

Related to the Haag duality is the local version of the "time slice property" (the QFT counterpart of the classical causal dependency property) sometimes referred to as "strong Einstein causality" $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})''$.

One of the most astonishing achievements of the algebraic approach is the DHR theory of superselection sectors [33] i.e. the realization that the structure of charged (nonvacuum) representations (with the unrestricted superposition principle being valid only within one representation) and the spacetime properties of the fields which are the carriers of these generalized charges including their spacelike commutation relation (leading to the particle statistics and to internal symmetry [34]) are already encoded in the structure of the Einstein causal observable algebra. The intuitive basis of this remarkable result (whose prerequisite is causal locality) is that one can generate charged sectors by spatially separating charges in the vacuum (neutral) sector and disposing of the unwanted charges at spatial infinity [26]. In higher spacetime dimensions the charge fusion structure turns out to be isomorphic to the tensor product operation of compact group representations; the framework does not exclude any compact group.

An important concept which especially in d=1+1 has considerable constructive clout is "modular localization". It is a consequence of the above algebraic setting if either the net of algebras has pointlike field generators, or if the one-particle masses are separated by spectral gaps so that the formalism of time dependent scattering can be applied [37]; in conformal theories this property holds automatically (in all spacetime dimensions). It rests on the basic observation [36] that a *standard pair* $(\mathcal{A}, \Omega)$ of a von Neumann operator algebra and a vector[6] gives rise to a Tomita operator S through its star-operation whose polar decomposition yield two modular objects, a 1-parametric subgroup of the unitary group of operators in Hilbert space $\Delta^{it}$ whose Ad-action defines the modular automorphism of $(\mathcal{A}, \Omega)$ whereas the angular part $J$ is the modular conjugation which maps $\mathcal{A}$ into its commutant $\mathcal{A}'$

$$\begin{aligned} S A\Omega &= A^*\Omega, \;\; S = J\Delta^{\frac{1}{2}} \\ J_W = U(j_W) &= S_{scat}J_0, \; \Delta_W^{it} = U(\Lambda_W(2\pi t)) \\ \sigma_W(t) &:= Ad\Delta_W^{it} \end{aligned} \tag{3}$$

The standardness assumption is always satisfied for any field theoretic pair $(A(\mathcal{O}), \Omega)$ of a $O$-localized algebra and the vacuum state (as long as $\mathcal{O}$ has a nontrivial causal disjoint $\mathcal{O}'$) but it is only for the wedge region $W$ that the modular objects have a physical interpretation in terms of the global spacetime symmetry group of the vacuum as specified in the second line (3), namely the modular unitary represents the $W$-associated boost $\Lambda_W(\chi)$ and the modular conjugation implements the TCP-like reflection along the edge of the wedge [35]. The third line is the definition of the modular group. Its usefulness results from the fact that it does not depend on the state vector $\Omega$ but only on the state $\omega(\cdot) = (\Omega, \cdot\Omega)$ which it induces. Another noteworthy fact is that the modular group $\sigma^{(\eta)}(t)$ associated with a different state $\eta(.)$ is unitarily equivalent to $\sigma^{(\omega)}(t)$ with a unitary $u(t)$ which fulfills the Connes cocycle property. The importance of this theory for local quantum physics results from the fact that it leads to the concept of *modular localization*[7], a new totally intrinsic (i.e. independent of field coordinatizations) scenario for field theoretic constructions which is different from the Lagrangian quantization schemes.

[6] Standardness means that the operator algebra of the pair $(\mathcal{A}, \Omega)$ act cyclic and separating on the vector $\Omega$.

[7] As opposed to classical localization via spacetime support properties of functions. Often one can construct intertwiner which transform quantum (modular) localization into classical localization (e.g. the u-v interwiners which lead from Wigner creation/annihilation operators to local point- or string-like covariant "fields"), but in general the relation between nets of spacetime localized algebras and their operator-distribution valued generators is quite subtle.

A good starting point for understanding the physical aspects and aims of modular localization is the Wigner particle representation theory. Localization in analogy to the Born probability interpretation in quantum mechanics is incompatible with relativistic covariance since there are simply no covariant localizing projection operators (even if one extends the Wigner space to the Fock space). A recent review of these facts and the physics behind the concept of modular localization which replaces the concept of localizing via projection operators can be found in [38]. The construction of modular localized subspaces of a positive energy Wigner space $H^{(1)}$ starts from the group theoretic definition of a wedge-affiliated S-operator $S_W^{(1)}$ by multiplying the (unbounded) analytically continued wedge affiliated boost $\Delta_W^{(1)} \equiv U_W^{(1)}(\Lambda_W(-\pi i))$ with the antiunitary involution $J_W^{(1)}$ which represents the reflection along the edge of the wedge as in (3) but without an operator algebra being present. As a result of the commutation of $\Lambda_W$ and $j_W$ and the antilinearity of $J_W^{(1)}$ this unbounded involutive antiunitary operator in $H^{(1)}$ fulfills all the properties of a Tomita S-operator and its $+1$ eigenspace defines a real subspace $K_W^{(1)}$ of the complex Wigner space. The sharpening of localization is achieved by intersecting $K_W^{(1)\prime}s$ or, what amounts to the same, via directly defining new S-operators by intersecting domains of definition of $S_W s$[41]. For finite spin representations the intersections associated with (compact) double cone regions $K_D^{(1)} = \cap_{W\supset D} K_W^{(1)}$ are nontrivial and "standard" (implying that $K_D^{(1)} + iK_D^{(1)}$ is dense in $H^{(1)}$), but for zero mass infinite spin representations as well as for massive d=1+2 anyonic spin representations these intersections are trivial and the smallest nontrivial and standard intersections are (noncompact) spacelike cones with semiinfinite stringlike cores. The important theorems on modular localization within a group representation setting can be found in [39].

The transition from modular localized subspaces to localized operator algebras of interaction-free systems is done in a functorial way using the Weyl (or CAR in case of halfinteger spin) operators which map the modular localized function spaces into spacetime-indexed operator algebras (generated by the images of the functor). The functorial relation between modular localized subspaces and localized von Neumann subalgebras[8] commutes with more stringent localization which is achieved by intersecting wedge spaces or wedge algebras. Point- and string-like generators (necessarily singular i.e. distribution-valued) can be constructed with the help of intertwiners (the analog of $u$ and $v$ spinors) which relate the modular (quantum) localization to the classical localization in terms of test function supports [40] [41] [42]. The concept of modular localization solves the age-old problem of the continuous spin Wigner representation by showing that the compact localization spaces in those cases (and also for d=1+2 anyons) are empty; in those cases the tightest possible localization is in (arbitrarily thin) spacelike cones associated with stringlike distributional generators. The application of modular localization to the Wigner one-particle spaces becomes especially simple for massless two-dimensional theories (since as a result of their decomposability into two chiral components the energy-momentum spectrum is not subject to a mass-shell restriction).

For interacting systems the construction of spacetime-indexed operator algebras looses its functorial relation with modular localization of Wigner particle states [38]. Instead of being geometrically defined in terms of Poincaré group representation theory, modular localized subspaces in Fock space are simply identified with the dense subspaces $\mathcal{A}(\mathcal{O})\Omega$ which the local operator algebras generate from the vacuum

[8] The $\Delta_W^{it}, J_W$ and $S_W$ operators in Fock space are related to their one-particle analogs by the rules of second quantization.

after they have been closed in the graph topology of the Tomita S-operator of the standard pair $(\mathcal{A}(\mathcal{O}),\otimes)$ i.e. $H(\mathcal{O})\equiv\overline{\mathcal{A}(\mathcal{O})\Omega}_{graph(S)}=\left\{\psi|\ |\psi|^2+|S\psi|^2<\infty\right\}$, $H(O)\overset{dense}{\subset}H$. As a result of geometric simplifications the application of modular theory to two dimensional theories leads to particularly powerful results.

For the later purpose of analyzing "thermal duality" we define what is meant by "Euclideanization" of the modular structure. We simply change the Hilbert space inner product by defining the following positive definite sesquilinear form [43] on the dense set of state vectors $L_{\mathcal{A}}\equiv\mathcal{A}\Omega$

$$\langle\Xi(A\Omega),\Xi(B\Omega)\rangle:=(A\Omega,\Delta^{\frac{1}{2}}B\Omega)=(\Omega,JAJB\Omega) \tag{4}$$

whose closure defines (thanks to the properties of the modular objects) the new "Euclidean" Hilbert space $H^E$ (the map $\Xi$ in $L_{\mathcal{A}}^E\equiv\Xi(L_{\mathcal{A}})$ denotes the Euclidean re-interpretation) on which this changed inner product leads to a new $^\dagger$Euclidean star algebra by starting from the formula

$$\begin{aligned}&\|\Xi(A)B\Omega\|_{H^E}^2=\left\|\Delta^{\frac{1}{4}}AB\Omega\right\|_H^2\leq\left\|\Delta^{\frac{1}{4}}A\Delta^{-\frac{1}{4}}\right\|_H^2\|B\|_{H^E}^2\\&D(\sigma_{-\frac{i}{4}})\equiv\left\{A\in\mathcal{A}\,|\,\Delta^{\frac{1}{4}}A\Delta^{-\frac{1}{4}}\in B(H)\right\},\\&\curvearrowright\Xi(D(\sigma_{-\frac{i}{4}}))\subset B(H^E)\end{aligned} \tag{5}$$

where the second line consists of the definition on a shared subalgebra $A_{sh}=D(\sigma_{-\frac{i}{4}})$ i.e. an algebra of bounded operators (without a star operation) which belongs to both the original setting and its Euclidean companion. It contains the dense subalgebra $\mathcal{A}_{an}$ of $\sigma_t$-analytic elements and affiliated pointlike field generators. Equipped with the original $^*$star, the von Neumann double commutant in $H$ is equal to the original operator algebra $A=(A_{sh})_*''$, while using instead the Euclidean $\dagger$star the double commutant in $H^E$ defines the Euclidean algebra $\mathcal{A}^E=(A_{sh})_\dagger''$.

It is easy to see that $(\mathcal{A}^E,\Omega^E)$ defines again a standard pair with $\Delta_E^{it}=\Delta^{it}$, whereas the old star becomes the new $J^E$ action and the old $J$-action the new star. In the physical $\mathcal{A}(W)$ situation one would phrase this interchange by saying that the $*$-conjugation (related to the physical charge-conjugation) and the $J$-reflection are interchanged. The use of the terminology "Euclidean" becomes clear if one specializes this formalism to chiral theories on the lightray; in that case the effect of the change of the inner product on the one-sided translation $U(x)$ of the right wedge (rather halfline) algebra $\mathcal{A}(0,\infty)$ leads to the formula

$$\Delta^{\frac{1}{4}}U(x)\Delta^{-\frac{1}{4}}=U^E(x)=\text{ ``}U(ix)\text{''} \tag{6}$$

(which has a meaning with respect to $\mathcal{A}_{sh}(0,\infty)$, the "starless" algebra) which mediates between $\mathcal{A}(0,\infty)$ and $\mathcal{A}^E(0,\infty)$. More details will be deferred to a separate paper.

A variant of this modular analog of the Osterwalder-Schrader property, which uses instead of a one- a two-sided compression $U(x)$ for $x>0$ a suitably defined 2-sided compression on $\mathcal{A}(0,\infty)$, is the crucial structure behind the thermal duality and the Verlinde relation in subsection 5.

A special property of d=1+1 Minkowski spacetime is the disconnectedness of the right/left spacelike region which leads to a right-left ordering structure. So in addition to the Lorentz invariant timelike ordering $x\prec y$ (x earlier than y) which exists in any spacetime dimension, there is an invariant spacelike ordering[9] $x<y$ (x to the left of y) in d=1+1; this opens the possibility of more general Lorentz-invariant

[9] The left/right ordering defines a class division of pairs (x,y) under causality-preserving changes.

spacelike commutation relation than those implemented by Bose/Fermi fields e.g. *plektonic* fields with a spacelike Artin braid group commutation structure. The appearance of such exotic statistics fields is not compatible with their Fourier transforms being on-shell creation/annihilation operators for Wigner particles, rather the states they generate from the vacuum contain in addition to the one-particle contribution a vacuum polarization cloud [8]. This close connection between new kinematic possibilities and interactions is one of the reasons why, different from higher dimensions (where interactions are prescribed by recipes based on local couplings of free fields, usually within the setting of Lagrangian quantization) low dimensional QFT offers an easier and more intrinsic access to the central issue of interactions. Although the operator-algebraic formulation is particularly well-suited to a more intrinsic approach, this does not mean that pointlike covariant fields have become obsolete. They only changed their role; instead of mediating between classical and quantum field theory in the (canonical or functional integral) setting of Lagrangian quantization, they are now universal generators of all local algebras and hence also of all modular objects which taken together generate an infinite dimensional noncommutative universal unitary group in the Hilbert space. Besides the implementors of global spacetime symmetries this universal modular group also contains (section 5) "partial diffeomorphisms" whose modular generated unitaries only act geometrically on localized subalgebras.

# 3 Boson/Fermion equivalence and superselection theory in a special model

The simplest and oldest but yet conceptually quite rich model is obtained (as first proposed by Pascual Jordan [9]) by using a 2-dim. massless Dirac current and showing that it may be expressed in terms of scalar canonical Bose creation/annihilation operators[10]

$$j_\mu =: \overline{\psi}\gamma_\mu\psi := \partial_\mu\phi,\, \phi := \int_{-\infty}^{+\infty}\{e^{ipx}a^*(p) + h.c.\}\frac{dp}{2\,|p|} \tag{7}$$

Although the potential of the current as a result of its infrared divergence is not a field in the standard sense of an operator-valued distribution in the Fock space of the a(p)$^*$ creation/annihilation operators [11], the formal exponential defined as the zero mass limit of a well-defined exponential free massive field (taken inside correlation functions)

$$: e^{i\alpha\phi(x)} := lim_{m\to 0}m^{\frac{\alpha^2}{2}} : e^{i\alpha\phi_m(x)} : \tag{8}$$

turns out to be a well-defined quantum (i.e. infrared finite) quantum field. Here the limit is understood in the sense of vacuum expectation values using the Wick combinatorics of the massive free field; the power in the pre-exponential mass factor is determined by the requirement that the most singular contribution from the Wick contraction stays finite. For example the leading singular part in $m \to 0$ of the two-point

[10] The bilinear formulae which relate these operators to the original Fermion operators can be found in [10].

[11] It becomes an operator after smearing with test functions whose Fourier transform vanishes at p=0.

functions behaves as

$$\left\langle : e^{i\alpha\phi_m(x)} :: e^{\mp i\alpha\phi_m(y)} : \right\rangle \sim m^{\mp\alpha^2} \left( \frac{1}{(x-y)^2_\varepsilon} \right)^{\pm\alpha^2} \tag{9}$$
$$\frac{1}{(\xi)^2_\varepsilon} := lim_{\varepsilon\to 0} \frac{1}{\xi^2 + i\varepsilon sign\xi_0} = lim_{\varepsilon\to 0} \frac{1}{\xi_+ + i\varepsilon} \frac{1}{\xi_- + i\varepsilon}$$

where in the last line the leading power has been re-written in lightray coordinates $\xi_\pm$ (with the correct positive energy $i\varepsilon$ -prescription). In order to maintain a finite zero mass limit one must use in (8) precisely that mass power which keeps all all correlations finite. Thanks to the general Wick combinatorics of exponential fields this leads to ($\xi_{ij} \equiv x_i - x_j$)

$$\left\langle \prod_i : e^{i\alpha_i \phi(x)} : \right\rangle = \begin{cases} \prod_{i<j} \left( \frac{-1}{(\xi_{+ij})_\varepsilon (\xi_{-ij})_\varepsilon} \right)^{\frac{1}{2}\alpha_i\alpha_j}, \sum \alpha_i = 0 \\ 0, \quad otherwise \end{cases} \tag{10}$$

i.e. those correlations which which do not obey the *charge selection rule* $\sum \alpha_i = 0$ vanish and the nonvanishing ones factorize into chiral components i.e. the model splits into two identical independent chiral theories. The additional presence in the vacuum expectation values of an arbitrary polynomial in the current $\prod_i j_{\mu i}(y_i)$ would not change these arguments which insures that the resulting zero mass limiting theory is a bona fide quantum field theory i.e. its system of Wightman functions is canonically associated (via the GNS construction) with an operator theory in a Hilbert space with a distinguished vacuum vector. There exists another physically more intuitive and intrinsic method (whose mathematical formulation is more involved) where one stays in the zero mass setting and obtains the charged sectors by splitting neutral states (belonging to the vacuum sector) and "dumping the unwanted compensating charge behind the moon" [26] (i.e. one uses spacial infinity as a wast-disposal). In that case one starts from smeared exponential $expij(f)$ of the current with smearing functions $f$ which are the smoothened version of a characteristic function $\chi(x,a)$, so that formally they represent exponentials of $\phi(x)$ smeared with bilocal function $\partial f$ being supported around $x$ (with positive values) and $a$ (with negative values) such that the total integral vanishes. The properly renormalized exponential

$$\frac{1}{Z(f)} e^{i\alpha j_0(f)} \tag{11}$$

has a finite limit for $a \to \infty$ in spacelike direction (within vacuum expectation values) precisely if the charge conservation among all finite endpoints $x_i$ in products of such smeared exponentials is maintained. The mechanism resembles the previous argument; the contributions from the contractions from the finite endpoints with the ends going to infinity lead to a vanishing result in case of no charge compensation between the finite ends (and approach a nontrivial finite limit with charge compensation). A conceptually very attractive method which determines the charge content without the disposal of the unwanted charge at infinity can be formulated in terms of inclusions which are canonically associated to disjoint two-intervals [31] (see next subsection).

The abelian current model and its associated charge-carrying exponential fields permit an extension to the compactified lightray $R \to \dot{R} \equiv S^1$ which is done with the help of the Cayley transformation

$$z = \frac{1+ix}{1-ix} \in S^1 \tag{12}$$

i.e. the Wightman fields which are operator-valued Schwartz distributions can be extended to a larger test function space which consists of smooth functions on the circle without the infinite order zero at $z = -1$ which corresponds to the fast decrase at $x = \pm\infty$). The structural property which places this extendibility on more general model-independent footing is conformal invariance; its more systematic exploration will be the subject of the two next subsections where it will become clear that the chiral decomposition into two lightray theories is a general property of two-dimensional conformal covariant theories.

There is a fine point in the compactified description namely the occurrence of a quantum mechanical zero modes in the Fourier decomposition of the circular description. It is not difficult to verify that their presence leads to a quantum mechanical pre-exponential factor for the Wick-ordered exponential fields which automatically enforce the $\alpha$-charge conservation. Hence in the rotational description the Wick contraction formalism holds in a standard way without having to add charge conservation by hand as in (10); they simply result from the zero mode quantum mechanics. In this approach the original chiral Dirac Fermion $\psi(x)$ from which the (chiral component of the current) was formed as a $j = \psi^*\psi$ composite re-appears as a charge-carrying exponential field for $\alpha = 1$ and illustrates the meaning of bosonization/fermionization. But note that this terminology has to be taken with a grain of salt in view of the fact that the bosonic current algebra only generates a superselected subspace of the space generated by the iterative application charge-carrying exponential fields i.e. although the Boson lives in the Fermion sector, the Fermion operator creates state vectors which are outside the Bose vacuum sector. Only in massive two-dimensional theories a complete bosonization/fermionization (in which the generated spaces are identical) can be achieved, a problem which is related to the irrelevance of statistics and the appearance of order/disorder variables in such massive models (see last subsection).

It is amusing to note that Jordan's treatment of fermionization had such a pre-exponential quantum mechanical factor. At this point it should however be clear to the reader that the physical content of Jordan's paper had nothing to do with its misleading title "neutrino theory of light" but rather was a special illustration about charge superselection rules in QFT long before this general concept was recognized.

A systematic approach which avoids pointlike fields in favor of spacetime indexed operator algebras can be formulated in terms of positive energy representation theory for the Weyl algebra[12] on the circle (which is the rigorous operator algebraic formulation of the abelian current algebra). It is the operator algebra generated by the exponential of a smeared chiral current (always with real test functions) with the following relation between the generators

$$\begin{aligned} &W(f) = e^{ij(f)},\ \ j(f) = \int \frac{dz}{2\pi i} j(z) f(z),\ [j(z), j(z')] = -\delta'(z - z'), \qquad (13) \\ &W(f)W(g) = e^{-\frac{1}{2}s(f,g)} W(f+g),\ \ W^*(f) = W(-f) \\ &\mathcal{A}(S^1) = alg\left\{W(f), f \in C_\infty(S^1)\right\}, \mathcal{A}(I) = alg\left\{W(f), suppf \subset I\right\} \end{aligned}$$

where $s(f,g) = \int \frac{dz}{2\pi i} f'(z) g(z)$ is the symplectic form which characterizes the Weyl algebra structure and the last line denotes the unique $C^*$ algebra generated by the unitary objects $W(f)$. A particular

[12] The Weyl algebra was not used in QFT at the time of Jordan's paper. By representation we mean here a regular representation in which the exponentials can be differentiated in order to obtain (unbounded) smeare current operators.

representation of this algebra is given by assigning the vacuum state on the generators

$$\langle W(f)\rangle_0 = e^{-\frac{1}{2}\|f\|_0^2},\ \|f\|_0^2 = \sum_{n\geq 1} n\,|f_n|^2 \qquad (14)$$
$$L_0 = \int \frac{dz}{2\pi i} T(z),\ T(z) =: j(z)^2 :$$

The norm in the first line leads to an inner product space which can be made into a Hilbert space by defining a complex structure[13] on the real space. In the last line we have written the circular Hamiltonian $L_0$ of the model in terms of its chiral energy-momentum tensor $T$. A more concrete method consists in starting with a Hilbert space representation $\mathcal{A}(S^1)_0 = \pi_0(\mathcal{A}(S^1))$ obtained by applying the GNS representation to this vacuum state functional. This representation has a positive energy operator given by the generator of rotations $L_0$ which is quadratic in the current. It is easy to check that the formula

$$\langle W(f)\rangle_\alpha := e^{i\alpha f_0}\,\langle W(f)\rangle_0 \qquad (15)$$
$$\pi_\alpha(W(f)) = e^{i\alpha f_0}\pi_0(W(f))$$

defines an inequivalent state i.e. one whose GNS representation for $\alpha \neq 0$ is unitarily inequivalent to the vacuum representation with positive energy $(L_0)_\alpha \equiv \pi_\alpha(L_0)$. The particular realization of the GNS representation of the $\alpha$-state in the second line is economical because in this way the inequivalent description becomes incorporated into the Hilbert space of the vacuum representation. For certain generalizations in the next section it is convenient to rephrase this result as the result of two steps, first a definition of an automorphism $\gamma_\alpha$ on the $C^*$-Weyl algebra $\mathcal{A}(S^1)$ and then the subsequent application of the vacuum state [44]

$$\langle W(f)\rangle_\alpha = \langle \gamma_\alpha(W(f))\rangle_0\,,\ \gamma_\alpha(W(f)) := e^{i\alpha f_0}W(f) \qquad (16)$$
$$\gamma_\alpha(W(f)) = \Gamma_\alpha W(f)\Gamma_\alpha^*,\ \Gamma_\alpha\Omega = \Omega_\alpha,\ \langle . \rangle_\alpha = (\Omega_\alpha, .\Omega_\alpha)$$

where in the second line the automorphism is implemented by a charge-carrying operator $\Gamma_\alpha$ which intertwines between the vacuum Hilbert space $H_0$ and the use of the same Hilbert space for the charged representation denoted by $H_\alpha$ (in order to indicate its different use). The charge-transfer operator $\Gamma_\alpha$ interwines the various copies of identical Hilbert spaces $H_\alpha$ and in particular relates the vacuum state to the ground state $\Omega_\alpha$ of the new sector (by definition $\Gamma_\alpha$ creates a rotational homogeneous charge distribution (i.e. a distribution without radial excitations) so that the full (in this case inseparable) Hilbert space becomes the direct (orthogonal) sum $H_{full} = \oplus H_\alpha$. Arbitrary charge distributions $\rho_\alpha$ of total charge $\alpha$ i.e. $\rho_\alpha[1] \equiv \int \frac{dz}{2\pi i}\rho_\alpha = \alpha$ are obtained in the form

$$\psi_{\rho_\alpha}^\zeta = \kappa(\rho_\alpha)W(\hat\rho_\alpha^\zeta)\Gamma_\alpha \qquad (17)$$

where $\kappa(\rho_\alpha)$ is a phase factor and the net effect of the test function in the Weyl operator is to modify the homogeneous charge distribution created by $\Gamma_\alpha$ in order to obtain $\rho_\alpha$ with the same total charge [44]. The necessary charge-neutral compensating test function $\hat\rho_\alpha^\zeta$ is uniquely determined in terms of $\rho_\alpha$ apart from a choice of one point $\zeta \in S^1$(the determining equation involves the $lnz$ function which needs the specification of a branch cut).

[13]The multiplication with i which characterizes a complex struture in the present case is multiplication with the number $\pm i$ of the $\pm$ Fourier components.

**Theorem 1** *The charge-carrying fields[14] for disjoint charge supports ($supp\rho_\alpha \perp supp\rho_\beta$) fulfill abelian braid group commutation relation and additive fusion laws*

$$\begin{aligned} \psi^\zeta_{\rho_\alpha}\psi^\zeta_{\rho_\beta} &= e^{\pm i\pi\alpha\beta}\psi^\zeta_{\rho_\beta}\psi^\zeta_{\rho_\alpha} \\ \psi^\zeta_{\rho_\alpha}\psi^\zeta_{\rho_\beta} &= e^{\pm i\frac{\pi}{2}\alpha\beta}\psi^\zeta_{\rho_\alpha+\rho_\beta} \end{aligned} \tag{18}$$

*where the $\pm$ signs depend on whether the path from $supp\rho_{\rho_a}$ to $supp\rho_\beta$ taken in positive (counterclockwise) direction crosses $\zeta$ or not. A change of the cut $\zeta$ leads to the appearance of a charge factor*

$$\psi^{\zeta_1}_{\rho_\alpha}\left(\psi^{\zeta_2}_{\rho_\alpha}\right)^* = e^{\pm i\pi\alpha\beta}e^{2\pi iQ\alpha} \tag{19}$$

*where the charge operator $Q$ is conjugate to $\Gamma$ in the sense. $QH_a = \alpha H_\alpha$ or $Q\Gamma_\alpha = \alpha\Gamma_\alpha Q$ i.e. is part of the zero mode structure.*

The second relation expresses the abelian fusion law of the model. Up to now the Hilbert space was the nonseparable Hilbert space of all charges and in order to get away from this unrealistic feature of our toy model we search for an argument which leads to charge quantization in a natural manner. It turns out that algebraic extension of the Weyl algebra which maintain commutativity for disjoint charge supports combined with a compatible restriction of the inseparable Hilbert space do the job

$$\begin{aligned} &\mathcal{A}_N = \cup_I \mathcal{A}_N(I), \mathcal{A}_N(I) = alg\left\{\psi^\zeta_{\rho_\alpha} \,|\, supp\rho_{\alpha_{gen}} \in I, \alpha_{gen} = \sqrt{2N}\right\}|_{H_{res}} \\ &H_{res} = \left\{\Psi \in H \,|\, e^{2\pi iQ\alpha}\Psi = \Psi\right\}, \, H_{res} = \oplus_{n=0}^{2N-1} H_n \end{aligned} \tag{20}$$

Clearly the vacuum space of the extended algebra $\mathcal{A}_N$ contains all integer multiples of the old locality-preserving generating charge $\alpha = \alpha_{gen}\mathbb{Z}$ (the charge neutral $\psi_{\rho_\alpha}\psi^*_{\rho'_\alpha}$ products lead back to the original Weyl algebra). The restricted Hilbert space $H_{res}$ is a orthogonal sum of new charges $Q = \frac{1}{\sqrt{2N}}\mathbb{Z}\,/\alpha_{gen}\mathbb{Z} \simeq \mathbb{Z}_{2N}$ i.e. consists of the dual (to the old) charge spectrum $\frac{1}{\sqrt{2N}}\mathbb{Z}$. (which has been "neutralized"). The effect of the mod counting is that the old charges are neutralized by enlarging the algebra (always with its local net structure) from $\mathcal{A}$ to $\mathcal{A}_N$, so that the superselection structure becomes finite (the model becomes "rational"). The charge-carrying fields in the new setting are also of the above form (17) but now the generating field carries the charge $\int\frac{dz}{2\pi i}\rho_{gen} = Q_{gen}$ which is a $\frac{1}{2N}$ fraction of the old $\alpha_{gen}$. Their commutation relations for disjoint charge supports are "braidal" (or "plektonic"[15] which sounds more in par with bosonic/fermionic). These objects considered as operators localized on $S^1$ do depend on the cut $\zeta$, but using an appropriate finite covering of $S^1$ this dependence is removed. So the field algebra $\mathcal{F}_{\mathbb{Z}_{2N}}$ (as opposed to the bosonic observable algebra $\mathcal{A}_N$) which they generate has its unique localization structure on a finite covering of $S^1$. An equivalent description which gets rid of $\zeta$ consists in dealing with operator-valued sections on $S^1$.

In abelian current algebras the transition from bounded Weyl-like operators to generating operator-valued distributional fields is especially simple; one just approaches the $\delta$-function charge distribution ("blip") at the origin $\alpha\delta(z-1)$ by a sequence of smooth functions $\rho_\alpha(z)$ and checks the existence of the

[14] It is costumary in the algebraic setting to use the word "field" for operators which (in contrast to the neutral observables) carry superselected charges and add the word pointlike if one is referring to its traditional use.

[15] In the abelian case the terminology "anyonic" enjoys widespread popularity, but in the present context the "any" does not go well with the present emphasis on charge quantization.

limit

$$\Phi_\alpha(\varphi) = lim_{\rho_\alpha \to \alpha\delta(z-1)} R_{\rho_\alpha} Ade^{i\varphi L_{0,\alpha}} \psi_{\rho_\alpha}^{\zeta=-1} \tag{21}$$
$$\left(\Gamma_\alpha \Omega, R_{\rho_\alpha} Ade^{i\varphi L_{0,\alpha}} \psi_{\rho_\alpha}^{\zeta=-1}\Omega\right) = 1$$

in words: the operators $\psi_{\rho_\alpha}^{\zeta=-1}$ which generate charged states with charges around the origin $\varphi = 0$ are translated so that their charge is concentrated around the angle $\varphi$ whereupon their charge distribution is compressed to a blip at $\varphi$ in such a way that their normalization (second line in (21)) is maintained (which leads to a formula for the renormalization factor $R_{\rho_\alpha}$). It is costumary to interpolate the $\delta$-function by a scaling limit $\lambda \searrow 0$ in which case the renormalization factor diverges with an inverse power related to the scale dimension of the resulting pointlike field [44].

The extension $\mathcal{A} \to \mathcal{A}_N$ which led to a "rational" (= finite number of sectors) charge superselection structure is a charge-quantization extension. Most other chiral models (next section) already come with a discrete charge spectrum. In both cases one can ask whether a model with discrete charge superselection spectrum allows (further) local extensions. For the abelian case at hand this would require the presence of another generating field of the same kind as above which belongs to an integer $N'$ and is relatively local to the first one. This is always possible if $N$ is divisible by a square, in fact the algebra $\mathcal{A}_N$ is maximal precisely if $N$ is of the form $N = p_1...p_k$ where $p_i$ are prime numbers. Whereas in abelian current models such question can be answered in terms of pedestrian computations, the generic case is conceptually much more challenging. In the next section we will return to these problems in a more general setting.

Before we pass to the issue of conformal invariance and the problems of general chiral models, we cannot resist to mention a simple yet somewhat surprising relation between the Schwinger model ($QED_2$ with massless Fermions), whose charges are screened, and the Jordan model, which has (liberated) charge sectors. Since the Lagrangian formulation of the Schwinger model is a gauge theory, the analog of the 4-dim. asymptotic freedom wisdom would suggest the possibility of charge liberation in the short distance limit of this model. This seems to contradict the statement that the intrinsic content of the Schwinger model after removing a classical degree of freedom[16] is the QFT of a free massive Bose field because such a simple free field is at first sight not expected to contain subtle informations about asymptotic charge liberation. But the massless limit as the potential of the free abelian current really does contain this information i.e. the observable part of the Schwinger model passes to the charge-liberated massless Jordan model as one can demonstrate in detail in the short distance limit [48]. This statement is truly intrinsic since it refers to the screened phase, unlike the 4-dimensional asymptotic freedom statement which is based on the perturbative phase instead of the physical quark confinement phase (the asymptotic freedom statement is the result of a consistency check falling short of a mathematical theorem). There are other properly renormalizable (i.e. not superrenormalizable as the Schwinger model) two-dimensional models in which one can prove the validity of asymptotic freedom in the physical phase, but the poor state of nonperturbative knowledge in d=1+3 is hampering an understanding of this issue in realistic cases beyond the level of a plausibility statement. These considerations show in addition that there is nothing

[16]In its original gauge theoretical form the Schwinger model has an infinite vacuum degeneracy. The removal of this degeneracy (restoration of the cluster property) with the help of the "$\theta$-angle formalism" leaves a massive free Bose field (the Schwinger-Higgs mechanism). As expected in d=1+1 the model only possesses this one phase, a characteristic feature of all two-dimensional non-lattice models.

intrinsic about a gauge theoretical formulation; in fact the gauge idea in the setting of quantum field idea is a computational device and not a physical principle since at least for selfinteracting vector mesons ("nonabelian gauge theory") renormalizability requirement only admits one perturbative model (the appearance of other physical (Higgs) degrees of freedom follows from consistency of perturbation theory) and where the situation already has a unique answer from the implementation of a quantum principle (renormalizability) no additional principle is needed. The gauge principle rather selects between several consistent classical field theories involving vector fields and follows from quantum renormalizability via quasiclassical approximations. The non-intrinsic nature and the absence of a quantum gauge principle is also implicit in many conjectures where a gauge theoretic formulation is expected to be dual to a non-gauge theory.

As a result of the peculiar nature of the zero mass limit of the derivative of the massive free field, Jordan's model is also closely related to the massless Thirring model (and the closely related Luttinger model for an interacting one-dimensional electron gas) whose massive version is in the class of factorizing models (see later section)[17]. Together with the massive version of the L-I QFT it shows two new (interconnected) properties which are characteristic for massive d=1+1 models: the absence of an intrinsic meaning of statistics [100] and the emergence of a disorder variable with a nonvanishing vacuum expectation value (disorder condensation).

The Thirring model proper is a special case in a large class of "generalized" (multi-coupling) multi-component Thirring models i.e. 4-Fermion interactions. Under this name they were studied in the early 70s [49][50][51] with the particular aim to identify massless subtheories for which the currents have a chiral decomposition and form current algebras.

It is interesting to look in more detail at the massive version of the Thirring model. The counterpart of the potential of the conserved Dirac current is the Sine-Gordon field, i.e. a composite field which in the attractive regime of the Thirring coupling obeys the so-called Sine-Gordon equation of motion. Coleman gave an argument [14] which however does not reveal the limitation in the size of the coupling[18]. A rigorous confirmation of the existence of a coupling range for Coleman's equivalence was recently given in the bootstrap-formfactor setting [52]. Two-dimensional massive models which have a continuous or discrete internal symmetry have "disorder" fields which are local fields (with respect to themselves) which implement a "half-space" symmetry on the charge-carrying field (acting as the identity in the other half axis). These are pointlike bosonic fields which live in the same Hilbert space as the charged fields, but only create the neutral part from the vacuum. Multiplication of disorder fields with the defining field of the model (leading short distance part of the product of the disorder operator with the charged field) generates "order" fields which act cyclically on the vacuum. The order/disorder fields have an interesting connection with phase transitions. Whereas in the lattice version the correlation functions [53] of the L-I model the system undergoes a second order phase transition as the temperature passes through the critical value, the mass parameter represents only the slope of temperature at criticality and lost its role of analytically connecting two phases; the only memory of the different phases in the QFT resulting in the scaling limit consists in the presence of a pair of order/disorder variables whose interchange in passing

[17] Another structural consequence of this peculiarity leads to Coleman's theorem [46] which connects the Mermin-Wagner No-Go theorem for two-dimensional spontaneous continuous symmetry breaking with these zero mass peculiarities.

[18] The current potential of the free massive Dirac Fermion (g=0) does not obey the Sine-Gordon equation [54].

from one phase to another has to be decreed as an additional rule. The resulting n-point order/disorder correlation functions of the L-I field theory can be represented in terms of order/disorder variables of a free Majorana field or as the (suitable defined) square root of the exponential disorder field of a free Dirac Fermion, both in the massive [54] as well as in the massless limit [55][56]. They are scalar Bose fields with a $Z_2$ "half-space" commutation relation between them. Whereas the massive scaling limit fields still have correlation functions which are order/disorder unsymmetrical, the conformal invariant zero mass limit leads to a symmetric situation where both variables carry superselected charges. The emergence of new charges in connection with the appearance of critical exponents of order/disorder fields in 2-dim. QFT is actually the content of a general theorem [57].

# 4 The general conformal setting and chiral theories

Chiral theories play a special role within the setting of conformal quantum fields. General conformal theories have observable algebras which live on compactified Minkowski space ($S^1$ in the case of chiral models) and fulfill the Huygens principle, which in an even number of spacetime dimension means that the commutator is only nonvanishing for lightlike separation of the fields. The fact that this rule breaks down for non-observable "would be" conformal fields (e.g. the massless Thirring field) was noticed at the beginning of the 70s and considered paradoxical at that time ("reverberation" in the timelike (Huygens) region). Its resolution led 1974/75 to two differently but basically equivalent concepts about globally causal objects. They are connected by the following global decomposition formula

$$A(x_{cov}) = \sum A_{\alpha,\beta}(x),\ A_{\alpha,\beta}(x) = P_\alpha A(x) P_\beta,\ Z = \sum e^{id_\alpha} P_\alpha \tag{22}$$

On the left hand side the field lives on the universal covering of the conformal compactified Minkowski space $\tilde{M}$. These are the Luescher-Mack fields [58] which "live" in the sense of quantum (= modular) localization on the universal covering spacetime (or a finite covering, depending on the model) and fulfill the global causality condition discovered by I. Segal [59]. In the presence of interactions they are highly reducible under the center of the covering group. The objects on the right hand side are the component fields which were introduced in [60] with the aim to have objects which live on the projection $x(x_{cov})$ i.e. on the (Dirac-Weyl compactified) Minkowski spacetime $\bar{M}$ of the laboratory instead of the "hells and heavens" of the covering; The connection is given by a decomposition formula into irreducible conformal blocks with respect to the center $\mathbf{Z}$ of the covering group $\widetilde{SO(2,n)}$ where $\alpha, \beta$ are labels for the eigenspaces of the generating unitary $Z$ of the abelian center $\mathbf{Z}$. Under central transformations these component fields transform with a numerical phase which is proportional to the difference of anomalous dimensions $d_\alpha - d_\beta$ [60] whereas the globally causal Luescher-Mack fields pick up an operator-valued phase. The composition formula is minimal in the sense that in general there will be a refinement due to the presence of additional charge superselection rules (and internal group symmetries) which have no bearing on the covering aspect of the Luescher-Mack fields and their algebraic commutation relations. Technically speaking the $A_{\alpha,\beta}(x)$ are operator-valued distributional sections ("conformal blocks") in the compactification of ordinary Minkowski spacetime. They are not Wightman fields since they annihilate the vacuum if the right hand projector $P_\beta$ differs from the projector onto the vacuum sector.

Note that the Huygens (timelike) region in Minkowski spacetime has an ordering structure $x \prec y$ or $x \succ y$ (earlier, later). In d=1+1 the topology allows in addition a spacelike left-right ordering $x \lesseqgtr y$. This together with the factorization of the group $\widetilde{SO(2,2)} \simeq \widetilde{PSL(2R)}_l \otimes \widetilde{PSL(2,R)}_r$ in particular $\mathbb{Z} = \mathbb{Z}_l \otimes \mathbb{Z}_r$ suggests a tensor factorization into chiral components and led to an extremely rich and successful construction program of two-dimensional conformal QFT as a two-step process: the classification of chiral theories on the lightray and the amalgamation of left-right chiral theories to two-dimensional local conformal QFT. The action of the covering chiral group on the lightray coordinates is through fractionally acting $PSL(2,R)$ Moebius transformations

$$x \to g(x) = \frac{ax+b}{cx+d},\, g = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in SL(2,R),\, i.e.\, detg = 1 \tag{23}$$

$$z \to g(z) = \frac{\alpha z+\beta}{\bar{\beta} z+\bar{\alpha}},\, g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1),\, i.e.\, |\alpha|^2 - |\beta|^2 = 1 \tag{24}$$

where the linear and circular descriptions are related through the Cayley transformation $z = \frac{1+ix}{1-ix}$.

The presence of an ordering structure permits the appearance of more general commutation relations for the above $A_{\alpha\beta}$ component fields namely

$$A_{\alpha,\beta}(x)B_{\beta,\gamma}(y) = \sum_{\beta'} R^{\alpha,\gamma}_{\beta,\beta'} B_{\alpha,\beta'}(y)A_{\beta',\gamma}(x),\;\; x > y \tag{25}$$

with numerical $R-$coefficients which (as a result of associativity and relative commutativity with respect to observable fields) represent the Artin braid group. Indeed, the DHR method to interpret charged fields as charge-superselection carriers tied by local representation theory to the bosonic local structure of observable algebras leads precisely to such a plektonic setting. With an appropriately formulated adjustment to observables fulfilling the Huygens commutativity, this could also be a possibility for the higher dimensional timelike structure. But whereas the plektonic lightlike structure is the only spacetime commutation imposition for chiral theories, a would-be timelike plektonic structure in higher dimensions has to coexist with the spacelike bosonic/fermionic statistics structure which appears to lead to a much more difficult point of departure for classifications&constructions than in the chiral case where a wealth of models with R-coefficients of their charge-carrying fields have been found. In the latter case the availability of infinite dimensional loop group and Diff(S) symmetries and the connection of the latter to the presence of the chiral stress-energy tensor give a rich supply of Lie-field generated observable algebras on which one has constructed the representations of the superselected charge sectors. In higher spacetime dimensions no such geometric infinite dimensional extensions of the finite dimensional global conformal vacuum symmetry is known. It has been observed that the Huygens principle in conjunctions with conformal invariance leads to quite strong restrictions on Wightman functions [61][62] which could help in the classification program and even suggest associated new algebraic structures. On the other hand the holographic lightfront projection leads to transversely extended chiral models which, although not quite as simple as chiral theories themselves, seem to be more susceptible to be analysed in terms of algebraic commutation structures (section 8).

Whereas the conformal block picture i.e. the objects on the right hand side of (22) naturally fits into a DHR approach in which one starts with a model of observable algebras on (compactified) Minkowski spacetime and constructs the so-called reduced field bundle (exchange algebra field sections), the globally causal objects in the left hand side (22) which are localized (in the sense of modular theory) on the covering spacetime suggests another approach which is more in the intrinsic spirit of group theoretical Wigner's particle representation setting. The guiding idea would be that the modular localization concept formulated globally on an appropriate n-sheeted covering space $\tilde{M}^{(n)}$ (for rational theories) within a representation theoretical setting could directly lead to global objects without going through the DHR analysis[19]. The latter has not been formulated on coverings and this deficiency probably leads to a somewhat artificial (in the sense of non-intrinsic) separation into outer (spacetime) and inner symmetries (see remarks after theorem at end of this section). Unfortunately a direct access to globally causal Luescher-Mack fields without going through the right hand side in (22) does not (yet?) exist.

From this presentation of the development of ideas about conformal QFT and chiral models in particular one may have obtained the impression that there is a straight line from the decomposition theory of the early 70s to the construction of interesting two-dimensional models, however this is not the way history of constructions of conformal QFTs developed[20]. The only examples known up to the appearance of the seminal Belavin-Polyakov-Zamolodchikov work (BPZ) [7] were the abelian current models of the previous subsection. The floodgates of conformal QFT were not opened by knowing an abstract setting of conformal block decomposition but by the BPZ discovery of "minimal models" and their connections to the already existing mathematics of Witt-Virasoro and Kac-Moody algebras. A crucial step in the understanding of the minimal models, in particular in what sense they are "minimal", was contained in a prior paper of Friedan, Qiu and Shenker [63]. That paper also showed for the first time that the positive energy representation category of the observable algebra generated by the energy-stress tensor cannot be encoded into a symmetry group. The FQS proof uses the setting of Verma module representations, but it is also possible to obtain the same conclusions from a standard field theoretic Hilbert space setting [64]. The combinatorial aspects of this new structures abstracted from these model observations were axiomatized in [65] and the origin of the Artin braid group structure as a new manifestation of Einstein causality in chiral field theory which led to "exchange algebras" was analyzed in [87]; in fact part of the motivation behind it was to connect the post BPZ development to what was known about conformal theories in the 70s [60]. This was followed by a systematic application of the DHR concepts to this new setting in [88][66].

There was an interesting, independent and much older idea for constructing models via representing new algebraic structures which in the course of time merged with the chiral conformal constructions. It goes back to a 1961 paper by O. W. Greenberg [67] who proposed to construct nontrivial examples of Wightman field theories instead of quantizing nonlinear field equations by studying "Lie fields" i.e. sets

[19] The DHR dichotomy between local observables and charge-carrying fields solves an important conceptual problem, but is not necessarily useful for model constructions; e.g. charge-carrying free fields are much simpler mathematical objects than their associated neutral observable algebras. All the known approximation schemes aim at the direct construction of field correlations.

[20] The algebra of chiral energy-stress tensors was known since the early 70s, first for the free Dirac field and subsequently as a general structural result [51]. A later (∼1976) manuscript of Luescher and Mack contained in addition to this result the beginnings of the c-quantization (the Ising case) but this project remained unfortunately unfinished and unpublished, see also [23]. All these early results were superseded by the 1984 work in [63].

of local fields $A_i(x)$ fulfilling the "Lie relation" (for simplicity for a set of Lorentz-scalar fields)

$$[A_i(x), A_j(y)] = c - number + \int C_{ij}^k(x,y,z)A_k(z)d^n z \tag{26}$$

In the days of axiomatic quantum field theory this subject led to several papers with inconclusive results [68]. The non-abelian chiral current algebras at the beginning of the 70s gave some obvious illustrations of this structure, but the more interesting case was that of the generic chiral stress-energy tensor which was proposed in [51] as illustration of a Lie field which closes upon itself. Of course in some general sense all chiral fields are Lie fields since the lightray locality only permits $\delta$-functions with a finite number of derivative multiplied with pointlike (composite) field generators of the same operator algebra, but here this terminology refers to the existence of a finite distinguished set of generators which close among themselves under commutation. The net result of this research is contained in a paper by Baumann [69] who proved that there are no nontrivial scalar Lie fields in higher spacetime dimensions i.e. $C_{ij}^k(x,y,z) \equiv 0$. Similar conclusions probably hold for tensor/spinor fields but there seems to be no proof. Examples of conformal Lie fields are the chiral current algebras and some of the so-called W-algebras (generalizations of the stress-energy algebra which contain additional fields without internal symmetry group multiplicities). Since massive Lie fields do not lead to scatteriong (not even in d=1+1), the interest in them within two-dimensional QFT is entirely limited to chiral models. Indeed we will see in section 5 that solvable (factorizable) massive two-dimensional theories are characterized by a very different algebraic structure. The Lie field structure of chiral current algebra is generally lost by processing these current algebras through reduced tensor products, orbifold constructions, coset constructions and (Longo-Rehren) extensions into other models, but it seems that all known models originate by such procedures from Lie field models.

In order to not get lost in the impressive wealth of detailed knowledge about chiral models and the associated two-dimensional conformal QFT, but also to avoid the opposite extreme of bothering the reader with too many conceptual generalities, I will try to keep a middle ground by presenting some salient points in connection with two families of models which illustrate some important structural points in concrete and pedestrian terms.

Let us start with a family which generalizes the abelian model of the previous section. Instead of a one-component abelian current we now take n independent copies. The resulting multi-component Weyl algebra has the previous form except that the current is n-component and the real function space underlying the Weyl algebra consists of functions with values in an n-component real vector space $f \in LV$ with the standard Euclidean inner product denoted by $(,)$. The maximal local extension now leads to $(\alpha, \beta) \in 2\mathbb{Z}$ i.e. an even integer lattice $\mathcal{L}$ in $V$, whereas the Hilbert subspace $H_{L^*}$ which ensures $\zeta$-independence is associated with the dual lattice $L^* : (\lambda_i, \alpha_k) = \delta_{ik}$ [45]. The resulting superselection structure (i.e. the $Q-$spectrum) corresponds to the finite group $L^*/L$. It offers the possibility of selfdual lattices $L^* = L$ i.e. two-dimensional QFT whose observables have no additional representations; a situation which only can occur in vector spaces V whose dimension is a multiple of 8 (the most famous case is the Leech lattice $\Lambda_{24}$ in $dimV = 24$ also called the "mooshine" model [47]). The observation that the root lattices of the Lie algebras of type $A, B$ or $E$ (example su(n) corresponding to $A_{n-1}$) also appear, suggests that the nonabelian current algebras associated to those Lie algebras can also be implemented. This turns out to be indeed true as far as the level 1 representations are concerned which brings us to the next family: the nonabelian current algebras of the mentioned Lie algebras of level k which are characterized by the commutation relation

$$[J_\alpha(z), J_\beta(z')] = if_{\alpha\beta}^{\gamma} j_\gamma(z)\delta(z-z') - \frac{1}{2}kg_{\alpha\beta}\delta'(z-z') \tag{27}$$

where $f$ are the structure constants of the underlying Lie algebra, $g$ is the Cartan-Killing form and $k$, the level of the algebra, must be integer in order that the current algebra can be globalized to a loop group algebra. The Fourier decomposition of this current algebra leads to the so called affine Lie algebras, a special family of Kac-Moody algebras. For k=1 this algebra can be constructed as bilinears starting from a multi-component chiral Dirac field; in addition there exists the mentioned possibility to construct it within the previous setting of abelian algebras by extending these algebras with certain charge-carrying ("vertex" algebra) operators. Level k representations can be constructed from tensor products of k level one currents by field theoretic reductions or directly by studying the representation theory of infinite-dimensional Kac-Moody Lie algebras[21]. Either way one finds that e.g. the SU(2) current algebra of level k has (together with the vacuum sector) k+1 sectors (inequivalent representations). The labelling of the different sectors is equivalent to the labelling of their ground states of the conformal Hamiltonian $L_0$. With a bigger group-theoretical effort one can construct the representation sectors, the generating charge-carrying fields (primary fields) including their R-matrices and the associated net of chiral operator algebras (indexed by intervals on the circle) which in the $SU(2)_k$ would be denoted by $\mathcal{A}_{SU(2)_k,n}$, n=0,...k and in the general semisimple case require a more complicated characterization (in terms of Weyl chambers).

Current algebras were introduced in the early 70s as a means to explore the multi-component multi-coupling Thirring model, in particular to find critical coupling values for which the model becomes conformally invariant [49][50][51] and could have interesting applications in the field theoretic treatment of critical phenomena. The question whether at such conformal points (the prerequisite being the vanishing of beta-functions) one can find the nonabelian analog of the Jordan bosonization received a positive answer when Witten [70] proposed a bosonic Lagrangian with a topological term (which set it apart from the standard perturbative Lagrangian quantization setting). Its name Wess-Zumino-Witten Lagrangian resulted from an analogy of its interaction terms in its Lagrangian group-valued field description with a 4-dimensional phenomenological Lagrangian used by Wess and Zumino.

As an important general message coming from two-dimensional solvable QFT it is worthwhile to note that even in those cases where the model permits baptizing it in terms of Lagrangian quantization (thus preparing the ground for a standard renormalized perturbation approach e.g. the massive Thirring model), the model cannot be fully solved in the Lagrangian setting but requires the algebraic approach. In the case of the WZW Lagrangian the correlation functions of the group-valued bosonic field are computed by identifying this field as a two-dimensional composite formed from combining the left/right current algebras and using the prior current algebra representation methods [17]. These calculations also show that there is no intrinsic physical meaning in topological aspects of Euclidean functional integral representations.

The construction of equivalence classes of irreducible positive energy representations for the minimal models is more tricky than that of current algebras. The algebraic structure of those models is given by

[21] The global exponentiated algebras (the analogs to the Weyl algebra) are called loop group algebras.

the commutation relation of the energy-momentum tensor

$$[T(z), T(z')] = i(T(z) + T(z'))\delta'(z - z') + \frac{ic}{24\pi}\delta'''(z - z') \tag{28}$$
$$c < 1 \curvearrowright c = c_m = 1 - \frac{6}{(m+2)(m+3)}, \; m = 1, 2....$$

whose Fourier decomposition yields the Witt-Virasoro algebra i.e. a central extension of the Lie algebra of the $Diff(S^1)$. The first two coefficients are determined by the physical role of $T(z)$ in connection with the generation of the Moebius transformations and the undetermined parameter $c > 0$ (the central extension parameter) is easily identified with the strength of the T two-point function whose form is completely fixed by Moebius invariance and dimT=2. Although the structure of the T-correlation functions resembles that of free fields (in the sense that the theory is known once one has specified the two-point function), the realization that if c<1 then it is necessarily quantized according to the second line in (28) came as a surprise one decade after the Lie-field structure of the energy-momentum tensor was unraveled (there is no such quantization for c$\geq$ 1)[22]. The admissable values for the existence of a Hilbert space representation are the $c_m$ values in (28) and the possible values for the conformal energy (the non-negative operator $L_0$) are

$$h_{p,q}(c_m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \begin{cases} 1 \leq p \leq m-1 \\ 1 \leq q \leq m \end{cases} \tag{29}$$

That there are really algebras representations $\mathcal{A}_{m,p,q}$ which fill these slots can be seen by constructing such models via a SU$(2)_k$ current coset construction which reduces the existence problems of these models to that of the simpler current algebras (which can be obtained by performing reductions on tensor product of free massless SU(2) Dirac fields). Constructing chiral models does generally not mean the explicit determination of the Wightman functions of their generating fields but a proof of their existence by demonstrating that these models are obtained from free fields by a series of controllable but often involved constructive steps as reduction of tensor products formation of orbifolds under group actions, coset constructions controllable extensions etc. The generating fields of the models are not obeying free field equations (are not "on-shell"). The cases where one can write down n-point functions of generating fields are very rare; in the case of the minimal family this is only possible for the Ising model. The reason for this is that by "doubling" the Ising model one connects to the exponential field of the Jordan model from where one can return to the already fairly complex chiral Ising n-point functions of order/disorder variables by drawing a square root in a suitable way [56]. This simplification through doubling works also for the massive Ising field theory [54]. It is even possible to construct a represenration for the Ising correlations on a 2-dimensional Euclidean lattice [53].

To show the power of inclusion theory for the determination of the charge content of theory let us look at a simple illustration in the context of the above multi-component abelian current algebra. The vacuum representation of the corresponding Weyl algebra is generated from smooth $V$-valued real functions on the circle modulo constant functions (i.e. with vanishing total integral) $f \in LV_0$. These functions equipped with the aforementioned complex structure generate a Hilbert space $H_1 = \overline{LV_0}$. The $I$-localized subalgebra is generated by the subspace of $I$-supported functions (class functions whose representing

[22] A similar quantization phenomenon was discovered by Vaughn Jones in the mathematical theory of subfactors of Jones index <4 but as far as I know the question about a possible direct relation remained without answer.

functions are constant in the complement $I'$)

$$\mathcal{A}(I) := alg\left\{W(f)|\, f \in LV_0,\, f = const\ in\ I'\right\} \tag{30}$$

The geometric one-interval Haag duality $\mathcal{A}(I)' = \mathcal{A}(I')$ (the commutant algebra equals the algebra localized in the complement) is simply a consequence of the fact that the symplectic complement in terms of $Im(f,g)$ consists of real functions in that space which are localized in the complement i.e. $K(I)' = K(I')$ in a self-explanatory notation. The answer to the same question for a double interval $I = I_1 \cup I_2$ of non-intersecting is more tricky but can be worked out in the same setting by a pedestrian calculation

$$\begin{aligned} & K((I_1 \cup I_2)') \subset K(I_1 \cup I_2)' \\ & \curvearrowright K(I_1 \cup I_2) \subset K((I_1 \cup I_2)')' \end{aligned} \tag{31}$$

The meaning of the left hand side is clear, these are functions which are constant in $I_1 \cup I_2$ with the same constant in the two intervals. A bit of thinking reveals that the symplectic complement on the right hand side consists of functions which are also constant there but now different constants are permitted. This statement translates via the functorial relation into a conversion of the Haag duality to an inclusion $\mathcal{A}(I_1 \cup I_2) \subset \mathcal{A}((I_1 \cup I_2)')'$. Physically the enlargement results from the fact that within the charge neutral vacuum algebra a charge split with one charge in $I_1$ and the compensating charge in $I_2$ for all values of the (unquantized) charge occurs. A more realistic picture is obtained if one allows a charge split to begin with, but one which is controlled by a lattice $f(I_2) - f(I_4) \in 2\pi L$ (where $f(I)$ denotes the constant value $f$ takes in that interval). Although imposing such a lattice structure destroys the linearity of the symplectic space underlying the Weyl algebra and hence the functorial relation between one-particle spaces and Weyl algebras, one can nevertheless define generalized Weyl generators which generate an operator algebra $\mathcal{A}_L(I_1 \cup I_2)$. It is easy to check that $\mathcal{A}_{L^*}(I_1 \cup I_2) \supset \mathcal{A}_L(I_1 \cup I_2)$ with $L^*$ being the dual lattice (which contains the original lattice) commutes with $\mathcal{A}_L((I_1 \cup I_2)')$, but in order to show $\mathcal{A}_L((I_1 \cup I_2)')' = \mathcal{A}_{L^*}(I_1 \cup I_2)$ one has to work a little bit harder since one cannot refer the algebraic to a one-particle spatial relation when lattices are involved. When we chose $L$ to be even integer we are back at the previous extension situation for which the charge structure is described by the finite group $G = L^*/L$. In fact using the conceptual framework of Vaughan Jones one can show that the two-interval inclusion is a Jones inclusion which is independent of the position of the disjoint intervals characterized by the group $G$; in particular the Jones index (a measure of the size of the bigger in terms of the smaller algebra is the Jones index

$$\begin{aligned} & ind\left\{\mathcal{A}_L(I_1 \cup I_2) \subset \mathcal{A}_L((I_1 \cup I_2)')'\right\} = |G| \\ & \mathcal{A}_L(I_1 \cup I_2) = inv_G \mathcal{A}_{L^*}(I_1 \cup I_2) \end{aligned} \tag{32}$$

There exists another form of this inclusion which is more suitable for generalizations. One starts from the quantized charge extended local algebra $\mathcal{A}_L^{ext} \supset \mathcal{A}$ described before in terms of an integer even lattice $L$ (which lives in the separable Hilbert space $H_{L^*}$) as our observable algebra. Again the Haag duality is violated and converted into an inclusion $\mathcal{A}_L^{ext}(I_1 \cup I_2) \subset \mathcal{A}_L^{ext}((I_1 \cup I_2)')'$ which reveals the same $L^*/L$ charge structure (it is in fact isomorphic to the previous inclusion). In the general setting (current algebras, minimal model algebras,...) this double interval inclusion is particularly interesting if the associated Jones index is finite. One finds [31]

**Theorem 2** *A chiral theory with finite Jones index* $\mu$

$$\mu = ind\,\{\mathcal{A}_{ext}((I_1 \cup I_2)) : \mathcal{A}(I_1 \cup I_2)\} \tag{33}$$

*for the double interval inclusion which is strongly additive and split is a rational (finite number of superselection sectors) theory and the statistical dimensions* $d_\rho$ *of its charge sectors are related to this Jones index through the formula*

$$\mu = \sum_\rho d_\rho^2 \tag{34}$$

Instead of going further into the zoology of models it may be more revealing to mention some of the algebraic methods by which they are constructed and explored. The already mentioned DHR theory provides the conceptual basis for converting the notion of positive energy representation sectors (equivalence classes of unitary representations) of the chiral model observable algebra $\mathcal{A}$ into endomorphisms $\rho$ of this algebra. This is an important step because contrary to group representations which have a natural (tensor product) composition structure, representations of operator algebras (beyond loop groups) do not come with a natural composition. The DHR theory of localized endomorphisms of $\mathcal{A}$ leads to fusion laws and an intrinsic notion of generalized statistics (for chiral theories: plektonic in addition to bosonic/fermionic). The chiral statistics parameter are complex numbers whose phase is related to a generalized concept of spin via a spin statistics theorem and whose absolute value (the inverse of the statistics dimension) generalizes the notion of multiplicities of fields known from the description of inner symmetries in higher dimensional standard QFTs. The different sectors may be united into one bigger algebra called the exchange algebra in the chiral context (the "reduced field bundle" of DHR) in which every sector occurs with multiplicity one and the statistics data are encoded into exchange (commutation) relations of charge-carrying operators ("exchange fields") [87][88]. Even though all properties concerning fusion and statistics are nicely encoded into this algebra, it lacks some cherished properties of standard field theory: there is no unique state–field relation i.e. no Reeh-Schlieder property; if a field whose source projection does not coalesce with the projection onto the vacuum sector hits the vacuum, it annihilates the latter. In operator algebraic terms, the local algebras are not factors. This poses the question of how to construct from the set of all sectors natural extensions (not necessarily local) with these desired properties. Despite numerous attempts using different concepts, no natural solution to this internal symmetry problem and an associated field algebra in analogy to the DR group symmetry [34] was found. Even the approach based on the concept of Quasi-Hopf quantum symmetries [89], which at least seems to be wide enough to cover all rational models, lacks intrinsicness and naturality[23] as a result of a not very attractive mixing of global with local aspects which causes non-localities in the relation of the gauge invariant observables to the charge-carrying quasi-Hopf objects. On the other hand it was found [96] that natural extensions can be characterized in operator algebraic terms by the existence of so called DHR triples $(\Theta, w, v)$ where the so-called dual canonical endomorphism $\Theta$ is an endomorphism

[23] All attempts are post factum i.e. none of them has been used in the construction of models. In fact they seem to be less useful than the reduced field bundle (the exchange algebra) which at least follows simple rules and does not create the mentioned problem.

of $\mathcal{A}$ which decomposes into the sector-associated irreducible DHR endomorphisms and $w, v$ two intertwining operators in $\mathcal{A}$ which fulfill specific interwining relations which assure the existence of a natural extension $\mathcal{A} \subset \mathcal{B}_{(\Theta,w,v)}$. But in general there is no extension which like the DHR field algebra combines all existing sectors into one object. In case of rational theories the number of such extensions is finite and in the aforementioned "classical" current algebra- and minimal- models they all have been constructed by this method [74][76], thus confirming and completing the previous incomplete less systematic constructions. The same method adapted to the chiral tensor product structure of d=1+1 conformal observables classifies and constructs all two-dimensional local (bosonic/fermionic) conformal QFT $\mathcal{B}_2$ which can be associated with the observable chiral input. It turns out that this approach leads to another of those pivotal numerical matrices which encode structural properties of QFT: the coupling matrix $Z$

$$\mathcal{A} \otimes \mathcal{A} \subset \mathcal{B}_2 \tag{35}$$
$$\sum_{\rho\sigma} Z_{\rho,\sigma}\rho(\mathcal{A}) \otimes \sigma(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{A}$$

where the second line is an inclusion solely expressed in terms of observable algebras from which the desired (isomorphic) inclusion in the first line follows by a canonical construction, the so-called Jones basic construction. The numerical matrix $Z$ is closely related to the so-called statistics character matrix and it has also a deep relation to the matrix S appearing in the SL(2,Z) modular character transformation (see also subsection 5). However unlike the construction of the DR field algebra, these extension methods generally do not lead to objects which incorporate all superselection sectors. Unlike the Luescher-Mack idea of aiming directly at globally causal fields on the covering spaces, the Longo-Rehren extension method is purely algebraic i.e. does not incorporate the global covering aspects in its present form.

The chiral extension problem is also closely related to the problem of amalgamating left and right chiral representations in order to arrive at local two-dimensional conformal algebras. Hence it is not surprising that also the construction of all two-dimensional models associated to c<0 chiral models has been successfully completed by these extension ideas [75].

Whether all the different construction ideas (coset and "orbifold" constructions starting from known models, extensions) are sufficient for a complete classification of chiral models is an open problem.

# 5 QFT in terms of modular positioning of "monade algebras"

QFT has been enriched by a the powerful new concept of modular localization which promises to revolutionize the task of (nonperturbative) classification and construction of models. It also provides an additional strong link between two-dimensional and higher dimensional QFT and admits a rich illustration for chiral theories. For a description of its history and aims, the reader is referred to [77][78][79]

It had been known for some time that under very general conditions (for wedge-localized algebras and interval localized algebras of chiral QFT no additional conditions need to be imposed) the localized operator algebras $\mathcal{A}(\mathcal{O})$ of AQFT are isomorphic to an algebra which belongs to a class which already appeared in the famous classification of factor algebras by Murray and von Neumann and whose special role was highlighted later in mathematical work by Connes and Haagerup. It is somewhat surprising that the full richness of QFT can be encoded into the relative position of a finite number of copies of this "monade"[24] within a common Hilbert space [80]. Chiral conformal field theory offers the simplest theoretical laboratory in which the emergence of the spacetime symmetry of the vacuum (the Moebius group) and the spacetime indexed (intervals on the compactified lightray) operator algebras can be analyzed by starting from 3 monades in a certain relative position of modular inclusion. A modular inclusion of two monades $(\mathcal{A}\subset\mathcal{B},\Omega)$ in a joint standard situation (common standard vector $\Omega$) has two modular groups. If the $\sigma_t^{\mathcal{B}}$ acts on the smaller algebra for $t<0$ as a one-sided compression $\sigma_t^{\mathcal{B}}(\mathcal{A})\subset\mathcal{A}$, the two unitaries $\Delta_{\mathcal{A},\mathcal{B}}^{it}$ modular groups generate a unitary representation of a positive energy spacetime translation-dilation group with the (Anosov) commutation relation

$$Dil(\lambda)U(a)Dil^{*}(\lambda)=U(\lambda a),\; Dil(e^{-2\pi t})=\Delta_{\mathcal{B}}^{it} \tag{36}$$

The geometrical picture which goes with this abstract modular inclusion is $\mathcal{B}=\mathcal{A}(I)\supset\mathcal{A}(I')=\mathcal{A}$ with the two intervals $I'\subset I$ having one endpoint in common so that the modular group of the bigger one ($\simeq Dil_I=$ Moebius transformation leaving $\partial I$ fixed) leaves this endpoint invariant and compresses $I'$ into itself by transforming the other endpoint of $\partial I'$ into $I'$. One can show that this half-sided modular inclusion ($\pm hsm$, $t\lessgtr 0$) actually forces the von Neumann algebras to be copies of the monade.

The simplest way to obtain the full Moebius group as a symmetry group of a vacuum representation is to require that the modular inclusion itself is standard[25] which means that in addition $\Omega$ is also standard with respect to the relative commutant $\mathcal{A}'\cap\mathcal{B}$.

**Theorem 3** *The observable algebras of chiral QFT are classified by standard hsm of two monades.*

The net of interval-indexed local observable algebras is obtained by applying the Moebius group to the original monade $\mathcal{A}$ or $\mathcal{B}$.

The reader may have wondered why we did not follow the classical analysis of conformal symmetry (based on transformations which leave the Minkowski metric invariant up to a spacetime dependent factor) which in d=1+1 leads to the infinite diffeomorphism group. Certainly all of the afore-mentioned models have energy-momentum tensors whose Fourier decomposition leads to the unitary implementation of $Diff(S^1)$. But there are also Moebius-invariant chiral models which do not originate from chiral decomposition of two-dimensional conformal theories but rather from holographic projections of higher dimensional QFT.

In passing it may be helpful to point out that most of the literature on chiral QFT is conceptually flawed on the meaning of two-dimensional conformal invariance and in particular about conformal invariance of chiral components. The textbook folklore claims that the spatial symmetry is described by a hypothetical group of "all analytic transformations $z\rightarrow f(z)$". This is incorrect since there is no such group (i.e. the functions which are analytic in a region of the complex plane do not form a Lie group or Lie algebra). The only group which describes the symmetry of the (compactified) plane is

[24] We borrow this terminology from the mathematician (co-inventer of calculus) and philosopher Gottfried Wilhelm Leibnitz; in addition to its intended philosophical content it has the advantage of being much shorter than the full mathematical terminology "hyperfinite type $III_1$ Murray-von Neumann factor". Instead of "a finite number of copies of the (abstract) monade", we will simply say "a finite number of monades"

[25] There are other equivalent algebraic assumptions about monades (hsm factorization, modular intersection) which are more convenient for higher dimensional generalizations of the monade generation of QFT.

the Moebius group. The correct group is $Diff(S^1) = \cup_{n=1}^{\infty} Diff_n(S^1)$ (of which the Moebius group $PSU(1,1) \equiv Diff_1(S^1)$ is the only vacuum-preserving subgroup). This is the group of *quasisymmetric* transformation, a subgroup of *quasiconformal* transformations which have a finite (increasing with n) distance from the Moebius group, where the distance is defined in a topology in terms of the Schwartz derivative [85]. Related to this conceptual flaw is the very unfortunate terminology of calling local chiral fields "holomorphic".The object behind the holomorphic properties of certain chiral correlation functions is the vacuum state; it is not a property of operators or algebras since it is immediatly lost in other states i.e. it would have been more sensible to consider "holomorphic" to be an attribute of the chiral vacuum. In particular there are no normalizable eigenstates of $Diff_n(S^1)$ for n>1; however, and this will be the main point of this section, there are *partially invariant modular states* which for fixed n have the same action as $Diff_n(S^1)$ but only on suitably chosen $Diff_n(S^!)$-invariant subalgebras.

The only known counterexamples of models which are Moebius invariant but lack the full $Diff(S^1)$ covariance can be excluded on the basis of two well motivated quantum physical properties: strong additivity and the split property [84]. So the question whether with these requirements the extension from vacuum preserving Moebius invariance to Diff(S) covariance is guarantied is a natural one. The fact that chiral structures do not only come from 2-dim. conformal QFT but also from holographic projections in higher dimensional massive QFT lends importance to this question. It is easy to see that if one assumes Diff(S) covariance then transformations $z \to z^{\kappa}, 0 < \kappa < 1$ (an angular down-scaling) have implementing isomorphisms between restricted algebras on whose localization region the transformation defines an invertible diffeomorphism between intervals (e.g. on open interval contained in S) since any such partial diffeomorphism may be completed to a global Diff(S) whose restricted automorphic action which leads to the isomorphism does not depend on how it was extended. Modular theory applied to the two algebras leads to a standard unitary implementation $U_I(\kappa)$ which transforms the modular invariant vacuum state $\Omega$ of the first algebra into a one-parameter family of standard vectors $\Omega_\kappa$ with respect to the image algebra $\mathcal{A}(I_k) = U_I(\kappa)A(I)U_I^*(\kappa)$. The last step consists in realizing that the modular group of $(\mathcal{A}(I_k), \Omega_k)$ is geometric and equal to the $\kappa$-transformed dilation group of the interval $I$. Hence the presence of an automorphic action of Diff(S) on the observable algebra results in a host of "partial geometric modular situations" which in contrast to the Moebius group only act geometrically if restricted to the relevant subalgebras. A particular physically attractive situation is obtained if the algebra has the split property[26]. In that case one can find a standard vector $\Phi$ on which the two-fold localized algebra $\mathcal{A}(I) \vee \mathcal{A}(J)$, with $I = (0, \frac{\pi}{2})$, $J = (\pi, \frac{3\pi}{2})$ being the two opposite quarter circles of the first and third quadrant, yields a partially geometric modular group which acts as $\mathrm{Dil}_2(e^{-2\pi t})$ with

$$z \to g_2(z) = \left( \frac{\alpha z^2 + \beta}{\gamma z^2 + \delta} \right)^{\frac{1}{2}} \tag{37}$$

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t \\ -\sinh 2\pi t & \cosh 2\pi t \end{pmatrix}$$

Clearly this $Dil_2$ transformation has 4 fixed points and leaves the doubly $I \cup J$ localized algebra invariant. It consists of $z \to z^2$ being followed by the Moebius dilation and the inverse of the first transformation

[26] A sufficient condition for the validity of the split property in chiral models is the finiteness of the particion function $tre^{-\beta L_0} < \infty$.

formally written as $z \to \sqrt{z}$. The split state on $\mathcal{A}(I) \vee \mathcal{A}(J) \simeq \mathcal{A}(I) \otimes \mathcal{A}(J)$ (this is the split isomorphism) is

$$\omega_\Phi(AB) = (\Phi, A\Phi)(\Phi, B\Phi),\ A \in \mathcal{A}(I), B \in \mathcal{A}(J) \tag{38}$$

According to the previous remarks the partial diffeomorphism $z \to z^2$ permits to re-write this state as $\omega(\hat{A})\omega(\hat{B})$ i.e. the product of vacuum expectation values of the images $\hat{A}, \hat{B} \in \mathcal{A}(0, \pi)$ which is left invariant under the $Dil(e^{-2\pi t})$ action and returns to the original form upon the inverse partial diffeomorphism. These remarks amount to the statement that the validity of the KMS condition (the criterion for a group to be the modular group of a state) with $Dil_2$ in the state in $\omega_\Phi$ is equivalent to the KMS condition with $Dil$ in $\omega \otimes \omega$, the last being true by the modular property of the Moebius dilation [81][82][83]. Again modular theory leads to a distinguished realization of the state $\omega_\Phi$ on $\mathcal{A}(I) \vee \mathcal{A}(J)$ by a vector in the vacuum Hilbert space. The split formalism introduces an unsymmetry between the representation of this algebra and that of its commutant which has the consequence that the modular action restricted to the commutant is not geometric (in order to obtain a geometric action one has to start with the commutant and go through the same steps). Note that the doubly localized algebras through their violation of Haag duality (which became replaced by an inclusion) were precisely those situations which revealed the charge content through their charge–anti-charge splitting in the previous subsection. Whereas the modular group of such situations can be made partially geometric by the choice of a suitable state, the modular conjugation cannot be geometric since it must carry the informations about the charge splitting.

The strong additivity property applied to the four quarter algebras which are fixed by the $Dil_2$ group is the equality $A = \vee_i A_i$ which permits to glue together the partial automorphisms to a $Dil_2$ automorphism of the global algebra. Automorphisms of global algebras are unitarily implementable since global algebras turn out to be type I operator algebras. Clearly by using Moebius subgroups in (37) with two fixed points in different positions one generates the diffeomorphism covariance $Diff_2(S)$ which is associated to the generators $L_{\pm 2}, L_0$. By generalizing the above construction to higher powers in $z$ and the corresponding inverse mappings one obtains partial modular vectors and partial isomorphisms which lead to partial geometric automorphisms (in the previous sense) associated with $Diff_n(S)$; in this way partial geometric modular theory generates $Diff(S^1)$. This gives for every local algebra $A(I)$ besides the vacuum another infinite set of partially geometric modular vectors which, different from the $Moeb$-invariant vacuum vector, change together with the change of the localization region. The adjective "partially geometric" refers to the fact that the modular group $\sigma_t^{\mathcal{A}(I),\omega_\Phi}$ restricted to $\mathcal{A}(I)$ acts like a diffeomorphism, but unlike the vacuum the modular state does not generate a globally geometric action. By the split property one can extend the construction to the more natural setting of n-fold localized states in $\mathcal{A}$.

The interesting question of whether assumptions about the existence of partially geometric modular states and groups can be rephrased in terms of a natural positioning of a finite number of monades in suitable joint modular states remains open.

This kind of problem has gained importance as a result of a recent discovery of Brunetti, Fredenhagen and Verch [29] (with important prior observations by Hollands and Wald) which permits to formulate Einstein's local covariance which underlies classical general relativity (physical equivalence of isometrically diffeomorphic manifolds) in the setting of curved spacetime QFT[27]. For the simplest case of free fields (Weyl algebras) BFV establish the validity of this new quantum local covariance requirement i.e. that the model which was local in the standard Minkowski sense also fulfills local covariance in the new sense. The quantum version of this new principle (which as mentioned contains the locality principle underlying standard Minkowski space QFT as a special case) adapted to the chiral setting amounts to the question whether Moebius covariant theories under reasonable local quantum physical assumptions are $Diff(S)$-covariant. As already mentioned this is of course the case in all models which possess a energy-stress tensors. Since the Moebius symmetry (in higher dimensions also the Poincaré- or conformal- symmetry) and the construction of Moebius invariant nets of local algebras can be fully encoded into the relative position of a finite number of monades, it would be very satisfying indeed to extend this algebraic setting to diffeomorphisms $Diff(S)$.

**It is hard to imagine how one can ever combine quantum theory and gravity without problematizing and understanding these still mysterious links between spacetime geometry, thermal properties and relative positioning of monades in a joint Hilbert space**. Chiral theory and d=1+1 conformal QFT offer certainly the simplest testing ground for these new ideas.

# 6 Euclidean rotational chiral theory and temperature duality

Euclidean theory associated with certain real time QFTs is a subject whose subtle and restrictive nature has been lost in many contemporary publications as a result of the "banalization" of the Wick rotation (for some pertinent critical remarks referred to in [86]). The mere presence of analyticity linking real with imaginary (Euclidean) time without establishing the subtle reflection positivity (which is necessary to derive the real time spacelike commutativity as well as the Hilbert space structure) is not of much physical use; one needs an operator algebraic understanding of the so-called Wick rotation.

The issue of understanding Euclideanization in chiral theories became particularly pressing after it was realized that Verlinde's observation on a deep connection between fusion rules and modular transformation properties of characters of irreducible representations of chiral observable algebras is best understood by making it part of a wider investigation involving angular parametrized thermal n-point correlation functions in the superselection sector $\rho_\alpha$

$$\begin{aligned}\langle\Phi(\varphi_1,..\varphi_n)\rangle_{\rho_\alpha,2\pi\beta_t} &:= tr_{H_{\rho_\alpha}} e^{-2\pi\beta_t\left(L_0^{\rho_\alpha}-\frac{c}{24}\right)}\pi_{\rho_\alpha}(\Phi(\varphi_1,..\varphi_n)) \\ \Phi(\varphi_1,..\varphi_n) &= \prod_{i=1}^{n}\Phi_i(\varphi_i) \\ \langle\Phi(\varphi_1,..\varphi_n)\rangle_{\rho_\alpha,2\pi\beta_t} &= \langle\Phi(\varphi_n+2\pi i\beta_t,\varphi_1,..\varphi_{n-1})\rangle_{\rho_\alpha,2\pi\beta_t}\end{aligned} \tag{39}$$

i.e. the Gibbs trace at inverse temperature $\beta = 2\pi\beta_t$ on observable fields in the representation $\pi_{\rho_\alpha}$. Gibbs states are special KMS states (states which fulfill the analytic property in the third line) whose zero point function is the partition functions. Such thermal states are (in contrast to the previously used ground states) *independent on the particular localization of charges* $loc\rho_\alpha$, they only depend on the equivalence

[27] It assures the "background independence" of the algebraic substrate and although this property by the very quantum nature of states does not permit to maintain it for individual states it does become transferred to the folium of a state [29].

class i.e. on the sector $[\rho_\alpha] \equiv \alpha$. These correlation functions[28] fulfill the following thermal duality relation

$$\langle \Phi(\varphi_1, ..\varphi_n)\rangle_{\alpha,2\pi\beta_t} = \left(\frac{i}{\beta_t}\right)^a \sum_\gamma S_{\alpha\gamma} \left\langle \Phi(\frac{i}{\beta_t}\varphi_1, ..\frac{i}{\beta_t}\varphi_n)\right\rangle_{\gamma,\frac{2\pi}{\beta_t}} \tag{40}$$
$$a = \sum_i dim\Phi_i$$

where the right hand side formally is a sum over thermal expectation at the inverse temperature $\frac{2\pi}{\beta_t}$ at the analytically continued pure imaginary values scaled with the factor $\frac{1}{\beta_t}$. The multiplicative scaling factor in front which depends on the scaling dimensions of the fields $\Phi_i$ is just the one which one would write if the transformation $\varphi \to \frac{i}{\beta_t}\varphi$ were a conformal transformation law. Before presenting a structural derivation of this relation which is based on a new Euclideanization using modular operator theory it is instructive to check this identity in the simple abelian current model of section3 which permits a calculation of the thermal correlation function. By a computation which is only slightly more involved than that in the appendix C of [44] one finds the following representation of the thermal Gibbs state two-point function in the sector $l$ of $\mathbb{Z}_{2N}$ (in the notation of (21))

$$\left\langle \Phi_{-\sqrt{2N}}(0)\Phi_{\sqrt{2N}}(\varphi)\right\rangle_{l,\tau} = \Theta_{2l,2N}(\sqrt{2N}\varphi,\tau,0)\times \tag{41}$$
$$\times \frac{1}{\eta(\tau)}\left[2isin\frac{1}{2}\varphi\prod_{\nu=1}^{\infty}\frac{(1-2e^{i2\pi\tau\nu}cos\varphi+e^{4\pi i\tau\nu})}{(1-e^{i2\pi\tau\nu})^2}\right]^{-\alpha^2}$$
$$tr_{H_l}e^{i\tau\left(L_0^\alpha-\frac{1}{24}\right)}e^{i\sqrt{2N}\varphi Q} = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}e^{i\pi\tau(n\sqrt{2N}+\frac{l}{\sqrt{2N}})^2+i\varphi\sqrt{2N}(n\sqrt{2N}+\frac{l}{\sqrt{2N}})} =$$
$$= \frac{1}{\eta(\tau)}\Theta_{2l,\sqrt{2N}}(\sqrt{2N}\varphi,\tau,0)$$

The first line contains the classical Jacobi theta-function and the $\eta(\tau)$ the Dedekind eta-function. Instead of the inverse temperature $2\pi\beta_t$ we have used the customary complex variable $\tau$ with $Im\tau = \beta_t$ in terms of which the modular $SL(2,\mathbb{Z})$ group covariance properties have the standard simple form[29]. The dependence on the $\mathbb{Z}_{2N}$ charge (the zero mode structure) is contained in the $\Theta$-function whereas the remainder is independent on the chosen $Z_{2N}$-model; in fact the Q-dependent part of the thermal two-point function can be separated which leads to the formula in the last two lines. The KMS property in terms of pointlike fields together with the circular nature of $\varphi$ yields the double periodicity in $\varphi$. All the expressions converge for $Im\tau > 0$ and the transformation properties under $SL(2,\mathbb{Z})$ whose generators are T: $\tau \to \tau + 1$ and S: $\tau \to -\frac{1}{\tau}$ follow from from those of $\Theta, \eta$ and the expression in the bracket. Under T transformations the bracket is invariant whereas $\eta$ and $\Theta$ are invariant up to a phase factor. Under S the $\Theta$ suffers a linear transformation

$$\sqrt{-i\tau}\Theta_{2l,2N}(\sqrt{2N}\varphi,\tau,0) = e^{2i\pi\frac{\sqrt{2N}\varphi}{\tau}}\sum_{p=1-N}^{N}\frac{e^{\frac{ipl}{N}}}{\sqrt{2N}}\Theta_{2p,2N}(\frac{\sqrt{2N}\varphi}{\tau},-\frac{1}{\tau},0) \tag{42}$$

from which one obtains the matrix S whereas the net effect of the multiplication factors including those from the S transformation of the bracket and the $\eta$ combine to the factor in (40). The simplicity of

[28] The conformal invariance actually allows a generalization to complex Gibbs parameters $\tau$ with $Im\tau = \beta$ which is however not neede in the context of the present discussion.

[29] In addition to higher-dimensional theories were locality and covariance permits to enlarge the KMS analytic complex strip to a larger tubular region involving the spatial coordinates [94], chiral theories even allow to complexify the value of the temperature $\beta$.

the model permits the calculation of general n-point functions with the result that only change in $\Theta$ consists in replacing the $\sqrt{2N}\varphi$ by $\sqrt{2N}\sum\pm\varphi_i$ where the sign depends on the sign of the charge and the number of + and - signs must be equal (charge neutrality). The calculation can be extended to the multi-current lattice models with the interesting possibility of encountering modular invariant functions in case of selfdual even lattices. Since the Gibbs states are not normalized, the character identities are actually the "zero-point function" part (i.e. $\Phi = 1$ with $a = 0$) of the above relation namely

$$\chi_\alpha(\tau) = \sum_\beta S_{\alpha\beta}\chi_\beta(-\frac{1}{\tau}) \tag{43}$$

involving a matrix $S$ already appeared in section 4.

Under certain technical assumption within the setting of vertex operators[30], Huang recently presented a structural proof [91] and it seems that his assumptions in that framework are equivalent to the standard rationality assumption (i.e. finite number of sectors in the operator-algebraic approach). As in the case of the above computational check, Huang's proof does not really reveal the deep local quantum physical principles which are behind the thermal duality relation.

The fact that the character relation is a special case of a relation which involves analytic continuation to imaginary rotational lightray coordinates suggests that one should look for a formulation in which the rotational Euclideanization has a well-defined operator-algebraic meaning. On the level of operators a positive imaginary rotation is related to the Moebius transformation $\tilde{\Delta}^{it}$ with the two fixed points $(-1,1)$ via the formula

$$e^{-2\pi\tau L_0} = \Delta^{\frac{1}{4}}\tilde{\Delta}^{i\tau}\Delta^{-\frac{1}{4}} = \tilde{\Delta}_c^{i\tau} \tag{44}$$

where $\Delta^{it}$ and $\tilde{\Delta}^{it}$ represents the $SL(2,R)$ Moebius subgroups with fixpoints $(0,\infty)$ resp. $(-1,1)$ and $\tilde{\Delta}_c^{i\tau}$ the $SU(1,1)$ subgroup with $z = (-i\frac{\pi}{2}, i\frac{\pi}{2})$ being fixed (the subscript $c$ denotes the compact picture description). which leaves $(0,\infty)$ fixed (the standard dilation). Note that $Ad\Delta^{\frac{1}{4}}$ acts the same way on $\tilde{\Delta}^{i\tau}$ as the Cayley transformation $AdT_c$, where the $T_c$ is the matrix which represents the fractional acting Cayley transformation

$$T_c = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \tag{45}$$

Ignoring for the moment domain problems for $\Delta^{\frac{1}{4}}$(to which we will return soon), the relation (44) gives an operator representation for the analytically continued rotation with positive imaginary part ($t > 0$) in terms of a Moebius transformation with real rapidity parameter. If we were to use this relation in the vacuum representation for products of pointlike covariant fields $\Phi$ where the spectrum of $L_0$ is nonnegative, we would with obtain with $\Phi(t) = e^{2\pi itL_0}\Phi(0)e^{-2\pi itL_0}$

$$\begin{aligned}\langle\Omega\left|\Phi_1(it_1)..\Phi_n(it_n)\right|\Omega\rangle^{ang} &= \langle\Omega\left|\Phi_1(t_1)_c..\Phi_n(t_n)_c\right|\Omega\rangle^{rap} \\ &= \omega_{2\pi}(\Phi_1(t_1)_c..\Phi_n(t_n)_c)^{rap}\end{aligned} \tag{46}$$

The left hand side contains the analytically continued rotational Wightman functions. As a result of positivity of $L_0$ in the vacuum representation this continuation is possible as long as the imaginary parts

[30] The Vertex framework is based on pointlike covariant objects, but unlike Wightman's formulation it is not operator-algebraic i.e. the star operation is not inexorably linked to the topology of the algebra as in $C^*$algebras of quantum mechanical origin. In addition it has no extension to higher-dimensional theories.

remain ordered i.e. $\infty > t_1 > ... > t_n > 0$. On the right hand side the fields are at their physical boundary values parametrized with the rapidities of the compact $\tilde{\Delta}_c^{it}$ Moebius subgroup of $SU(1,1)$. Note that this rapidity interpretation implies a restriction since the rapidities associated with $x = th\frac{t}{2}$ cover only the interval $(-1,1)$. The notation in the second line indicates that this is a KMS state at modular temperature $\beta_{mod} = 1$ ($\beta_{Hawking} = 2\pi\beta_{mod} = 2\pi$) in agreement with the well-known fact that the restriction of the global vacuum state to the interval (-1,1) becomes a state at fixed Hawking-Unruh temperature $2\pi$. Note that only the physical right hand side is a Wightman *distribution in terms of a standard* $i\varepsilon$ *boundary prescription*, whereas the left hand side is an *analytic function (i.e. without any boundary prescription).* **This significant conceptual (but numerical harmless) difference is responsible for the fact that in the process of angular Euclideanization of chiral models the KMS condition**[31] **passes to a periodicity property and vice versa**.

At this point it is helpful to look at an analogous situation within the setting of the of the Osterwalder-Schrader Euclideanization. It is well-known that this is the natural setting for the formulation of the Nelson-Symanzik duality. Its thermal version is analogous to our problem. It says that a two-dimensional QFT which obeys a KMS condition is Nelson-Symanzik dual (interchange of space with imaginary time) to a ground state theory in a periodic box. This of course appears almost a tautology if one in the setting of the Feynman-Kac representation so that the Osterwalder-Schrader Euclideanization leads to a bona fide classical statistical mechanics. In the case one starts with a periodic quantization box (or rather interval) the use of the Feynman-Kac presentation even suggests a stronger form the Nelson-Symanzik symmetry: the thermal box (interval) at length $L$ and KMS state (which is even Gibbs) at temperature $\beta$ is (generalized) Nelson-Symanzik dual to a system in which $L$ and $\beta$ are interchanged.

Hence the analogy with the generalized Nelson-Symanzik situation suggests to start from a rotational thermal representation in the chiral setting. For simplicity let us first assume that our chiral theory is one of those special selfdual lattice Weyl-like models in section 4 which have no other positive energy representation except the vacuum in which case the statistics character matrix is trivial i.e. $S = 1$ in the above matrix relation relation (40). Assume for the moment that the Gibbs temperature is the same as the period namely $\beta_{mod} = 1$. According what was said about the interchange of KMS with periodicity in the process of angular Euclideanization we expect the selfdual relation

$$\begin{aligned}
&\langle \Omega_1 | \Phi(it_1)...\Phi(it_n) | \Omega_1 \rangle^{rot} = (i)^{ndim\Phi} \left\langle \Omega_1^E \right| \Phi^E(t_1)...\Phi^E(t_n) \left| \Omega_1^E \right\rangle^{rot} \\
&\langle \Omega_1 | \Phi(t_1)...\Phi(t_n) | \Omega_1 \rangle^{rot} \equiv tr(\Omega_1, \Phi(t_1)...\Phi(t_n)\Omega_1),\ \Omega_1 \equiv e^{-\pi L_0} \\
&\Phi^E(t_1)^\dagger \equiv \tilde{J}\Phi^E(t_1)\tilde{J} = \Phi^E(-t_1)^*,\ \left[\tilde{J}, L_0\right] = 0
\end{aligned} \tag{47}$$

where the analyticity according to a general theorem about thermal states [92] limits the $t's$ to the unit interval and requires the ordering $1 > t_1 > ... > t_n > 0$. Thermal Gibbs states are conveniently written in the Hilbert space inner product notation with the help of the Hilbert-Schmidt operators $\Omega_1 \equiv e^{-\pi L_0}$, in which case the modular conjugation is the action of the Hermitian adjoint operators from the right on $\Omega_1$ [26]. Since the KMS and the periodicity match crosswise, the only property to be checked is the positivity of the right hand side i.e. that the correlations on the imaginary axis are distributions which fulfill the Wightman positivity. Here the label $E$ on $\Phi(t_1)$ denotes the Euclideanization in the sense of

[31] Contrary to popular believes KMS is not equivalent to periodicity in time but it leads to such a situation if the the involved operators commute inside the correlation function (e.g. spacelike separated observables).

the change of inner product and star operation as presented at the end of section 2 (4). For this we need the star conjugation associated with $\tilde{J}$ which interchanges the right with the left halfcircle which because of $L_0 = H + \tilde{J}H\tilde{J}$ commutes with $L_0$. In that case the modular group of $\Phi^E(t_1) = \Phi(it)$ is $e^{-2\pi t L_0}$ and the modular conjugation is the Ad action of $\tilde{J}$ which changes the sign of $t$ as in the third line (47). Whereas the modular conjugation in the original theory maps a vector $A\Omega_1$ into $\Omega_1 A^*$ with the star being the Hermitean conjugate, the Euclidean modular conjugation is $A^E\Omega_1^E \rightarrow \Omega_1^E \tilde{J} A^E \tilde{J} \equiv \Omega_1^E \left(A^E\right)^\dagger$. This property is at the root of the curious selfconjugacy (47). .

There are two changes to be taken into consideration if one passes to a more general situation. The extension to the case where one starts with a $\beta$ Gibbs state which corresponds in the Hilbert-Schmidt setting to $\Omega_\beta = e^{-\pi\beta L_0}$ needs a simple rescaling $t \rightarrow \frac{1}{\beta}t$ on the Euclidean side in order to maintain the crosswise correspondence between KMS and periodicity. Since the Euclidean KMS property has to match the unit periodicity on the left hand side, the Euclidean temperature must also be $\frac{1}{\beta}$ i.e. the more general temperature duality reads

$$\langle\Omega_\beta|\Phi(it_1)...\Phi(it_n)|\Omega_\beta\rangle^{rot} = \left(\frac{i}{\beta}\right)^{ndim\Phi} \left\langle \Omega^E_{\frac{1}{\beta}} \right| \Phi^E(t_1)...\Phi^E(t_n) \left| \Omega^E_{\frac{1}{\beta}} \right\rangle^{rot} \tag{48}$$

The positivity argument through change of the star-operation remains unaffected. This relation between expectation values of pointlike covariant fields should not be interpreted as an identity between operator algebras. As already hinted at the end of section 2 one only can expect a sharing of the analytic core of two different algebras whose different star-operations lead to different closure. In particular the above relation does not represent a symmetry in the usual sense.

The second generalization consists in passing to generic chiral models with more superselection sectors than just the vacuum sector. As usual the systems of interests will be rational i.e. the number of sectors is assumed to be finite. In that case the mere matching between KMS and periodicity does not suffice because all sectors are periodic as well as KMS and one does not know which sectors to match. A closer examination (at the operator level taking the Connes cocycle properties versus charge transportation around the circle into account) reveals that the statistics character matrix $S$ [93] enters as in (40) as a consequence of the well-known connection between the *invariant* content (in agreement with the sector $[\rho]$ dependence of rotational Gibbs states) of the circular charge transport and the statistics character matrix [88]. For those known rational models for which Kac-Peterson characters have been computed, this matrix $S$ turns out to be identical to the Verlinde matrix $S$ which diagonalizes the fusion rules [95] and which together with a diagonal phase matrix $T$ generates a unitary representation of the modular group $SL(2,Z)$[32]. Confronting the previous zero temperature situation of angular Euclidean situation with the asymptotic limit of the finite temperture identity, one obtains the Kac-Wakimoto relations as an identity between the temperature zero limit and the double limit of infinite temperature (the chaos state) and short distances on the Euclidean side.

This superselection aspect of angular Euclideanization together with the problem in what sense this modular group $SL(2,Z)$ can be called a new symmetry is closely related to a more profound algebraic

[32]Whereas relativistic causality already leads to an extension of the standard KMS $\beta$-strip analyticity domain to a $\beta$-tube domain [94], conformal invariance even permits a complex extension of the temperature parameter to $\tau$ with $Im\tau > 0$. For this reason the chiral theory in a thermal Gibbs state can be associated with a torus in the sense of a Riemann surface, but note that in *no physical sense* of localization this theory lives *on* a torus.

understanding of the relation between the analytic cores of the two algebras and requires a more thorough treatment which we hope to return to in a separate publication.

Modular operator theory is also expected to play an important role in bridging the still existing gap between the Cardy Euclidean boundary setting and those in the recent real time operator algebra formulation by Longo and Rehren [96].

# 7 "PFG", factorizing models and generators of wedge algebras

In contrast to conformal two-dimensional QFT the DHR where the DHR superselection structure is important for the classification and construction of models, the issue of statistics looses its physical relevance in massive two-dimensional models[33]. The main reason for this unexpected and somewhat peculiar aspect is the fact that statistics ceases to be an intrinsic attribute of two-dimensional particles (they are rather statistical "schizons" [100]) and the commutation relation between fields can be changed at will (e.g. from bosonic to fermionic or even plektonic) by passing to other fields in the same Fock space. It is easy to see that the Fock space of two-dimensional Bosons can be described in terms of fermionic or even "anyonic" particle creation/annihilation operators [100]. The same result holds for pointlike localized massive fields in x space. Since the argument for pointlike covariant x-space fields is less obvious it may be helpful to give a brief indication. Starting from a free massive Dirac field one constructs the (pseudo)potential (or field strength) of the conserved current (as in section 3) $j_\mu(x) = \varepsilon_{\mu\nu}\partial^\nu\phi$ which is bilinear in the fermion creation/annihilation operators. It is spacelike bosonic with respect to itself but has a spacelike step function commutation relation relativ to the Dirac field; hence by mutiplying its exponential with strength $\alpha$ (section 3) with the Dirac field and Wick-ordering one obtains a pointlike field $\psi^{(\alpha)}(x)$ and its conjugate which form a complex $\alpha$-anyonic field. In contradistiction to the massless case where this construction would have led to a charge-carrying field which transforms into different (orthogonal) charge sector, all $\alpha$-anyonic fields (including a bosonic field for a special choice of $\alpha$) act cyclically on the vacuum and generate the same Hilbert space. The manifestation of this observation within the DHR setting is well-known [102]: the (global) gauge-invariant subalgebra of field algebras with an internal symmetry do not fulfill Haag duality, and those observable algebras which do satisfy Haag duality cannot have a nontrivial charge superselection structure. Although it is possible to define the observable algebras in such a way that the different $\alpha$-values can be interpreted as different superselection sectors, this viewpoint is not natural and in particular it is not useful in the construction of massive models.

The $\psi^{(\alpha)}$-model illustrates another property which goes against some popular folklore concerning the connection of conformal algebras with their massive counterpart: there is no algebraic correspondence between operators in the massive theory with those in their massless scaling limit; rather the massive algebra contains many more operators which have no conformal counterpart i.e. they converge to zero in the massless limit done analogously as in section 3. To illustrate this "thinning out" phenomenon of the scaling limit, one only needs to look at the trajectory of the $\psi^{(\alpha)}$ as $\alpha$ increases. The short distance singularity increases and the anyonic commutation behavior is periodic with period 1 (in suitable

[33] This can be traced back to the absence of a compact (rotation-like transformation) i.e. the chiral rotation (as well as the chiral tensor factorization) is lost in the presence of a massive particle.

normalization convention for $\phi$); if $\psi^{(\alpha_{red})}$ denote the anyonic fields in the interval $0 \leq \alpha_{red} < 1$, the field for a generic $\alpha$-value $\alpha = \alpha_{red} + n$ differs from $\psi^{(\alpha_{red})}$ by a local operator (even degree in the free Fermions); in more sophisticated terminology: the massive anyonic Borchers class is generated by a reduced element and the even part of the free fermion Borchers class. The process does not lead to the expected massless $\frac{\alpha^2}{2}$ trajectory rather the mass power matched to the expected short-distance dimension leads to a a vanishing limit as a consequence of the appearance of cumulative mass singularities (the same phenomenon which is responsible for the restriction of the Sine-Gordon-Thirring model model equivalence outside a certain range of coupling strength).

The non-intrinsic nature of statistics in d=1+1 massive QFT is related to the new phenomenon of the appearance of order/disorder- and quantum soliton- fields [101][102]. However the important structural aspect of two-dimensional massive theories which has led to the construction of an impressive wealth of model has been scattering theory of massive particles and its connection with integrability (and not the structure of commutators between spacelike separated fields).

The origin of these developments can be traced back to two ideas which attracted a lot of attention during the 60s and 70s. On the one hand there was the quite old idea to bypass the "off-shell" field theoretic approach to particle physics (in particular strong interactions) in favor of a pure on-shell S-matrix setting which, as the result of the elimination of short distances via the mass-shell restriction would be free of ultraviolet divergencies. This idea was enriched in the 60s by the crossing property which in turn led to the bootstrap idea as a (highly nonlinear) selfconsistent method for the determination of the S-matrix. The protagonists of this S-matrix bootstrap program placed themselves in a totally antagonistic position with respect to QFT so that the strong return of QFT in the form of gauge theory undermined the credibility of the bootstrap approach. On the other hand there were rather convincing quasiclassical calculations on certain two-dimensional QFTs as the Sine-Gordon model suggesting that they were quantum integrable systems and that in particular their quasiclassical mass spectrum was exact [103]. These provocative observations asked for a structural explanation beyond quasiclassical approximations and it became soon clear that the natural setting was that of the fusion of boundstate poles of unitary crossing symmetric purely elastic S-matrices; first in the special context of the Sine-Gordon model [104] and afterwards as a general classification and computation program from which factorizing S-matrices can be determined by solving well-defined equations for the elastic 2-particle S-matrix [105]. This line of research led finally to a general program of a bootstrap-formfactor construction of so-called d=1+1 factorizable models [106][107][108]. This formfactor program uses the very ambitious original S-matrix bootstrap idea in the limited context of a d=1+1 S-matrix Ansatz in which $S$ factorizes into 2-particle elastic components $S^{(2)}$. A consequence of this simplification is that the classification and calculation of factorizing S-matrices can be separated from the problem of the construction of the associated off-shell QFT. Hence the S-matrix bootstrap becomes the first step in a bootstrap-formfactor program, followed by a second step which consists in calculating generalized formfactors of fields and operators[34] beyond that of the identity operator (whose formfacors correspond to the S-matrix entries). Of course such a two-step approach is limited to factorizable models; for more general models the construction of the S-matrix cannot be separated from the general formfactor construction. Following an idea of Swieca,

[34]The formfactors of an operator are defined as its matrix elements between multi-particle bra out- and ket in- vectors.

the irrelevance of statistics is expected to manifest itself already on the level of the S-matrix in form of a factorization into a dynamical and a (rapidity-independent) "statistical" part. This would permit to reduce a formfactor-bootstrap program with exotic commutation relation to one with Boson/Fermion fields.

This nonperturbative bootstrap-formfactor approach for factorizing models produced a steady stream of new models and it continues to be an important innovative area of research. Our interest in exploring this approach lies in the potential messages it contains with respect to a mass-shell based field theoretic constructions without the "classical crutches" and ultraviolet problems which characterize the Lagrangian quantization setting. In important step in this direction would be an intrinsic and systematic understanding of this subclass of models within the conceptual setting of QFT.

Limiting our interest to QFT with a mass gap we automatically secure the validity of the powerful time-dependent (LSZ) scattering theory. Let us in addition make the standard assumption that the Fock space of asymptotic multi-particle states is equal to the total Hilbert space (asymptotic completeness). To keep the notation simple, we imagine that we are dealing with an interacting theory of just one kind of particle. Let $G$ be a (generally unbounded) operator affiliated with the local algebra $\mathcal{A}(\mathcal{O})$. We call such $G$ a vacuum-**p**olarization-**f**ree **g**enerator (PFG) affiliated with $\mathcal{A}(\mathcal{O})$ (denoted as $G\eta\mathcal{A}(\mathcal{O})$) if the state vector $G\Omega$ (with $\Omega$ the vacuum) is a one-particle state without any vacuum polarization admixture [79]. By definition such PFGs are (unbounded) on-shell operators and it is well-known that the existence of a *subwedge-localized* PFG forces the theory to be interaction-free, i.e. the local algebras possess free field generators. However, and this is the surprising fact, this link between localized PFG and absence of interactions breaks down if one admits wedge regions. In that case modular theory guaranties the existence of wedge generators without vacuum polarization; but only if these PFG are "tempered" (well-defined on a translation invariant domain) [97] they have useful properties in the setting of time-dependent scattering theory. The restriction implied by this additional requirement can be shown to only permit theories with a purely elastic S-matrix; it has been known for a long time that this is only possible in d=1+1 where such theories have been investigated since the late 70s within the so-called S-matrix bootstrap program [99]. In fact one can show that the elastic two-dimensional S-matrices coming from local QFT are necessarily described by two-particle S-matrices; all the higher elastic contributions factorize into two-particle contributions and the latter are classified by solving equations for functions in the rapidity variable which incorporate unitarity, analyticity and crossing [105].

The second surprise is that the Fourier transforms of the wedge algebra-generating tempered PFGs are identical to operators introduced at the end of the 70s by the Zamolodchikovs (their algebraic properties were spelled out in more detail by Faddeev). Although the usefulness of this new algebraic structure in the bootstrap formfactor program was immediately recognized, its conceptual position within QFT was not clear since despite the similarity of these objects to free field creation/annihilation operators the Z-F operators are distinctively different from those of the incoming or outgoing free fields of scattering theory. In the simplest case the Z-F algebra relation are of the form ($\theta$= momentum space rapidity)

$$
\begin{aligned}
Z(\theta)Z^*(\theta') &= S^{(2)}(\theta-\theta')Z^*(\theta')Z(\theta)+\delta(\theta-\theta') \\
Z(\theta)Z(\theta') &= S^{(2)}(\theta'-\theta)Z(\theta')Z(\theta) \\
\phi(x) &= \frac{1}{\sqrt{2\pi}}\int (e^{ip(\theta)x(\chi)}Z(\theta)+h.c.)d\theta
\end{aligned} \tag{49}
$$

where the notation for the structure functions $S^{(2)}$ already preempts their physical interpretation as the two-particle S-matrix (which can be derived [98]). The last line defines a covariant field which, although not being pointlike local, turns out to be wedge-like localized[35] i.e. it commutes with its "modular opposite" $J\phi(x)J$ where $J = S_{scat}J_0$, and $S_{scat}$ is the scattering operator in Fock space and $J_0$ the TCP operator of the free incoming particles. This interpretation of the Z-F algebra operators in terms of localization concepts turns out to be a valuable guide for the construction of tighter localized algebras $\mathcal{A}(D)$ associated with double cone regions by computing intersections of wedge algebras whose generating operators turn out to be infinite series in the $Z's$ with coefficient functions which are generalized formfactors. The problem of demonstrating the existence of a nontricial QFT associated with the algebraic structure (49) of the wedge algebra generators is then encoded into a nontriviality statement ($\mathcal{A}(D) \neq C\mathbf{1}$) for the double cone intersections; the fact that the Z-F algebra is different from that of free field creation/annihilation operators has the consequence that the operators in the intersection have infinitly many vacuum polarization components (connected formfactors) involving all particle numbers. **In this way the problem of existence of nontrivial QFTs becomes disconnected from the inexorable short-distance problems of the standard approach** [79].

Factorizing models are presently the testing ground for new ideas on the age-old unsolved problem of existence of interacting QFT [98] i.e. on whether the principles of QFT and the concepts used to implement them continue to be mathematically consistent in the presence of interactions. It is remarkable that this construction program leads to a derivation of those recipes (crossing for formfactors kinematical pole relation,...) which were used in a more or less ad hoc fashion [108] in the standard formulations of the bootstrap-formfactor program from first principles.

It is profitable to pause for a moment and ask the question in what sense these findings are related to the Coleman-Mandula theorem which states (on the basis of analytic properties of the scattering matrix) that apart from the case of graded symmetries which needs a special consideration, a non-trivial entanglement (trivial = tensor product) of spacetime with inner symmetry for theories with a mass gap is only possible in d=1+1. The main theorem in [97] further qualifies this statement by linking it to the existence of tempered PFGs which are a generalization of free fields. Free fields and tempered PFGs have in common that they possess in addition to Poincaré invariance infinitely many additional other conserved charges which are not really inner. But to cite massive free field as examples for a nontrivial entanglement of spacetime symmetry with something else is a bit awkward from a physical viewpoint and this extends to the factorizing model. It would be physically more appropriate to say that the two-dimensional exception allowed by the Coleman-Mandula is related to special vacuum polarization properties expressed by the existence of tempered wedge-localized PFGs. The vacuum polarization properties are fundamental on the

[35] The $x$ continues to comply with the covariant transformation law but it is not a point of localization i.e. the smearing with wedge supported test functions $\phi(f)$ does not lead to an improvement in localization if one reduces the support of $f$.

quantum level whereas the presentation in terms of integrability in the sense of infinitely many conserved charges is a formulation which requires classical hindsight.

The recognition that the knowledge of the position of a wedge-localized subalgebra $\mathcal{A}(W)$ with $\mathcal{A}(W)' = \mathcal{A}(W')$ within the full Fock space algebra $B(H)$ together with the action of the of the Poincaré group in $B(H)$ on the $\mathcal{A}(W))$ determines the full net of algebras $\mathcal{A}(\mathcal{O})$ via intersections

$$\mathcal{A}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) \tag{50}$$

is actually independent of spacetime dimensions and factorizability. But only in d=1+1 within the setting of factorizable models one finds simple generators for $\mathcal{A}(W)$ which permit the computation of intersections. Outside of factorizing models, wedge generators cannot be expected to have such a simple relation to in/out fields and one may have to take recourse to a perturbative approach, starting with the free fields as wedge generators and taking for $S_{scat}$ (which fixes the position of the commutant) a lowest order tree graph expression and afterwards recursively computing corrections to the wedge generators and $S_{scat}$ in the spirit of the Epstein-Glaser iteration using the fact that $S_{scat}$ enters the definition of the commutant. Since one is not aiming at a perturbation theory of pointlike localized fields but rather of wedge-localized generators, there should be no short distance problem (inasmuch as there is non in the Z-F wedge generators); in this way one may have the chance to see the **true intrinsic frontiers of QFT** according to its own physical principles beyond those created by the use of short distance singular pointlike localized fields already at the beginning of the computation (which leads to the renormalizable/nonrenormalizable dichotomy whose fundamental significance is unknown.

# 8 Wedge localized algebras and holographic lightfront projection

With the particle picture outside factorizing theories being made less useful by de-localization through interaction induced vacuum polarization, it is encouraging to note the existence of another constructive idea also based on modular inclusion and intersections which does not require the very restrictive presence of wedge-localized PFGs.. This is the holographic projection to the lightfront. In d=1+1 it maps a massive (non-conformal) QFT to a chiral theory on the lightfront (lightray) $x_- = 0$ in such a way that the global ambient algebra on Minkowski spacetime $\mathcal{A}(M) = B(H)$ and its global holographic lightfront projection $\mathcal{A}(LF) = B(H)$ coalesce[36], but the local substructure (the spacetime-indexed net) is radically different; the only algebra (besides the global) which is shared between the lightfron spacetime indexing and the ambient spacetime indexing is the wedge-localized algebra which is identical to the algebra of their upper lightfront boundary $\mathcal{A}(W) = \mathcal{A}(LF(W))$.

Using concepts of modular theory (modular inclusions and modular intersections of wedge algebras) one can construct the local structure [79] (i.e. the local algebraic net) and identify the subgroup $G(LF)$ of the Poincaré group which is the symmetry group of the holographic lightfront projection. Whereas

[36]Lightfront holography also works for higher-dimensional conformal theories, the d=1+1 conformal models are the only exception (already their classical version requires the knowledge of both upper and lower boundary characteristic data to fix the wave function inside the wedge).

some of the ambient Poincaré symmetries are evidently lost (in d=1+1 the translation leading away from the lightray), the holographic projection is also symmetry-enhancing in the sense that the rotational symmetry of the Moebius group associated with the compactified lightray (and according to subsection 4 also the infinite dimensional Diff($S^1$) group) becomes geometric. These symmetries are already present in the ambient theory, but they are not noticed because they acts there in a nonclassical fuzzy manner and hence escapes the standard quantization approach [79].

Whereas the holographic lightfront projection exists in every spacetime dimension, the setting of d=1+1 factorizing models presents a nice theoretical laboratory to study the intricate exact relation between massive models and their chiral projection in the context of mathematically controllable surrounding. Those chiral observables, which appear as the holographic projection of factorizable massive models, have the property of admitting generators with simple Z-F algebraic creation/annihilation properties and a covariant transformation property under the full two-dimensional Poincaré group. It is clear that a chiral theory specified in terms of such P-covariant operators leads (in analogy to free fields) a unique natural *holographic inversion* (but without guaranty of its existence) from a chiral theory to a massive two-dimensional ambient theory. But not having access to this additional knowledge, the relation of ambient theories to their holographic projection is not expected to be one-to-one. As in the case of the canonical equal time formalism, one rather expects that the specification of a kind of "Hamiltonian" propagating *in the* $x_-$ *direction* is necessary for a unique holographic inversion.

The holographic relation between chiral models and factorizing theories is different from Zamolodchikov's perturbative identification and classification of factorizing theories starting from a perturbation of chiral models changes. In the latter case the representation space of the zero mass limit deviates from that of the ones in which the different members of the original massive universality class live. The specification of the chiral perturbation in conjunction with the restriction to the factorizable members of the universality class may make them singletons, but they nevertheless are different theories living in different Hilbert spaces. Zamolodchikov's successful approach is not based on any one-to one correspondence between massless fields and their would be massive counterpart i.e. there is no "mass dressing operation"[37]. What matters is that those fields which are lost in the chiral limit are composites of those massive fields which persist in that limit. In holography on the other hand there is no loss of algebraic structure, only a radical change of spacetime indexing of the algebraic substrate between the original ambient theory and its lightfront holographic projection. An intuitive useful analogy is that to stem cells which can be grown into different organs; the abstract algebraic substrate (e.g. the abstract Weyl algebra) can be converted into different spacetime-indexed algebraic nets. It is interesting to note that this picture is precisely the idea which underlies the recently discovered local covariance principle for QFT in curved spacetime [29].

It already was alluded to that the entire issue of statistics of particles looses its physical relevance for 2-dim. massive models; they can be changed without affecting the physical content [100]. Instead such notions as order/disorder fields and solitons take their place. In such a world it would be possible to rewrite a Mendeleev periodic table of elements in terms of Bosons. As the mass approaches zero ($\sim$ short distance limit) "confined" charges become liberated and the situation changes to the one in conformal theory where the chiral (possibly anomalous) spins (= dimensions) are uniquely related to the statistic

[37] The number of massless fields in their local equivalence class is smaller since some massive fields vanish in the limiting theory as can be shown in examples.

(commutation relations) by the conformal spin-statistics theorem.

Since the classification of Z-F algebras is a structurally simpler (possibly computationally more complicated) program than that of chiral observable algebras, it may very well turn out that method of holographic lightray projection of factorizable theories may be useful for a more intrinsic constructive approach to chiral observable algebras.

There are many additional observations on factorizing models which, although potentially important for more profound understanding of QFT (e.g. renormalization group flows, the meaning of the c-parameter in energy-momentum commutation relations outside of the chiral setting, the thermodynamic Bethe Ansatz) which have not yet reached their final conceptual placement which identifies them as special two-dimensional manifestations of general concepts of QFT.

# 9 Concluding remarks

In order to present two-dimensional models as a theoretical testing ground for the still unfinished project of QFT (which was initiated more than three quarters of a century ago by Pascual Jordan's "Quantelung der Wellenfelder" [109], and in particular to illustrate his later plea for a formulation without "classical crutches"), we have used the three oldest models proposed by Jordan, Lenz-Ising and Schwinger as paradigmatic role models. The conceptual messages they reveal allow to analyze and structure the vast contemporary literature on low dimensional QFT and expose the achievements as well as the unsolved problems in a comprehensible manner without compromising their depth and complexity (which even their protagonists were not aware of).

The new way of viewing QFT with the help of modern developments in algebraic QFT tested in the setting of 2-dim. QFT becomes most apparent if one looks at the changes in the way one treats the problem of proving the existence of models of QFT. The old measure theoretical approach ($P\Phi_2$) required intermediate regularizations and was limited by the requirement of short distance behavior to low-dimensional models. It shares with the new approach the presence of some free field or free field-like reference structure from which the construction starts (e.g. the tensor product reduction of free massless Dirac fields for level k current algebras, the Zamolodchikov-Faddeev algebra creation/annihilation operators for factorizing models0. But in contrast to the old constructive QFT the free reference structure becomes modified by a sequence of very nontrivial steps (tensor product reduction, coset- and orbifold construction, extensions and intersections of operator algebras) which finally lead to very nontrivial objects whose short distance behaviour is far removed from that of the starting free fields, so that the original auxiliary Fock space structure is of not much use and becomes replaced by a reduced Hilbert space description which is more intrinsic to the resulting algebraic structure. In this process of construction one learns more about the physical content of models than in the old approach to constructive QFT which was limited to near canonical dimensions. Different from this old constructive approach [19], there is never any need to go outside the principles of local QFT in intermediate steps (as regularization of short distance singularities, recovering Poincaré invariance only in the infrared limit on functional measures); to the contrary, the modular-based construction depends entirely on maintaining sharp covariant localization properties in every step of the computation. As a result the old plagues of short distance divergencies and their control are gone and instead one has to face the new problem of deciding whether certain intersections of operator algebras are nontrivial ($\neq C\mathbf{1}$). Even the use of singular pointlike fields in the bootstrap-formfactor program does not cause any short distance problem as long as one only works with formfactors and avoids correlation functions. An intrinsic indication of the distance to free theories (i.e. the presence of interactions) is the interaction-caused vacuum polarization[38] which entails the absence of subwedge-localized polarization-free one-particle states in the setting of massive factorizing models. The very existence of models whose local algebras only admit generating fields whose short distance singularities are worse than those of the superrenormalizable $P\Phi_2$ models shows that the ultraviolet problems of QFT are not intrinsic but were forced upon quantum field theorists because they entered QFT via (Lagrangian) quantization and had to do their calculations with rather singular pointlike covariant fields and their correlation functions[39]. The new approach to QFT should aim at a "modular perturbation theory" for generators of wedge-localized algebras and come to tighter localizations by the process of intersections, with the pointlike generators being a convenient coordinatization for the computed net but not to be used via their correlation functions in the actual calculations. Such an approach should reproduce the results of standard approach in case of renormalizable interactions (in the standard sense), and, what is more important, by problematizing old recipes it should tell us something about the true frontiers of perturbative iterations which are set by the principles of localizable QFT. In this context it is interesting to observe that already the encoding of (m,s) Wigner representation into modular semiinfinite string-localized fields (still singular objects) already improves the short-distance behaviour by transferring part of the singularity to the fluctuating spacelike direction (a localization point in a one-dimension lower de Sitter spacetime) [110].

It is expected that two-dimensional QFTs will continue to play a crucial role in the future development of these new aspects of QFT either directly (e.g. holographic projections$\rightarrow$ generalized chiral theories) or more indirectly as a testing ground for new concepts.

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