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Real Structures in Clifford Algebras and Majorana Conditions in Any Space-time

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Abstract

The Weyl representation is used to analyze Clifford algebras and Majorana conditions in any spacetime. An index labeling inequivalent Γ-structures up to orthogonal conjugations is introduced.

Inequivalent charge-operators in even-dimensions, invariant under Wick rotations, are considered. The hermiticity condition on free-spinors lagrangians is presented. The constraints put by the Majorana condition on the free-spinors dynamics are analyzed. Tables specifying which spacetimes admit lagrangians with non-vanishing kinetic, massive or pseudomassive terms (for both charge-operators in even dimensions) are given. The admissible free lagrangians for free Majorana-Weyl spinors are fully classified.

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1 Introduction.

The theory of Clifford algebras is an old subject which has been extensively investigated both in the mathematical and in the physicists' literature. For obvious reasons physicists mainly dealt with the theory of spinors in Minkowskian or Euclidean spacetimes [1, 2]. Nevertheless in some papers [3–6] spinors in pseudoeuclidean spacetimes with arbitrary signature (t, s), $D = t + s$ being the dimensionality of the spacetime, have been analyzed. In particular [4] can be regarded as the reference work on the subject since it presents a rather complete list of results in this topic.

The development of supergravities and superstring theories which emphasize the Kaluza-Klein aspect of compactification to lower dimensional space led investigating properties of spinors (and supersymmetries) in arbitrary dimensional spaces. However, apart some special papers as the ones previously recalled, the great majority of works were still devoted to standard-signature spacetimes. The question of providing a physical interpretation for the extra-times somehow masked the fact that from a strictly mathematical point of view consistent superstring theories can be formulated in exotic signatures (like e.g. $5 + 5$). This negative attitude towards exotic signatures seems at present time changing and their possible physical implications find increasing attentions (see e.g. [7]). Various reasons are at the basis of this shift of attitude. Some recent works [8] for instance pointed out the existence of dualities relating theories formulated in different signatures. On the other hand the still-mysterious M-theory suggests that we need investigating along all possible directions. A rather formal argument can also be invoked, a reasonable demand for any possible theory which could claim to be a genuine "theory of everything" is that the signature of the spacetime should be determined by the properties of the spacetime itself rather than imposed a priori. The Minkowskian signature should therefore be selected after confrontation with the other signatures.

Motivated by the above considerations in this paper we analyze the real Clifford structures and Majorana conditions in any signature spacetime with arbitrary dimensions. The analysis here presented is based on the Weyl realizations of Clifford algebras. The technique employed allows to recover the results of [4] in a considerably simplified manner. Besides that, extra-informations, not presented in [4], are gained. As an example discussed in the text we mention the correct choice of the charge operator which preserves the Majorana condition under a Wick rotation to the Euclidean.

A list of further topics here discussed, some of them we are not aware to be found elsewhere, is the following. An index is introduced to label and discriminate among classes of inequivalent Clifford algebras up to orthogonal conjugations. Such index could in principle be relevant to physical applications whenever some kind of reality conditions are imposed on the fields.

Moreover the compatibility of the Majorana condition with the free massive equations of motion is thoroughly investigated. The conditions upon free lagrangians for Majorana spinors in order to be non-vanishing, hermitian and charge-conjugated are presented.

Explicit and easy-to-consult tables of spacetimes supporting massive (or pseudomassive in the even-dimensional case) Majorana spinors are provided. The list of results here presented is more complete than the one given in reference [6].

The scheme the present work is as follows: the next section is devoted to notations and preliminary results. In section 3 the Weyl representation is introduced. Even-dimensional inequivalent charge operators, invariant under Wick rotations, are constructed. The index labeling Γ-structures up to orthogonal conjugation is discussed in section 4. The hermiticity condition on free-spinors lagrangians is discussed in section 5. Section 6 presents an exhaustive list of the constraints put by the Majorana condition on the free-spinors dynamics, both at the level of the equations of motion and of the action. Tables specifying which spacetimes admit lagrangians with non-vanishing kinetic, massive or pseudomassive terms (for both charge-operators in even dimensions) are given. Finally in section 7 the problem of determining the admissible free lagrangians (kind of terms, non-vanishing conditions, type of coefficients) for Majorana-Weyl spinors in even-dimensional spacetimes is fully solved.

2 Notations and preliminary results.

Let $\eta^{\mu\nu}$ be the (pseudo)-euclidean metric associated to an $M^{t,s}$ generalized Minkowski space-time with t time-directions and s space-directions. The space-time dimension being $D = t + s$. In the following we will denote as time (space) directions those which are related to the $+$ (and respectively $-$) sign in $\eta^{\mu\nu}$.

A Γ-structure associated to the $M^{t,s}$ spacetime is a matrix-representation of the Clifford algebra generators Γ^μ ($\mu = 1, ..., D$) satisfying the anticommutation relations

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu \quad = \quad 2\eta^{\mu\nu}\cdot \mathbf{1}_\Gamma \tag{1}$$

The representation is realized by $2^{[\frac{D}{2}]}\times 2^{[\frac{D}{2}]}$ matrices ($[\frac{D}{2}]$ denoting the integral part of $\frac{D}{2}$) which can be further assumed to satisfy the unitarity requirement

$$\Gamma^{\mu\dagger} \quad = \quad \Gamma^{\mu -1} \tag{2}$$

A tracelessness condition holds

$$tr\Gamma^\mu \quad = \quad 0 \tag{3}$$

for any μ.

According to the fundamental Pauli theorem [9] the above matrix-representation is uniquely realized up to unitary conjugation.

Notice that choosing the $(+)$ sign in the r.h.s. of (1) is a matter of convention. The opposite choice (i.e. r.h.s. $\equiv -2\eta^{\mu\nu}$) is admissible. However it is just sufficient to look at results and tables obtained for an (s,t)-signature within the $+$ convention since the results for the $-$ convention-case are immediately recovered by interchanging t and s: $t \leftrightarrow s$. It should be therefore clear that the complete set of solutions for (s,t) spacetimes are recovered from the tables produced below only after such (t,s)-"dualization" has been taken into account.

The introduction of lagrangians and charge conjugations for the spinor fields require the presence of three (only two of them mutually independent) unitary matrices, denoted in the literature as A, B, C, associated to each one of the three conjugations (hermitian, complex-conjugation and transposition respectively) acting on the Γ^μ-matrices, according to

$$\begin{aligned} A\Gamma^\mu A^\dagger &= (-1)^{t+1}\Gamma^{\mu\dagger} \\ B\Gamma^\mu B^\dagger &= \eta\Gamma^{\mu}* \\ C\Gamma^\mu C^\dagger &= \eta(-1)^{t+1}\Gamma^{\mu T} \end{aligned} \tag{4}$$

As discussed later η, as well as ε introduced below, is a sign (± 1) specifying the assignment of a $\{\Gamma^\mu, A, B, C\}$ structure up to unitary transformations. η and ε will be explicitly computed in the next section. The introduction of η as defined in (4) corresponds to the standard convention in the literature.

The equation relating A, B and C can be expressed through

$$C = B^T A \tag{5}$$

with the transposed matrix B^T satisfying

$$B^T = \varepsilon B \tag{6}$$

An useful form of restating the above equation is

$$B^* B = \varepsilon \cdot \mathbf{1} \tag{7}$$

The A-matrix can be expressed through the position

$$A = \prod_{i=1,\ldots,t} \Gamma^i \tag{8}$$

where the product (the order is irrelevant since A, B, C can always be determined up to an arbitrary phase) is restricted to time-like Γ-matrices, i.e. those satisfying the ${\Gamma^i}^2 = +\mathbf{1}$ equation (conversely the spacelike Γ-matrices are those belonging to the complementary set satisfying ${\Gamma^j}^2 = -\mathbf{1}$).

The A-matrix allows constructing in generic flat spacetimes $M^{t,s}$ the conjugated $\overline{\psi}$ spinor as $\overline{\psi} = \psi^\dagger A$, and generalizes the Γ^0-matrix of the standard Minkowskian spacetime.

The matrix C corresponds to the charge-conjugation matrix, while B is employed in introducing the charge-conjugated spinors ψ^c according to

$$\psi^c = B^\dagger \psi^* \tag{9}$$

Quantum mechanical states are rays in a Hilbert space. A physical spinorial state can be equally well described by a spinor transformed via an unitary matrix U, $\psi \mapsto U\psi$.

It is easily proven that under such a unitary transformation Γ^μ, A, B, C are mapped as follows

$$\begin{aligned} \Gamma^\mu &\mapsto U\Gamma^\mu U^\dagger \\ A &\mapsto UAU^\dagger \\ B &\mapsto U^* B U^\dagger \\ C &\mapsto U^* C U^\dagger \end{aligned} \tag{10}$$

Notice that the unitary transformations acting upon B, C *do not* coincide with their unitary conjugations.

If we introduce the notion of a $\{\Gamma^\mu, A, B, C\}$-structure assignment associated to a given spacetime $M^{t,s}$ and we look for inequivalent classes of such assignments under the (10) transformations, we easily realize that the η, ε signs introduced above label inequivalent classes of assignments. Indeed η, ε can be equivalently introduced in a unitary-invariant trace form as

$$\begin{aligned} tr(B^* B) &= \varepsilon \cdot tr\mathbf{1} \\ tr(B\Gamma^\mu B^\dagger {\Gamma_\mu}^*) &= \eta D \cdot tr\mathbf{1} \end{aligned} \tag{11}$$

where the convention on the repeated indices is understood.

With a slight abuse of language we can say that η, ε label inequivalent choices of charge-conjugations.

In an even-dimensional spacetime ($D = 2n$) we can introduce a timelike generalized Γ^5 matrix (i.e. the matrix generalizing the one associated to the ordinary Minkowskian spacetime), through the position

$$\Gamma^5 = (-1)^{\frac{s-t}{4}} \prod_{\mu=1,\ldots,D} \Gamma^\mu \tag{12}$$

The sign is chosen in order to guarantee ${\Gamma^5}^2 = \mathbf{1}$.

Let us conclude this section by presenting some further useful identities

$$\begin{aligned} A^\dagger &= (-1)^{\frac{t}{2}(t-1)} A \\ A^* &= \eta^t B A B^\dagger \\ A^T &= \eta^t (-1)^{\frac{t}{2}(t-1)} C A C^\dagger \end{aligned} \tag{13}$$

and

$$C^T = \varepsilon \eta^t (-1)^{\frac{t}{2}(t-1)} C \tag{14}$$

3 Clifford algebras and the Majorana condition.

The allowed values for the signs η, ε labelling inequivalent $\{\Gamma^\mu, A, B, C\}$-structures associated to any given spacetime have been computed in [4]. A very efficient and much simpler method of computing η, ε is at disposal by explicitly using a Γ-structure in a Weyl representation. The choice of working within a Weyl representation can always be done and, due to the fundamental property that Γ-structures are all equivalent up to unitary conjugation [9], by no means affects the generality of the results so obtained. More than just reproducing previous results the computation through a Weyl representation encodes further information. Indeed we will show that Weyl-represented C charge-conjugation operators are left unchanged under a Wick rotation. When inequivalent charge-conjugation operators are present the tables provided below inform which charge-conjugations should be correctly chosen in performing analytical continuation to let's say the Euclidean space.

A Weyl representation of an even-dimensional ($D = 2n$) Γ-structure is defined by the property that the Γ^μ matrices are all symmetric or antisymmetric under transposition ($\Gamma^\mu = \pm {\Gamma^\mu}^T$). Moreover the number of symmetric equal the number of antisymmetric Γ^μ matrices ($= n$).

In odd dimensional spacetimes a further symmetric matrix, the Γ^5 introduced in (12) is presents.

A Wick rotation of a timelike $\overline{\mu}$ direction into a spacelike direction is represented on Γ-matrices by the rescaling $\Gamma^{\overline{\mu}} \mapsto i\Gamma^{\overline{\mu}}$, while the remaining Γ-matrices are left unchanged. Clearly the symmetric or antisymmetric character of the $\Gamma^{\overline{\mu}}$ matrix is not affected by a Wick rotation.

In a Weyl-represented even-dimensional Γ-structure we can introduce two inequivalent charge operators (i.e. realizing inequivalent $\{\Gamma^\mu, A, B, C\}$ assignments, see the discussion in the previous section) C_S and C_A defined as follows

$$\begin{aligned} C_S &= \prod_{i_S=1,\ldots,n} \Gamma^{i_S} \\ C_A &= \prod_{i_A=1,\ldots,n} \Gamma^{i_A} \end{aligned} \tag{15}$$

the products being restricted to symmetric (and respectively antisymmetric) Γ-matrices. As in the definition of the matrix A (8), the ordering of the products is irrelevant. Please notice that the index (S or A) labeling C reflects the construction, via symmetric or antisymmetric matrices, of the corresponding charge-conjugation operator and not its (anti)-symmetry property which is expressed by formula (14). ¿From (8) and (15) we obtain the relation $C_A \equiv C_S\Gamma^5$.

Clearly C_S and C_A are left invariant by Wick rotations up to an arbitrary phase, implying the convenience of the Weyl basis in discussing such an issue.

In odd-dimensional spacetimes a charge operator C can be introduced by using both formulas in (15). Due to the presence in this case of the extra Γ^5 among the symmetric matrices, the two definitions indeed collapse into a single one (modulo an arbitrary phase), recovering the well-known result that there exists a unique $\{\Gamma^\mu, A, B, C\}$-assignment, up to unitary transformations, in odd dimensions.

We recall that the A matrix is defined in (8), while $B_{S,A}$ are introduced from (5) as

$$B_{S,A} = A \cdot {C_{S,A}}^T \tag{16}$$

If we take into account the fact that timelike Γ-matrices are hermitian, it is just a matter of tedious but straightforward computations to check for both (S, A)-cases, which (± 1)-signs correspond to η_S, η_A, as well as ε_S, ε_A, introduced in the formulas (4) and (6).

In an (s, t) even-dimensional spacetime we obtain the following table

♠	0	2	4	6
η_S	$+$	$-$	$+$	$-$
η_A	$-$	$+$	$-$	$+$
ε_S	$+$	$+$	$-$	$-$
ε_A	$+$	$-$	$-$	$+$

(17)

where the even values characterizing the columns correspond to

$$X \quad = \quad s-t \quad mod \quad 8 \tag{18}$$

A similar table can be produced for odd-dimensional spacetimes. In this case no splitting between the S, A-cases is produced

♠	1	3	5	7
η	$-$	$+$	$-$	$+$
ε	$+$	$-$	$-$	$+$

(19)

As above the columns are marked by X given by (18).

Another sign, denoted by ξ and important for later considerations, is introduced through the position

$$B\Gamma^5 B^\dagger \quad = \quad \xi\Gamma^5 \tag{20}$$

where Γ^5 is the timelike extra-Γ matrix given in (12). We obtain

♠	0	2	4	6
ξ	$+$	$-$	$+$	$-$

(21)

Here as well columns correspond to X given in (18).

The Majorana reality condition on spinors is a constraint on the charge-conjugated spinor ψ^c introduced in (9), imposed to satisfy

$$\psi^c \quad = \quad \psi \tag{22}$$

Such a constraint can be consistently set only when $\varepsilon = +1$ (due to the combined result of applying the complex conjugation on (9) and the formula (7)).

One of the consequences read from the table (17) is the well-known result that Majorana spinors do not exist in the euclidean 4-dimensional space.[1]

The table (17) is useful for another purpose. It allows reading which choice of the C_S, C_A charge-conjugation operators should be adopted to mantain a Majorana reality condition if a Wick analytical continuation is performed. Indeed in the 0-column both the C_S and C_A charge-conjugation operators are consistent with the Majorana condition. Therefore e.g. the $(2,2)$ spacetime supports inequivalent Majorana spinors, based either on C_S or on C_A; conversely for the standard $(3,1)$-Minkowskian spacetime the Majorana condition is only defined w.r.t. C_S. Recalling the property that C_S, C_A are left unchanged

[1] Enlarged reality conditions which are applicable when $\varepsilon = -1$, like the $SU(2)$-reality condition proposed be Wetterich [3], will not be discussed in the present paper.

by Wick rotation, it turns out that only the $(2,2)$ C_S-Majorana spinors are Wick related to the $(3,1)$ Majorana spinors.

An euclidean space which supports Majorana spinors is the 10-dimensional one. We obtain the two following chains of Wick-related Majorana spinors:

$$\begin{array}{lcl} C_S & : & (10,0) \rightarrow (9,1) \rightarrow (6,4) \rightarrow (5,5) \rightarrow (2,8) \rightarrow (1,9) \\ C_A & : & (0,10) \rightarrow (1,9) \rightarrow (4,6) \rightarrow (5,5) \rightarrow (8,2) \rightarrow (9,1) \end{array} \tag{23}$$

The three potentially ambiguous cases are $(9,1)$, $(5,5)$ and $(1,9)$ which present both kinds of Majorana spinors.

4 Inequivalent real Clifford-Weyl structures.

Let a Clifford Γ-structure in a Weyl basis be denoted a Clifford-Weyl structure. In this section we provide an answer to the question: how many inequivalent real Clifford-Weyl structures do exist? To provide a solution we introduce an appropriate index labeling inequivalent structures.

The mathematical formulation of the problem is better phrased as finding the classes of equivalence of Γ-matrices up to orthogonal conjugation

$$\forall \mu, \quad \Gamma^\mu \quad \mapsto \quad O\Gamma^\mu O^T \tag{24}$$

with O $2^{[\frac{D}{2}]} \times 2^{[\frac{D}{2}]}$ real-valued, orthogonal ($OO^T = O^T O = \mathbf{1}$) matrices.

We already mentioned that the fundamental Pauli theorem guarantees that Γ-matrices are uniquely represented up to unitary conjugation; they however fit into different classes when just orthogonality is concerned.

One could ask whether this well-posed mathematical problem has sensible physical consequences. Indeed, as far as quantum mechanics is concerned, equivalent descriptions are provided by unitary-transformed states in a given Hilbert space. However, if some reality condition has to be imposed, it may well restrict the class of allowed transformations to be the orthogonal ones. Indeed this happens when e.g. the Majorana reality condition is imposed on spinors. Later we comment more on that.

The above mathematical problem finds the following solution.

Let an index I be defined for a D-dimensional (s,t)-spacetime ($D = s+t$) through the position

$$I \;=\; \frac{1}{2^{([\frac{D}{2}]+1)}} \cdot tr(\Gamma^\mu {\Gamma_\mu}^*) \tag{25}$$

The sum over repeated indices is understood. The normalization is chosen for a matter of convenience and as before $[\frac{D}{2}]$ denotes the integral part of $\frac{D}{2}$.

I is clearly left invariant by orthogonal transformations (24) while it is affected by unitary conjugations of Γ-matrices. It can be therefore used to label inequivalent classes of Γ-matrices up to orthogonal conjugation.

In a Weyl basis I can be easily computed. Indeed, as previously recalled, in such a basis Γ-matrices are either symmetric or antisymmetric. $\Gamma^{\mu *}$ coincides with Γ^μ up to a sign which is determined by both the time-like or space-like character of the μ direction, as well as the (anti)-symmetry nature of Γ^μ. It is a matter of straightforward computations to check the following results.

i) Let us consider at first an even ($D = 2n$) dimensional spacetime. We denote as t_A (s_A) the number of time-like (space-like) directions associated to antisymmetric Γ-matrices. The number of symmetric timelike (spacelike) matrices is therefore $t - t_A$ ($s - s_A$). In a Weyl basis the equality $t_A + s_A = n$ holds. For a Weyl assignment with t_A antisymmetric timelike matrices the index I takes the value

$$I = t - 2t_A \tag{26}$$

Let us introduce m given by $m = \min(s,t)$. We are free to choose among $m+1$ different Weyl assignments, $t_A = 0,...,m$, each one leading to a different value for I and therefore inequivalent under orthogonal conjugations. Indeed we obtain $m+1$ possible values for I,

$$-m + 2j \quad , \qquad j = 0,...,m \tag{27}$$

in the even-dimensional case.

ii) Let $D = 2n+1$ ($s+t = 2n+1$) be an odd-dimensional spacetime ($s+t = 2n+1$). An extra (the generalized-Γ^5 matrix here denoted Γ^{2n+1}) symmetric matrix is present w.r.t. the previous case. It could be associated either to a time-like or to a space-like direction according to the sign

$$\Gamma^{2n+1\,2} = \kappa = \pm 1 \tag{28}$$

Let as before t_A be the number of antisymmetric timelike Γ-matrices. The computation of the index I follows the same steps. We obtain

$$I = t - 2t_A - \tfrac{1}{2}(1-\kappa) \tag{29}$$

Notice that for a fixed (s,t)-spacetime the parity of I is determined by the sign of κ:

$$(-1)^I = \kappa(-1)^t \tag{30}$$

or conversely

$$\kappa = (-1)^{t+I} \tag{31}$$

which implies that the timelike or spacelike character of Γ^{2n+1} cannot be reverted by orthogonal conjugations. As before let us put $m = \min(s,t)$.

κ can be arbitrary chosen unless $m = 0$. I can assume $2m+1$ different values labeling corresponding inequivalent classes. For $m \neq 0$ they are given by

$$\begin{array}{l} m - 2j - \frac{1}{2}(1-\kappa) \\ \kappa = \pm 1, \qquad j = 0,1,...,m - \frac{1}{2}(1+\kappa) \end{array} \tag{32}$$

For $m = 0$ either we have $I = +1$ in the $(2n+1,0)$ case or $I = -1$ in the $(0,2n+1)$ case.

Let us discuss now a possible application of the above construction to the Majorana reality condition. A standard result (see [4]) states that for $\varepsilon = 1$, i.e. the consistency requirement for the Majorana condition, the B matrix introduced in (5) can be unitary-transformed (10) to the identity matrix

$$\exists U \quad s.t. \quad U^* B U^\dagger = \mathbf{1} \tag{33}$$

The choice $B \equiv \mathbf{1}$ corresponds to the so-called Majorana representation ($\psi^c = \psi^*$). The orthogonal transformations are the unitary transformations acting on B and preserving the Majorana representation

$$U^* \mathbf{1} U^\dagger = \mathbf{1} \quad \Rightarrow \quad U U^T = U^T U = \mathbf{1} \tag{34}$$

¿From (4) in the Majorana representation we have $\Gamma^{\mu *} = \eta \Gamma^\mu$, so that the index I takes the value

$$I \;\; = \;\; \eta D \tag{35}$$

In this particular case the information furnished by the index I is reduced to the same information provided by η, ε. The logics behind is however different. η, ε label inequivalent classes under unitary transformations of a richer $\{\Gamma^\mu, A, B, C\}$-structure, while I corresponds to inequivalent classes of orthogonal transformations of just a Γ-structure (in the Majorana realization). It deserves a careful investigation to determine whether for other choices of reality conditions which can select physical fields (such as the $SU(2)$-Majorana condition on spinors) the index I can refine the standard classification and be physically meaningful. In a different but related context we already found [10] a physical application where the index plays a non-trivial role.

5 Free Hermitian actions.

The most general lagrangian involving free spinorial fields is given by

$$\alpha \cdot \overline{\psi} \Gamma^\mu \partial_\mu \psi + \beta \cdot \overline{\psi} \psi + \gamma \cdot \overline{\psi} \Gamma^\mu \Gamma^5 \partial_\mu \psi + \delta \cdot \overline{\psi} \Gamma^5 \psi \tag{36}$$

The third (pseudokinetic) and the fourth (pseudomassive) term involve the Γ^5 matrix defined in (12) and are present in even D-dimensional spacetimes only.

The transposition acting on anticommuting fields ζ, ψ satisfies

$$(\zeta \cdot \psi)^T \;\; = \;\; -\psi^T \cdot \zeta^T \tag{37}$$

while the hermitian conjugation can be conventionally defined, without losing generality, according to

$$(\zeta \cdot \psi)^\dagger \;\; = \;\; \psi^\dagger \cdot \zeta^\dagger \tag{38}$$

(as for the complex conjugation, it follows from (37) and (38)).

Demanding the hermiticity of the action, i.e. of the (36) lagrangian, fixes unambiguously the nature, real or imaginary, of the coefficients in (36). Straightforward computations lead to the table

♠	0	1	2	3
α	**R**	**I**	**I**	**R**
β	**R**	**R**	**I**	**I**
γ	**I**	**I**	**R**	**R**
δ	**R**	**I**	**I**	**R**

(39)

the columns are labeled by $t \quad mod \quad 4, \quad t$ being the number of time-like dimensions.

The table above is useful e.g. in finding mass-shell properties. Indeed in the case of a theory involving, let's say, massive and/or pseudomassive terms, the mass-shell condition reads as follows

$$p^2 = \frac{1}{\alpha^2}(\delta^2 - \beta^2) \tag{40}$$

where α, β, δ enter (36). The hermiticity requirement (39) allows to set e.g. in the $(3,1)$ Minkowski spacetime $\alpha = i$, $\beta = m$, $\delta = im_5$, with m, m_5 real, so that $p^2 = m^2 + {m_5}^2$ is necessarily positive. In a $(2,2)$-spacetime we only need to change the definition of β, which must be imaginary, by setting $\beta = im$. The mass-shell condition reads $p^2 = {m_5}^2 - m^2$. A vanishing value can be found even for $m, m_5 \neq 0$ provided that $m = m_5$.

6 Majorana constraints on the dynamics.

In this section we analyze the constraints put by the (9) Majorana condition on the dynamics of free spinors.

From the (36) lagrangian we derive the equation of motion

$$\alpha\Gamma^\mu\partial_\mu\psi + \beta\psi + \gamma\Gamma^\mu\Gamma^5\partial_\mu\psi + \delta\Gamma^5\psi = 0 \tag{41}$$

The above equation of motion is compatible with the (9) Majorana condition provided the coefficients are constrained to satisfy

$$\begin{aligned}
\alpha^* &= \chi\cdot(\eta\alpha)\\
\beta^* &= \chi\cdot\beta\\
\gamma^* &= \chi\cdot(\eta\xi\gamma)\\
\delta^* &= \chi\cdot(\xi\delta)
\end{aligned} \tag{42}$$

where η, ξ have been introduced in (4) and (6) respectively. The common factor χ, as far as the equation of motion alone is concerned, is an arbitrary phase

$$|\,\chi\,|^2 = 1 \tag{43}$$

The derivation of the (41) equation of motion from a lagrangian puts further constraints. Each Majorana-constrained $\mathcal{L}_i$ $(i = 1, ..., 4)$ term appearing in (36), in order to be non-vanishing, must be symmetric, i.e.

$${\mathcal{L}_i}^T = \mathcal{L}_i \tag{44}$$

For an (s,t)-spacetime $(s+t=D)$ this so happens when the following signs assume the $+1$ value:

i) for the kinetic term the sign is λ given by

$$\lambda \;=\; -\varepsilon\eta^{t+1}(-1)^{\frac{t}{2}(t+1)} \tag{45}$$

ii) for the massive term, μ

$$\mu \;=\; -\varepsilon\eta^{t}(-1)^{\frac{t}{2}(t-1)} \tag{46}$$

iii) for the pseudokinetic term, λ_5

$$\lambda_5 \;=\; \lambda(-1)^{\frac{D}{2}} \tag{47}$$

iv) for the pseudomassive term, μ_5

$$\mu_5 \;=\; \mu(-1)^{\frac{D}{2}} \tag{48}$$

The following tables, specifying which spacetimes support the existence of non-vanishing kinetic and massive terms, can be produced. For even-dimensional spacetimes we have

♠	0	2	4	6
λ_S	$1,2$	$0,1$		
λ_A	$0,1$			$1,2$
μ_S	$2,3$	$1,2$		
μ_A	$1,2$			$2,3$

(49)

Some comments are in order. The columns are labeled by $X = s-t \quad mod \quad 8$. The index S or A is referred to the corresponding charge-conjugation (either C_S or C_A). The entries are evaluated only when $\varepsilon = 1$ (Majorana consistency requirement); the $\sharp$'s in the entries specify for which number of t time-like directions

$$t \;=\; \sharp \quad mod \quad 4 \tag{50}$$

the sign in the associated row assumes the $+1$ value.

Similarly, for odd-dimensional spacetimes, we have

♠	1	3	5	7
λ	0,1			$1,2$
μ	$1,2$			$2,3$

(51)

(same meaning for the symbols).

The next question to be answered is whether the action associated to the (36) lagrangian admits a charge-conjugation which allows to consistently introduce the Majorana condition. This point has been raised in [6]. The existence of a charge conjugation requires

$$\mathcal{L}^* = \mathcal{L} \tag{52}$$

and is automatically guaranteed from the non-vanishing condition $\mathcal{L}^T = \mathcal{L}$ once assumed the hermiticity of the action ($\mathcal{L}^\dagger = \mathcal{L}$), i.e. when the coefficients are chosen to satisfy the table (39).

It turns out that the phase χ appearing in (43) is no longer arbitrary now but fixed to be

$$\chi = -\eta^t \tag{53}$$

From the (49) and (51) tables above we can extract some particular results, e.g. that massive lagrangians for Majorana spinors exists in

$i)$ $t = 1 \quad mod \quad 4$ spacetimes (for $\eta = -1$) when

$ia)$ $s - t = 0 \quad mod \quad 8$ (for the C_A charge-operator),

$ib)$ $s - t = 2 \quad mod \quad 8$ (for the C_S charge-operator),

$ic)$ $s - t = 1 \quad mod \quad 8$,

as well as in

$ii)$ $t = 2 \quad mod \quad 4$ (for $\eta = +1$) when

$iia)$ $s - t = 0 \quad mod \quad 8$ (for the C_S charge-operator),

$iib)$ $s - t = 6 \quad mod \quad 8$ (for the C_A charge-operator),

$iic)$ $s - t = 7 \quad mod \quad 8$.

The role of s, t can be interchanged as recalled in section 2.

In the case of odd-dimensional spacetimes the table (51) provides further information. Kinetic (K) or massive (M) terms are only allowed in D-dimensional spacetimes according to

$$\begin{array}{llll} D = 1 & mod & 8 & \{K\} \\ D = 3 & mod & 8 & \{K, M\} \\ D = 5 & mod & 8 & \{M\} \\ D = 7 & mod & 8 & \{...\} \end{array} \tag{54}$$

Up to $D = 11$ dimensions the list of odd-dimensional spacetimes supporting Majorana spinors is given by

$$\begin{array}{rlllll} \{K, M\} & : & (2,1), & (10,1), & (9,2), & (6,5) \\ \{K\} & : & (1,0), & (9,0), & (8,1), & (5,4) \\ \{M\} & : & (2,3) & & & \\ \{...\} & : & (7,0), & (4,3) & & \end{array} \tag{55}$$

For even-dimensional spacetimes (up to $D = 10$) an useful table can be written

♠	2	4	6	8	10
0	$S-KP$		$A+$	$S+$ $A-K$	$S-KP$
1	$S+KP$ $A-KM$	$S-KMP$		$A+K$	$S+KP$ $A-KM$
2	$A+KM$	$S+KMP$ $A-$	$S-$		$A+KM$
3	•	$A+$	$S+$ $A-$	$S-$	
4	•		$A+$	$S+$ $A-K$	$S-KP$
5	•	•		$A+K$	$S+KP$ $A-KM$
6	•	•	$S-$		$A+KM$
7	•	•	•	$A+$	
8	•	•	•	$S+$ $A-K$	$S-KP$
9	•	•	•	•	$S+KP$ $A-KM$
10	•	•	•	•	$A+KM$

(56)

It contains the following informations. Columns are labeled by D, rows by t. Each entry is evaluated for $\varepsilon = 1$. The presence of S or A denotes if the corresponding charge-operator defines a Majorana spinor. The sign $(\pm)$ represents the corresponding value of η. The presence of K, M, P denotes if the kinetic (K), massive (M) or pseudomassive (P) term in the (36) lagrangian can be non-vanishing. These last two terms have been evaluated only when the corresponding kinetic term is nonzero. The pseudokinetic term has not been inserted here since its physical interpretation is problematic (due to the presence of negative-normed states, which however can be eliminated if projected out, as in the Majorana-Weyl case discussed in the next section).

Since the same results are repeated $mod \quad 8$, both in s and in t, the following compact information can be extracted. Majorana spacetimes with a non-vanishing kinetic term can be found in

$$\begin{aligned} D &= 0 \quad mod \quad 8: \quad \{A, K\} \\ D &= 2 \quad mod \quad 8: \quad either \quad \{S, KP\} \quad or \quad \{A, KM\} \\ D &= 4 \quad mod \quad 8: \quad \{S, KMP\} \end{aligned} \tag{57}$$

In $D = 2, 10, \ldots$ either a massive or a pseudomassive term could be present, according to the choice of the charge-conjugation operator. Simultaneous presence of massive and pseudomassive terms is allowed in $D = 4, 12, \ldots$ dimensions only.

7 The Majorana-Weyl conditions.

In order to make this paper self-consistent we review in this section the status of Majorana-Weyl spinors and present a complete list of results.

In $D = 2n$ even-dimensional spacetimes the projectors

$$P_{R,L} \equiv (\frac{\mathbf{1} \pm \Gamma^5}{2}) \tag{58}$$

(where Γ^5 has been introduced in (12)) allow defining chiral (Weyl) spinors $\psi_{R,L}$ as

$$\psi_{R,L} = P_{R,L}\psi \tag{59}$$

Majorana-Weyl spinors, satisfying both the condition (9) and the projection (59), can be consistently defined (see [4]) in spacetimes such that

$$s - t = 0 \quad mod \quad 8 \tag{60}$$

therefore in particular in all (n, n) spacetimes. Up to 10 dimensions the remaining spacetimes supporting Majorana-Weyl spinors are the euclidean $(8, 0)$ space and the minkowskian $(9, 1)$ spacetime.

In any spacetime satisfying (60) Majorana-Weyl spinors can be introduced for both C_S and C_A charge-conjugation operators. The list of results presented below holds in both cases.

Let us first recall that $\overline{\psi} = \psi^T C$ under the condition (9) and that moreover the $C_{S,A}$ charge-operator is respectively block-diagonal or block-antidiagonal according if n is even or odd. As a consequence kinetic (K) and massive (M) terms can either mix (denoted in such case as K_{xy}, M_{xy}) chiralities or not (K_{xx}, M_{xx}). We can write

$$\begin{aligned} K_{xx} &\equiv {\psi_{R,L}}^T C\Gamma^\mu\partial_\mu\psi_{R,L} \\ M_{xx} &\equiv {\psi_{R,L}}^T C\psi_{R,L} \end{aligned} \tag{61}$$

and

$$\begin{aligned} K_{xy} &\equiv {\psi_R}^T C\Gamma^\mu\partial_\mu\psi_L + \lambda{\psi_L}^T C\Gamma^\mu\partial_\mu\psi_R \\ M_{xy} &\equiv {\psi_R}^T C\psi_L + \mu{\psi_L}^T C\psi_R \end{aligned} \tag{62}$$

The "mixed" terms K_{xy}, M_{xy} can always be chosen to be non-vanishing. It is sufficient for this purpose to conveniently fix the relative sign between the two terms in the r.h.s. of (62). This is done in (62), the two signs λ and μ coincide with their values given in (45) and (46) respectively.

Conversely the K_{xx} and M_{xx} terms could be identically zero according to the (anti-)symmetry properties of C.

Let us now introduce $\upsilon = (-1)^n$. From the previous remarks on the block-character of C we have that

$$\begin{aligned} K_{\upsilon=+1} \equiv K_{xy} \quad &, \quad M_{\upsilon=+1} \equiv M_{xx} \\ K_{\upsilon=-1} \equiv K_{xx} \quad &, \quad M_{\upsilon=-1} \equiv M_{xy} \end{aligned} \tag{63}$$

The most general free lagrangian for Majorana-Weyl spinors in $D = 2n$ dimensions can be expressed as

$$\mathcal{L} = \alpha K_v + \beta M_v \tag{64}$$

The formula (63) specifies which kind of kinetic and which kind of massive term could appear in $D = 2n$. The coefficients α, β are either real or imaginary according to the table (39).

The last feature to be computed is in which dimensions the K_{xx}, M_{xx} terms are not identically vanishing. The final results can be summarized in the following table, which presents the types of allowed kinetic and massive terms in accordance with the dimensionality D of the spacetime

$$\begin{array}{llll} D = 0 & mod & 8, & \{K_{xy}\} \\ D = 2 & mod & 8, & \{K_{xx}, M_{xy}\} \\ D = 4 & mod & 8, & \{K_{xy}, M_{xx}\} \\ D = 6 & mod & 8, & \{M_{xy}\} \end{array} \tag{65}$$

The list of results presented in this section removes any possible ambiguities and completely determines all features of the free actions for Majorana-Weyl spinors in any spacetime.

8 Conclusions.

This paper has been devoted to discuss real structures in Clifford algebras and Majorana conditions in any space-time. The Weyl representation for Clifford algebras has been employed to analyze Γ-structures and Majorana spinors. An index, which to our knowledge has not been discussed before at least in the physicists' literature, has been introduced. It classifies Γ-structures up to orthogonal conjugations.

For what concerns Majorana spinors, some of the issues here discussed have not been considered in previous papers. We can mention e.g. the interplay between the hermiticity condition, the charge-conjugation and the non-vanishing condition for the (36) lagrangian. The different role played by the even-dimensional charge operators C_S, C_A (15), invariant under Wick rotations, is another example.

We have furnished a series of tables presenting an exhaustive list of results concerning Majorana and Majorana-Weyl spinors. They include in particular the non-vanishing conditions in any given space-time for kinetic, massive and pseudomassive terms, in association with each charge operator C_S, C_A (in even dimensions), as well as the type of coefficients (39) and the kind of terms (in the Majorana-Weyl case) entering the free lagrangian (36).

One of our main motivations for presenting here such a systematic list of results concerns their relevance in analyzing supersymmetries in generic pseudoeuclidean spacetimes. Their connection with supergravities, strings, brane dynamics, etc., could be explored (for a recent review of this topic in standard Minkowskian spacetimes see e.g. [11]). In the introduction we mentioned why this issue could be important. Problems like Kaluza-Klein compactifications, dimensional reductions, analytical continuations to the euclidean spaces, are among those which have to be addressed.

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References

[1] F. Gliozzi, J. Scherk and D. Olive, *Nucl. Phys.* **B122** (1977) 253.

[2] J. Scherk, in *Recent Developments in Gravitation*, ed. M. Lévy and S. Deser, Plenum Press, New York (1979).

[3] C. Wetterich, *Nucl. Phys.* **B211** (1983) 177.

[4] T. Kugo and P. Townsend, *Nucl. Phys.* **B221** (1983) 357.

[5] M. F. Sohnius, *Phys. Rep.* **128** (1985) 39.

[6] M.A. De Andrade, *Mod. Phys. Lett.* **A 10** (1995) 161.

[7] I. Bars, C. Deliduman and O. Andreev, *Phys. Rev.* **D 58** (1998) 066004; I. Bars, *Phys. Rev.* **D 58** (1998) 066006; I. Bars, Preprint hep-th/9810025.

[8] C.M. Hull, "Duality and the Signature of Space-Time", Preprint QMW-PH-98-30, he-th/9807127.

[9] J.J. Sakurai, "Advanced Quantum Mechanics", Addison Wesley 10^{th} printing (1967) USA.

[10] L.P. Colatto, M.A. De Andrade and F. Toppan, "Matrix-Spacetimes and a $2D$ Calculus in Any Even Dimension", Preprint CBPF-NF-063/98, hep-th/9810145.

[11] P.C. West, "Supergravity, Brane Dynamics and String Duality", Preprint KCL-MTH-98-55, hep-th/981101